Nonlinear sigma models for (super-)spin chains

1. What's a nonlinear sigma model (NLσM)?

1.1 Classical field theory

1.1.1 Symmetric spaces

1.1.2 Energy/action functional

1.1.3 Global G-symmetry & conserved current

1.1.4 Some useful formulas

1.2 Quantum field theory

1.2.1 Lattice definition/regularization.

1.2.2 Universality hypothesis

1.2.3 Main results/conjectures

1.2.4 Perturbative renormalization group

2. Antiferromagnetic quantum spin chains

2.1 Motivation / background

2.2 Heisenberg model

2.3 G/K spin-coherent states

2.4 From spin chain to NLσM

3. Superspin chain

3.1 Exterior algebra

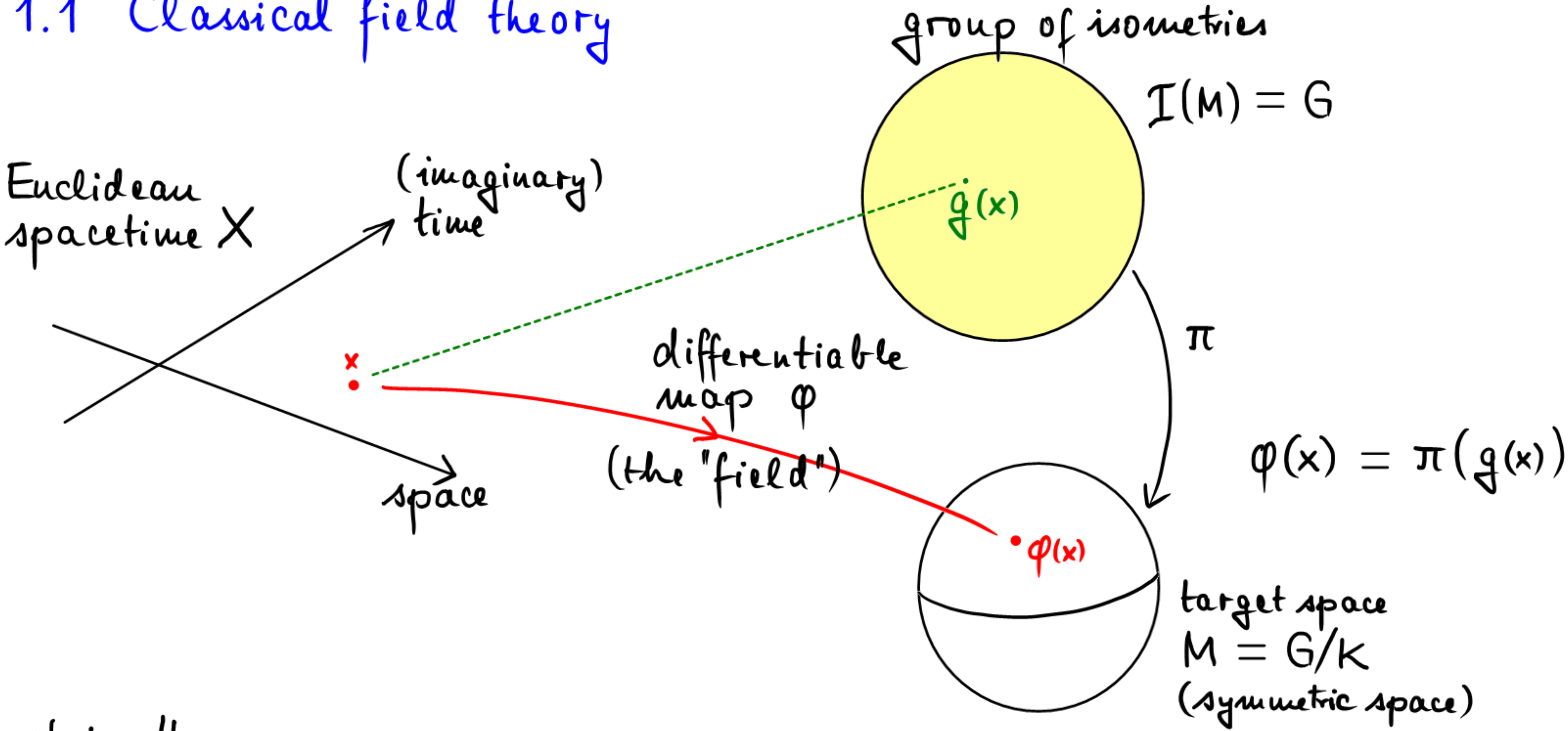
3.2 Bosonization formula

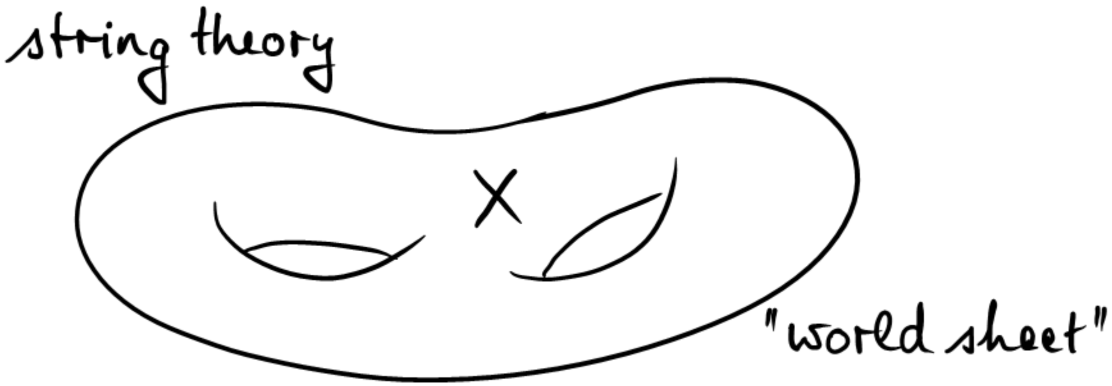
3.3 A glimpse of super...

3.4 Random band matrices and NLσM

3.5 Suggestion

1.1 Classical field theory





1.1.1 Symmetric spaces

Riemann tensor: $R^i{}_{jkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^m_{lj}\Gamma^i_{km} - \Gamma^m_{kj}\Gamma^i_{lm}$

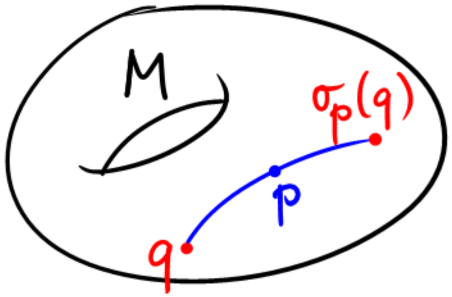
Def.: a locally symmetric space is a Riemannian manifold M with covariantly constant curvature: $\nabla R = 0$.

Ex. 1: the round two-sphere $M = S^2$, $dl^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

Ex. 2: the set $M = Gr_n(\mathbb{C}^N)$ of complex n-planes in $\mathbb{C}^N$ (endowed with the $U(N)$-invariant geometry).

Def.: a *globally* symmetric space is a Riemannian manifold M s.t.

for all $p \in M$ the geodesic inversion extends to an isometry $\sigma_p : M \to M$.



Fact. $M = G/K$ with $K = \mathrm{Fix}_G(\theta)$ for $\theta : G \to G$ (Cartan involution).

