

# Riemannian symmetric superspaces and their origin in random-matrix theory

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Gaussian random-matrix ensembles defined over the tangent spaces of the large families of Cartan's symmetric spaces are considered. Such ensembles play a central role in mesoscopic physics, as they describe the universal ergodic limit of disordered and chaotic single-particle systems. The generating function for the spectral correlations of each ensemble is reduced to an integral over a Riemannian symmetric superspace in the limit of large matrix dimension. Such a space is defined as a pair  $(G/H, M_r)$ , where  $G/H$  is a complex-analytic graded manifold homogeneous with respect to the action of a complex Lie supergroup  $G$ , and  $M_r$  is a maximal Riemannian submanifold of the support of  $G/H$ . © 1996 American Institute of Physics. [S0022-2488(96)00710-4]

## I. INTRODUCTION

The mathematics of supersymmetry, though conceived and developed in elementary particle theory, has been applied extensively to the physics of disordered metals during the past decade. Improving on earlier work by Wegner,<sup>1,2</sup> Efetov<sup>3</sup> showed how to approximately map the problem of calculating disorder averages of products of the energy Green's functions for a single electron in a random potential, on a supersymmetric nonlinear  $\sigma$  model. Later it was shown<sup>4</sup> that the same nonlinear  $\sigma$  model describes the large- $N$  limit of a random-matrix ensemble of the Wigner–Dyson<sup>5</sup> type. Since then, Efetov's method has evolved into a prime analytical tool in the theory of disordered or chaotic mesoscopic single-particle systems. Competing methods are limited either to the diffusive regime (the impurity diagram technique), or to isolated systems in the ergodic regime (the Dyson–Mehta orthogonal polynomial method), or to quasi-one-dimensional systems (the DMPK equation). In contrast, Efetov's method is applicable to isolated and to open systems in the diffusive, ergodic, localized, and even ballistic regime, to both spectral correlations and transport properties, and it can, in principle, be used in any dimension. This versatility has engendered a large body of nontrivial applications, many of which are outside the range of other methods. Of these, let me mention (i) the Anderson transition on a Bethe lattice,<sup>6–8</sup> (ii) localization in disordered wires,<sup>9–13</sup> (iii) multifractality of energy eigenstates in two dimensions,<sup>14–16</sup> (iv) weak localization and conductance fluctuations of chaotic billiards strongly coupled to a small number of scattering channels,<sup>17,18</sup> and, most recently, (v) a theoretical physicist's proof of the Bohigas–Giannoni–Schmit conjecture for chaotic Hamiltonian systems.<sup>19,20</sup>

In spite of these manifest successes, Efetov's supersymmetry method has been ignored (for all that I know) by mathematical physicists. This is rather unfortunate for several reasons. First, an infusion of mathematical expertise is needed to sort out some matters of principle and promote the method to a rigorous tool. Second, various extensions of currently available results seem possible but have been hindered by the lack of mathematical training on the part of the condensed matter theorists applying the method. And third, the geometric structures underlying Efetov's nonlinear  $\sigma$  models are of exquisite beauty and deserve to be studied in their own right. Part of the reason why neither mathematicians nor mathematical physicists have monitored or contributed to the devel-

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opment, may be that there does not exist a concise status report that would appeal to a mind striving for clarity and rigor. Hence the first, and very ambitious, motivation for getting started on the present paper was to make an attempt and partially fill the gap.

Another objective is to report on a recent extension of the supersymmetry method to random-matrix theories beyond the standard Wigner–Dyson ones. In her study of Anderson localization in the presence of an A–B sublattice symmetry, Gade<sup>21</sup> noticed that the manifold of the nonlinear  $\sigma$  model is promoted to a larger manifold at zero energy. The same phenomenon occurs in the chiral limit of the QCD Dirac operator at zero virtuality.<sup>22</sup> For several years it remained unclear how to handle this enlargement of the manifold in the supersymmetric scheme. (Gade used the replica trick instead of supersymmetry.) The key to solving the problem can be found in a paper by Andreev, Simons, and Taniguchi,<sup>23</sup> who observed that what one needs to do is to avoid complex conjugation of the anticommuting variables. In the present paper I will elaborate on this observation and cast it in a concise mathematical language. Moreover, I will show that the same technical innovation allows one to treat the random-matrix theories that arose<sup>24,25</sup> in the stochastic modeling of mesoscopic metallic systems in contact with a superconductor.