Ex.: $M = \mathcal{U}$ ($\mathcal{U}$ compact Lie group).

$G = \mathcal{U} \times \mathcal{U}$, $\quad \theta(u_L, u_R) = (u_R, u_L)$.

$K = \mathrm{Fix}_G(\theta) \cong \mathcal{U}$.

$\pi : G \to G/K = M$, $\quad (u_L, u_R) \mapsto u_L u_R^{-1}$.

G-action on M: $\quad u \mapsto u_L u u_R^{-1}$.

1.1.2 Energy/action functional.

$$S[\varphi] = \int_X \|D\varphi\|^2 .$$

In local coordinates $\xi^i : M \to \mathbb{R}$, $d\xi^i(e_j) = \delta^i_j$, $\varphi^i = \xi^i \circ \varphi$, $x^\mu : X = \mathbb{R}^d \to \mathbb{R}$, one has $D\varphi = dx^\mu \frac{\partial \varphi^i}{\partial x^\mu} e_i$

and $$S[\varphi] = \int_X d^dx \; g^{\mu\nu}_{(X)}(x) \, \frac{\partial \varphi^i}{\partial x^\mu}(x) \, \frac{\partial \varphi^j}{\partial x^\nu}(x) \; g^{(M)}_{ij}(\varphi(x))$$

where $g^{(M)}_{ij} = (e_i, e_j)_M$ and $g^{\mu\nu}_{(X)} = (dx^\mu, dx^\nu)_X$.

1.1.3 Global G-symmetry.

$$(g\varphi)(x) := g(x) \cdot \varphi(x).$$

If $g(x) = g_0 \in G$ is constant, then $S[\varphi] = S[g\varphi]$ for any φ.

Conserved current. Let $Y: X \to \mathrm{Lie}(G)$ be differentiable, and for $t \in [-\varepsilon, \varepsilon]$ exponentiate to $e^{tY}: X \to G$. Then

$$\left.\frac{d}{dt}\right|_{t=0} S[e^{tY}\cdot\varphi] = \int_X dY \wedge J_\varphi = \int_X d^dx\, \partial_\mu Y^a J^\mu_a$$

where J_φ is $(d-1)$ form on X with values in $\mathrm{Lie}(G)^*$.

If φ is critical point of S
($\Leftrightarrow$ solution of classical eqs of motion $\delta S = 0$),
then $\boxed{dJ_\varphi = 0}$ (conservation of current).

1.1.4 Some useful formulas.

A. Differential of exponential map: $\mathrm{Lie}(G) \longrightarrow G$.

$$\left.\frac{d}{dt}\right|_{t=0} e^{-X} e^{X+tY} = \frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)} Y$$

$$= Y - \frac{1}{2}[X,Y] + \frac{1}{6}[X,[X,Y]] - \dots$$

B. Cartan decomposition of Lie algebra.

Cartan involution $\theta : G \longrightarrow G$, $\theta(g_1 g_2) = \theta(g_1)\theta(g_2)$,

induces $\theta_* : \mathrm{Lie}(G) \longrightarrow \mathrm{Lie}(G)$, $\theta_*([X,Y]) = [\theta_*(X), \theta_*(Y)]$.

$\leadsto$ $\mathrm{Lie}(G) = E_{+1}(\theta_*) \oplus E_{-1}(\theta_*) =: \mathfrak{k} \oplus \mathfrak{p}$

with $[\mathfrak{p},\mathfrak{p}] \subseteq \mathfrak{k}$ and $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$.

Hence for $X, Y \in \mathfrak{p}$ one has $\left(\frac{1-e^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)} Y\right)_{\mathfrak{p}} = \frac{\sinh \mathrm{ad}(X)}{\mathrm{ad}(X)} Y$.

C. Geometry of G/K in (Riemann) normal coordinates.

Let $o \equiv eK \in G/K$, and note $T_o(G/K) = \mathfrak{p}$.

G-invariant metric $= \mathrm{Tr}\,(g^{-1}dg)^2_{\mathfrak{p}} \equiv \varkappa$.

Parametrize points : $\quad q = e^X \cdot o \equiv e^X K \quad (X \in \mathfrak{p})$.

Tangent vectors : $\quad v = \frac{d}{dt}\Big|_{t=0} e^{X+tY} \cdot o \quad (Y \in \mathfrak{p})$.

$$\varkappa_q(v,v) = \varkappa_o(e^{-X}\cdot v, e^{-X}\cdot v) = \mathrm{Tr}\left(\frac{\sinh \mathrm{ad}(X)}{\mathrm{ad}(X)}\, Y\right)^2.$$

G-invariant measure $= \mathrm{Det}\, \frac{\sinh \mathrm{ad}(X)}{\mathrm{ad}(X)}\Big|_{\mathfrak{p}\to\mathfrak{p}} \cdot d\mathrm{vol}(X)$.

Exercises

A.1 Argue that the round two-sphere is a globally symmetric space.

A.2 Argue that $Gr_n(\mathbb{C}^N) \cong U(N) / U(n) \times U(N-n)$.

A.3 For $G = SU(2)$ with Cartan involution $\Theta(u) = \sigma_3 u \sigma_3$ compute the decomposition of Lie $SU(2)$ into Θ_*-even/odd parts.

B. For a compact Lie group G of matrices g with Cartan involution Θ consider the NLσM with target space $M = G/K$ and energy functional

$$S = \frac{1}{4} \int_X d^dx \, \mathrm{Tr}\, \partial_\mu Q \, \partial_\mu Q^{-1} \quad \text{where} \quad Q(x) = g(x)\, \Theta(g(x))^{-1}.$$

Verify the following claims:

B.1 S indeed depends only on $\varphi(x) = \pi(g(x)) \equiv g(x) K \in G/K$.

B.2 In terms of $g(x)$, S has the expression $S = -\int_X d^dx \, \mathrm{Tr}\,(g^{-1}\partial_\mu g)_{\mathfrak{p}}^2$.

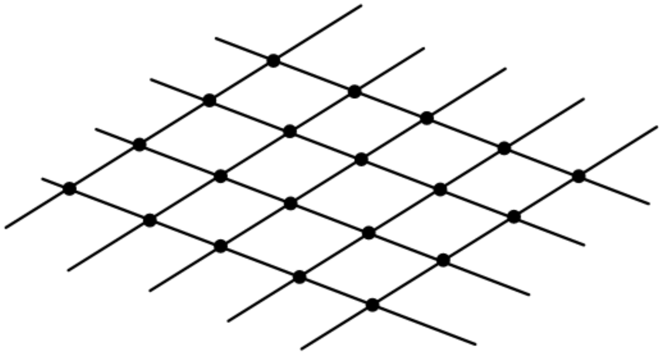
B.3 The conserved current is $J^\mu = -2\, g\, (g^{-1}\partial^\mu g)_{\mathfrak{p}}\, g^{-1}$.

(Cartan decomposition $\mathrm{Lie}\, G \ni Y = Y_{\mathfrak{k}} + Y_{\mathfrak{p}}$.)

1.2 Quantum field theory

One wants to take averages (with weight $e^{-S/\hbar}$) over all maps (?!)

1.2.1 Lattice definition/regularization.



(finite) cubic lattice $\Lambda = \mathbb{Z}_L^d$

$|\Lambda| = L^d, \quad L \to \infty.$

Discretized maps
$$\varphi : \Lambda \longrightarrow M, \qquad x \longmapsto \varphi_x .$$

Require S_Λ to be local, Λ-symmetric, G-invariant.