An outline of the basic mathematical structure is as follows. Consider a homogeneous space  $G/H$ , where  $G$  and  $H$  are complex Lie supergroups, and regard  $G/H$  as a complex-analytic  $(p, q)$ -dimensional supermanifold in the sense of Berezin–Kostant–Leites.<sup>26,27</sup> To integrate its holomorphic sections, select a closed, oriented, and real  $p$ -manifold  $M_r$  contained in the support  $M = G_0/H_0$  of the supermanifold. The natural (invariant) supergeometry of  $G/H$  induces a geometry on  $M_r$  by restriction. If this geometry is Riemann and  $M_r$  is a symmetric space, the pair  $(G/H, M_r)$  is called a Riemannian symmetric superspace. This definition will be shown to be the one needed for the extension of the supersymmetry method beyond Wigner–Dyson. The difficulties disordered single-particle theorists had been battling with were caused by the fact that the exact sequence

$$0 \rightarrow \text{nilpotents} \rightarrow G/H \rightarrow M \rightarrow 0,$$

does not, in general, reduce to an exact sequence of sheaves of *real-analytic* sections terminating at the Riemannian submanifold  $M_r$ .

When integrating the invariant holomorphic Berezin superform on  $G/H$ , one must pay careful attention to its coordinate ambiguity. This subtle point is reviewed in Sec. II A. After a brief reminder of the procedure of Grassmann-analytic continuation (in Sec. II B), the complex Lie supergroups  $Gl(m|n)$  and  $Osp(m|2n)$  (in Sec. II C), and Cartan’s symmetric spaces (in Sec. II E), the details of the definition of Riemannian symmetric superspaces are given in Sec. II F. Table II lists the large families of these spaces.

Section III, the largest of the paper, treats the Gaussian random-matrix ensemble defined over the symplectic Lie algebra  $sp(N)$ , by an adaptation of Efetov’s method. A simple example (Sec. III A) illustrates the general strategy. Details of the method, including a complete justification of all manipulations involved, are presented in Secs. III B–III F. Theorem 3.3 expresses the Gaussian ensemble average of a product of  $n$  ratios of spectral determinants as a superintegral. Theorem 3.4 reduces this expression to an integral over the Riemannian symmetric superspace  $Osp(2n|2n)/Gl(n|n)$  with  $M_r = (SO^*(2n)/U(n)) \times (Sp(n)/U(n))$ , in the limit  $N \rightarrow \infty$ .

According to Cartan’s list, there exists 11 large families of symmetric spaces. Ten of these correspond to universality classes that are known to describe disordered single-particle systems in the ergodic regime.<sup>24,25</sup> The class singled out for detailed treatment in Sec. III describes mesoscopic normal-superconducting hybrid systems with time-reversal symmetry broken by a weak magnetic field. The remaining nine classes are briefly discussed in Sec. IV. Each of them is related, by the supersymmetry method, to one of the large families of Riemannian symmetric superspaces of Table II. A summary is given in Sec. V.