This still leaves a vast number of possible choices ...

$$S_{\Lambda,\alpha}[\varphi] = \alpha_1 \sum_{\substack{x,y\in\Lambda^2 \\ \text{neighbors}}} \mathrm{dist}^2(\varphi_x, \varphi_y) + \ldots$$

parameters $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\infty)$.

Gibbs measure $= \frac{1}{Z} e^{-\frac{1}{\hbar} S_{\Lambda,\alpha}[\varphi]} \prod_{x \in \Lambda} d\mathrm{vol}(\varphi_x)$

is a G-invariant density on $M^{|\Lambda|}$;

$\hbar \equiv T$ (Planck's constant, or temperature).

Observables $A, B : M \to \mathbb{R}$.

Correlation functions: $\langle A(\varphi_x) B(\varphi_y) \rangle_{S_{\Lambda,\alpha}} =: C_{A,B}(x, y; \alpha)$.

1.2.2 Universality hypothesis (rough statement).

In the continuum limit (α_1 large) there exists a function $\xi(\alpha)$ (called the correlation length) and for each pair A, B a function $\gamma_{A,B} : \mathbb{R}_+^2 \to \mathbb{R}$ such that $C_{A,B}(x, y; \alpha) \overset{\text{C.L.}}{=} \gamma_{A,B}(|x-y|; \xi(\alpha))$.

(In this sense the multitude of lattice theories is governed by a single master theory — the nonlinear sigma model.)

1.2.3 Main results / conjectures

- Mermin–Wagner–Coleman Theorem.

  Let $d = 1,2$ and M compact. If $A : M \to \mathbb{R}$ is orthogonal to the constants $\left(\int_M A \, d\mathrm{vol}_M = 0\right)$, then

  $$\langle A(\varphi_x) A(\varphi_y) \rangle \longrightarrow 0 \quad \text{for } |x-y| \longrightarrow \infty .$$

- Mass gap conjecture.

  For $d = 1, 2$ and M compact with positive curvature correlations decay exponentially: $\langle A(\varphi_x) A(\varphi_y) \rangle \sim e^{-c_A |x-y|/\xi}$ for all temperatures.

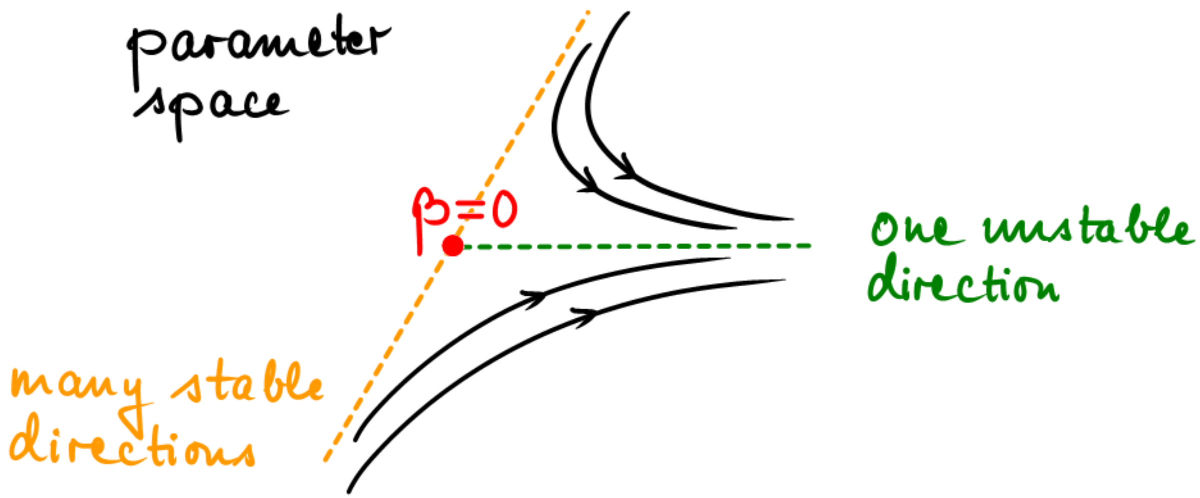
- In $d \geq 3$ spontaneous symmetry breaking occurs for low temperatures.

1.2.4 Perturbative renormalization group (RG)

— K.G. Wilson: $\exists$ vector field (the "beta function") in parameter space s.t. $\left(\frac{\partial}{\partial s} + \beta^i(\alpha)\frac{\partial}{\partial \alpha^i}\right) C_{A,B}(e^{-s}x, e^{-s}y; \alpha) = 0$.

dynamical-systems viewpoint: $\dot{\alpha}^i = \beta^i(\alpha)$.

— Universality occurs near zeroes of β (or fixed points of the RG flow): there, the flow is contracting in all directions but very few (just one for $d=1,2$ and M compact).



– β-function of nonlinear models in $d=2$

(M Riemannian manifold with metric $\frac{1}{T} g_{ij}$):

$$\frac{d}{ds}\left(\frac{1}{T} g_{ij}\right) = -Ric_{ij} - \frac{T}{2} R_{iklm} R_{jklm} + \mathcal{O}(T^2).$$

– M compact symmetric space (choose $g_{ij} \equiv Ric_{ij}$):

$$\frac{d}{ds}\left(\frac{1}{T}\right) = -1 + \mathcal{O}(T)$$

$\rightsquigarrow \left(\frac{\partial}{\partial s} + \beta(1/T)\frac{\partial}{\partial(1/T)}\right) e^{s}\xi(T) = 0 \quad \rightsquigarrow \quad \xi(T) \sim e^{1/T}.$

2. Antiferromagnetic quantum spin chains

2.1 Motivation / background

- Provide a physical example realizing NLσM.
- Key words: topological quantum matter, classification, interacting topological insulators, AKLT spin chain, Haldane phase, matrix product states ...
- Phenomenology:

Ferromagnetic spin chains have low-energy excitations with dispersion $\varepsilon(k) \sim k^2$.

Antiferromagnetic spin chains (naively) have $\varepsilon(k) \sim |k|$ (relativistic dispersion) $\rightsquigarrow$ NLσM (Haldane).

(c.f. Affleck, Les Houches 1988)

2.2 Heisenberg model

V highest-weight irrep for (compact) Lie group G.

$\cdots \otimes V \otimes V^* \otimes V \otimes V^* \otimes \cdots$ Hilbert space of spin chain.

$\{X_a\}$ orthonormal basis of $\mathrm{Lie}(G)$.

$$\mathrm{Cas}_2 = \sum_a X_a X_a \in \mathcal{U}(\mathrm{Lie}(G)).$$

$$H = -\sum_n \sum_a X_a^{(V)}(2n)\left(X_a^{(V^*)}(2n-1) + X_a^{(V^*)}(2n+1)\right)$$

Hamiltonian of antiferromagnetic quantum spin chain ("Heisenberg").

2.3 G/K spin-coherent states

$\Theta : G \longrightarrow G$ Cartan involution

$|v_0\rangle \in V$ highest-weight vector for G-action.

Let $\mathbb{C}\cdot|v_0\rangle$ be one-dimensional repn for $K = \mathrm{Fix}_G(\Theta)$.

$\{\mathbb{P}\, g|v_0\rangle\}_{g\in G} \cong G/K$ the G-orbit of coherent states.

Resolution of the identity.

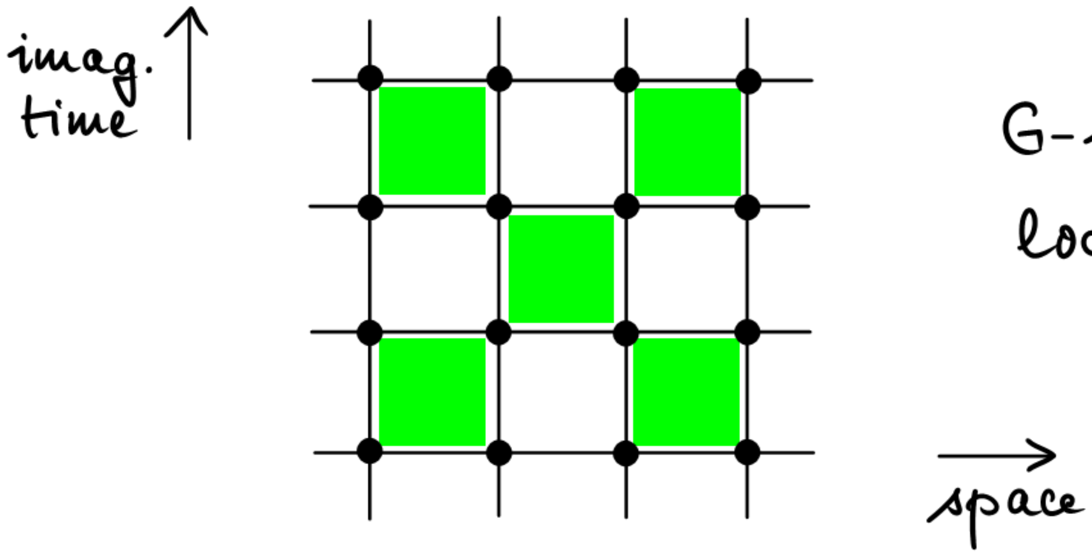
For G compact there exists a G-invariant density dg_K s.th.

$$\mathbb{1}_V = \int_{G/K} dg_K \, g|v_0\rangle\langle v_0| g^{-1}.$$

Proof as an exercise.

2.4 From spin chain to NLσM

Use the Feynman–Trotter–Suzuki method to convert the quantum partition function $Z = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr} \left(e^{-\frac{\beta}{L} H}\right)^L$ (and correlation fcts) into an integral over many copies of G/K.



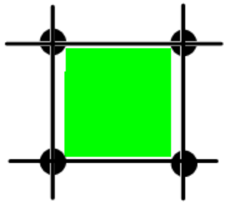
G-invariance ✓

locality ✓

Continuum limit = NLσM ?

Control achievable only in a semiclassical limit:

↝ scale the highest weight $\left(|v_0\rangle \to |N v_0\rangle \right)$ and consider $N \to \infty$ based on $\cdots \otimes |N v_0\rangle_V \otimes |-N v_0\rangle_{V^*} \otimes |N v_0\rangle_V \otimes |-N v_0\rangle_{V^*} \otimes \cdots$

Two spins. 

$$H_2 = -\frac{1}{N}\sum_a X_a^{(V)} X_a^{(V^*)}$$

For N large, approximate $G/K \times G/K \overset{!}{=} T(G/K) = G \times_K \mathfrak{p}$,

i.e. parametrize coherent states as

$$g\, e^{Y/2}|N v_0\rangle_V \otimes g\, e^{-Y/2}|-N v_0\rangle_{V^*} \equiv |g; Y\rangle = |gk; k^{-1} Y k\rangle .$$

Matrix element: $\langle g\, e^{Z/2}; Y' |\, e^{-\frac{\beta}{L} H_2} \,|\, g\, e^{-Z/2}; Y\rangle$

$$= \exp\left(\frac{\beta}{2L} N \,\mathrm{Tr}\,(Y^2 + Y'^2) + iN\, \omega_0(Z, Y+Y') + \dots \right).$$

Kähler form
Berry curvature

Integration over Y, Y' gives $e^{\frac{L}{\beta} N \,\mathrm{Tr}\, Z^2 + \dots}$

Haldane conjecture (N even).

3. Superspin chain

3.1 Exterior algebra.

$V = \mathbb{C}^N$ complex vector space, V^* dual space.
(the linear fcts on V)

Def.: The Grassmann algebra $\Lambda(V^*) = \bigoplus_{k=0}^{N} \Lambda^k(V^*)$ is the associative algebra generated by $\mathbb{C} \equiv \Lambda^0(V^*)$ and $V^* \equiv \Lambda^1(V^*)$ with relations $\alpha\beta + \beta\alpha = 0$ for any $\alpha, \beta \in V^*$.

Note: $\omega^2 = 0$ for $\omega \in \Lambda^{odd}(V^*)$.

The inner product of $v \in V$ with $\alpha \in V^*$ extends to an odd derivation $\iota(v) : \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*)$,

$$\iota(v)(\alpha\beta) = (\iota(v)\alpha)\beta + (-1)^l \alpha\,(\iota(v)\beta), \quad \alpha \in \Lambda^l(V^*).$$

Note: $\iota(v)\iota(v') + \iota(v')\iota(v) = 0$ and $\iota(v)^2 = 0$.

Fermi integral is a linear mapping $\Omega : \Lambda(V^*) \longrightarrow \mathbb{C}$
with "translation" invariance: $\Omega[\iota(v)\Psi] = 0$.

$\rightsquigarrow$ $\Omega[\Psi] = \iota(e_1)\cdots\iota(e_N)\Psi$ for some basis $e_1, \dots, e_N$ of V.

Remark. $\Omega \in \Lambda^N(V) \cong \mathbb{C}$.

Change the notation. $\xi^1, \dots, \xi^N$ dual basis of V^*;
$\iota(e_j) \equiv \frac{\partial}{\partial \xi^j}$ and $\Omega[\Psi] = \frac{\partial}{\partial \xi^1}\cdots\frac{\partial}{\partial \xi^N}\Psi \equiv \int_F \Psi$.

Pfaffian. For $A : V \times V \longrightarrow \mathbb{C}$ skew

let $\xi A \xi \equiv \xi^i A(e_i, e_j) \xi^j \in \Lambda^2(V^*)$.

$$\mathrm{Pf}(A) := \int_{\mathcal{F}} e^{\frac{1}{2}\xi A \xi}$$

$$= \begin{cases} 0 & N \text{ odd} \\ \dfrac{2^{-N/2}}{(N/2)!} \sum\limits_{\pi \in S_N} \mathrm{sign}(\pi)\, A(e_{\pi(N)}, e_{\pi(N-1)}) \cdots A(e_{\pi(2)}, e_{\pi(1)}) & N \text{ even} \end{cases}$$

Determinant as Gaussian integral.

For $B : V \longrightarrow V$ linear let $\bar{\xi} B \xi \equiv \bar{\xi}_i B^i_{\ j} \xi^j \quad (\bar{\xi}_i = e_i)$.

Then $$\mathrm{Det}\, B = \prod_{i=1}^{N} \frac{\partial^2}{\partial \xi^i \partial \bar{\xi}_i} e^{\bar{\xi} B \xi}.$$

3.2 Bosonization formula (Od).

Let $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^N) \cong \mathbb{C}^N \otimes (\mathbb{C}^n)^*$; $V^* = \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^n)$.

Write $\psi(\bar{\xi}^i_a, \xi^b_j) \equiv \psi(\bar{\xi}, \xi)$ for $\psi \in \Lambda(V^* \oplus V)$.

Thm. Assume $U(N)$-invariance: $\psi(\bar{\xi}, \xi) = \psi(\bar{\xi} u^{-1}, u\xi)$. Then

(i) $\exists\ \Psi : \mathrm{End}(\mathbb{C}^n) \xrightarrow{\text{entire}} \mathbb{C}$ s.th. $\psi(\bar{\xi}, \xi) = \Psi(\bar{\xi} \cdot \xi)$, and

(ii) $\exists\ c_{n,N} \in \mathbb{C}$: for any such Ψ one has

$$\int_F \psi = c_{n,N} \int_{U(n)} \Psi(u)\, \mathrm{Det}^{-N}(u)\, du.$$

Haar measure

Idea of proof for (ii). Let $({}^{h}\psi)(\xi,\bar{\xi}) := \psi(h_L\xi, \bar{\xi}h_R^{-1})$.

$$\Omega_1[{}^{h}\psi] \equiv \int_{\mathcal{F}} {}^{h}\psi = \frac{\mathrm{Det}^N(h_L)}{\mathrm{Det}^N(h_R)}\, \Omega_1[\psi].$$

$$\Omega_2[{}^{h}\psi] \equiv c_{n,N}\int_{\mathcal{U}(n)} \Psi(h_L u h_R^{-1})\, \mathrm{Det}^{-N}(u)\, du = \frac{\mathrm{Det}^N(h_L)}{\mathrm{Det}^N(h_R)}\, \Omega_2[\psi].$$

Exercises

C.1 Check that $\mathrm{Tr}\,(g^{-1}dg)_{\mathfrak{p}}^2 \equiv \mathfrak{x}$ really defines a G-invariant metric tensor on G/K.

C.2 For G compact and V irreducible there exists a G-invariant density dg_K s.th.

$$\mathbb{1}_V = \int_{G/K} dg_K\; g|v_0\rangle\langle v_0|\, g^{-1}.$$

C.3
$$\mathrm{Pf}\,(A) = \begin{cases} 0 & N \text{ odd} \\ \dfrac{2^{-N/2}}{(N/2)!}\displaystyle\sum_{\pi\in S^N} \mathrm{sign}(\pi)\; A(e_{\pi(N)}, e_{\pi(N-1)})\cdots A(e_{\pi(2)}, e_{\pi(1)}) & N \text{ even}\end{cases}$$

C.4 Prove the bosonization identity and determine $c_{n,N}$ for $n=1$:

$$\int_{\mathcal{F}} \psi = c_{1,N}\int_{\mathcal{U}(1)} \Psi(u)\, u^{-N} du.$$

3.3 A glimpse of super

Supertrace. $V = V_0 \oplus V_1$ $\quad \mathbb{Z}_2$-graded vector space

$$\mathrm{STr}: \ \mathrm{End}(V) \longrightarrow \mathbb{C}, \quad X \longmapsto \mathrm{Tr}_{V_0} X - \mathrm{Tr}_{V_1} X .$$

Superdeterminant.

supermatrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$; $\quad$ A, D have matrix entries from Λ^{even},
B, C from Λ^{odd}.

If D is invertible,

$$\mathrm{SDet}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\mathrm{Det}(A - BD^{-1}C)}{\mathrm{Det}(D)} = \frac{\mathrm{Det}(A)}{\mathrm{Det}(D - CA^{-1}B)} .$$

Berezin integral = Fermi integral followed by ordinary integration.

Example: $\displaystyle \int_{\mathbb{R}^{2|2}} f = \int_{\mathbb{R}^2} dx\,dy \, \frac{\partial^2}{\partial\xi\,\partial\eta} f(x, y; \xi, \eta).$

Most features of ordinary analysis carry over (mutatis mutandi) to superanalysis, e.g. the substitution rule (for compact supports).

Example. Superbosonization identity $(n_0 = n_1 = 1\,;\ 0d)$

Complex variables $\varphi^i, \bar\varphi_i$ and anticommuting variables $\psi^i, \bar\psi_i$ $(i = 1, \ldots, N)$.

$$p = \bar\varphi \cdot \varphi\,, \quad \xi = \bar\varphi \cdot \psi\,, \quad \eta = \bar\psi \cdot \varphi\,, \quad q = \bar\psi \cdot \psi\,.$$

$$F\begin{pmatrix} p & \xi \\ \eta & q \end{pmatrix} = \left(F_0 + \xi F_1 + \eta F_2 + \xi\eta F_3\right)(p,q).$$

$$\mathrm{SDet}^N \begin{pmatrix} p & \xi \\ \eta & q \end{pmatrix} = \frac{p^N}{q^N} - N \xi \eta \frac{p^{N-1}}{q^{N+1}}\,.$$

$$\pi^N \int_{\mathbb{C}^N} \prod_{i=1}^N d\varphi^i d\bar\varphi_i \prod_{i=1}^N \frac{\partial^2}{\partial\psi^i \partial\bar\psi_i} F\begin{pmatrix} \bar\varphi\cdot\varphi & \bar\varphi\cdot\psi \\ \bar\psi\cdot\varphi & \bar\psi\cdot\psi \end{pmatrix}$$

$$= \frac{1}{(N-1)!} \int_0^\infty \left(\partial_q^N F_0(p,q) - p\,\partial_q^{N-1} F_3(p,q)\right)\Big|_{q=0} p^{N-1} dp$$

$$= \int_{\mathbb{R}_+ \times U(1)} DM\ \mathrm{SDet}^N(M)\, F(M) \quad \text{if} \quad DM = (2\pi i)^{-1} dp\, dq\, \frac{\partial^2}{\partial\xi\,\partial\eta}\,.$$

3.4 Random band matrices and NLσM

For a quasi-1D random matrix model of length L and band width W the approximation by a NLσM

$$\sim \exp\left(-\xi \int_0^L dx\, \mathrm{STr}\, \partial_x Q\, \partial_x Q^{-1}\right)$$

predicts (i) $\xi = \mathrm{const} \cdot W^2$,

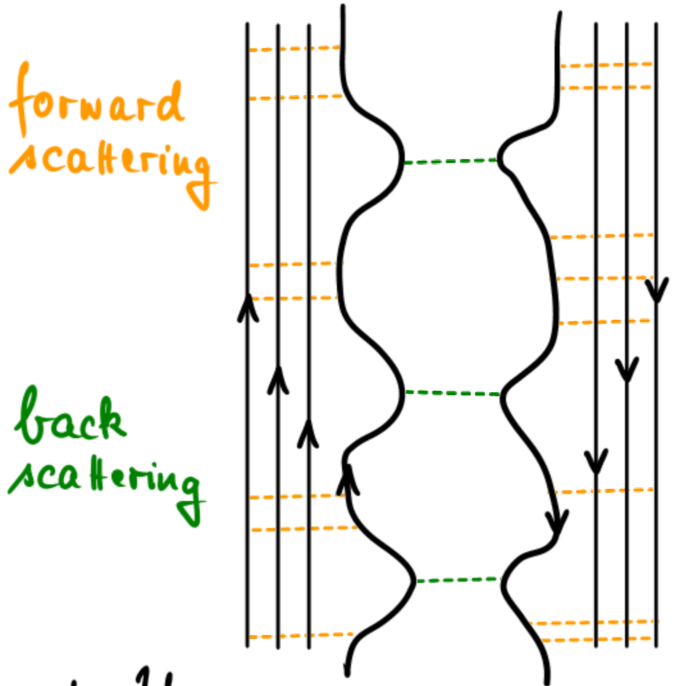
(ii) crossover from Wigner–Dyson statistics for $L/\xi \ll 1$

to Poisson statistics for $L/\xi \gg 1$.

Challenge: establish this rigorously!

3.5 Suggestion

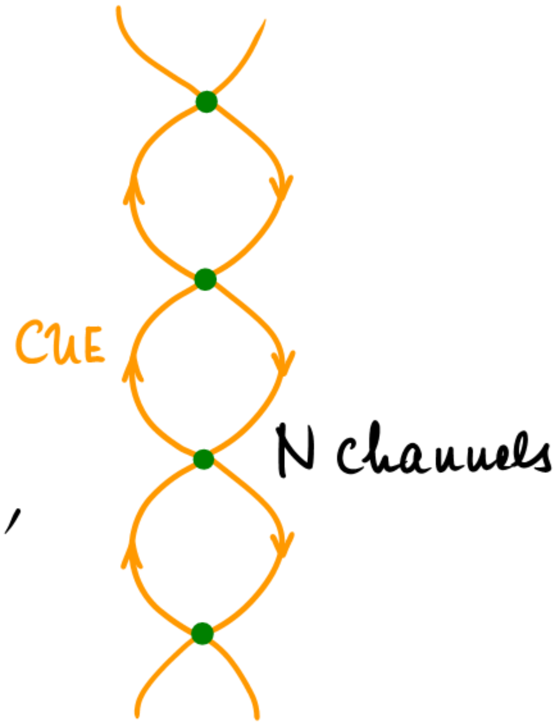
Consider a quantum Hall bar ("chiral" edges) with a sequence of constrictions:



Model by discrete-time evolution operator $\mathcal{U}$.

$$\mathcal{U} = \mathcal{U}_{back}\, \mathcal{U}_{forw}$$

Heuristically (at least) this translates into an "antiferromagnetic" two-superspin problem on $V \otimes V^*$.



NLσM predicts localization length $\xi \sim N$ and crossover between Wigner-Dyson and Poisson regime.

Attractive features: (i) Laboratory realization exists, (ii) No "massive" modes (NLσM approximation exact).