## II. RIEMANNIAN SYMMETRIC SUPERSPACES

### A. The Berezin integral on analytic supermanifolds

Let  $A(U)$  denote the algebra of analytic functions on an open subset  $U$  of  $p$ -dimensional real space. By taking the tensor product with the Grassmann algebra with  $q$  generators one obtains  $A(U) \otimes \Lambda(\mathbb{R}^q)$ , the algebra of analytic functions on  $U$  with values in  $\Lambda(\mathbb{R}^q)$ . Multiplication on  $\Lambda(\mathbb{R}^q)$  is the exterior one, so the algebra is supercommutative (or graded commutative). The object at hand serves as a model for what is called a real-analytic  $(p, q)$ -dimensional supermanifold (or graded manifold<sup>28</sup>) in the sense of Berezin, Kostant, and Leites (BKL);<sup>26,27</sup> which, precisely speaking, is a sheaf of supercommutative algebras  $\mathcal{A}$  with an ideal  $\mathcal{N}$  (the nilpotents), such that  $M \simeq \mathcal{A}/\mathcal{N}$  is an analytic  $p$ -manifold and on a domain  $U \subset M$ ,  $\mathcal{A}$  splits as  $A(U) \otimes \Lambda(\mathbb{R}^q)$ . The global sections of the bundle  $\mathcal{A} \rightarrow M$  are called superfunctions, or functions for short.  $M$  is called the underlying space, or base, or support, of the supermanifold.  $M$  will be assumed to be orientable and closed ( $\partial M = 0$ ).

The calculus on analytic supermanifolds is a natural extension of the calculus on analytic manifolds. Functions are locally expressed in terms of (super-)coordinates  $(x; \xi) := (x^1, \dots, x^p; \xi^1, \dots, \xi^q)$ , where  $x^i$  ( $\xi^j$ ) are even (resp., odd) local sections of  $\mathcal{A}$ . If  $(x; \xi)$  and  $(y; \eta)$  are two sets of local coordinates on domains that overlap, the transition functions  $y^i = f^i(x; \xi)$  and  $\eta^j = \varphi^j(x; \xi)$  are analytic functions of their arguments and are consistent with the  $\mathbb{Z}_2$  grading of  $\mathcal{A}$ .

In what follows the focus is on the theory of integration on analytic supermanifolds. Recall that on  $p$ -manifolds the objects one integrates are  $p$ -forms and their transformation law is given by

$$dy^1 \wedge \dots \wedge dy^p = dx^1 \wedge \dots \wedge dx^p \text{Det} \left( \frac{\partial y^i}{\partial x^j} \right).$$

The obvious (super-)generalization of the Jacobian  $\text{Det}(\partial y^i / \partial x^j)$  is the Berezinian<sup>29</sup>

$$\text{Ber} \left( \begin{array}{c} y, \eta \\ x, \xi \end{array} \right) := \text{SDet} \left( \begin{array}{cc} \frac{\partial y^i}{\partial x^j} & \frac{\partial y^i}{\partial \xi^j} \\ \frac{\partial \eta_i}{\partial x^j} & \frac{\partial \eta_i}{\partial \xi^j} \end{array} \right),$$

where  $\text{SDet}$  is the symbol for superdeterminant. Guided by analogy, one postulates that an integral superform ought to be an object  $\tilde{D}$  transforming according to the law

$$\tilde{D}(y, \eta) = \tilde{D}(x, \xi) \text{Ber}(y, \eta/x, \xi). \quad (1)$$

A natural candidate would seem to be

$$D(x, \xi) := dx^1 \wedge \dots \wedge dx^p \otimes \partial_{\xi^1} \dots \partial_{\xi^q},$$

which is a linear differential operator taking superfunctions  $f$  into  $p$ -forms  $D[f]$  ( $\partial_{\xi^i}$  denotes the partial derivative with respect to the anticommuting coordinate  $\xi^i$ ). The  $p$ -form  $D[f]$  can be integrated in the usual sense to produce a number. However, the transformation law for  $D(x, \xi)$  turns out to be not quite (1), but rather

$$D(y, \eta) = D(x, \xi) \text{Ber}(y, \eta/x, \xi) + \beta. \quad (2)$$

An explicit description of the term  $\beta$  on the right-hand side, here referred to as the *anomaly*, was first given by Rothstein.<sup>30</sup> It is nonzero whenever some even coordinate functions are shifted by nilpotent terms. Its main characteristic is that on applying it to a superfunction  $f$ , one gets a  $p$ -form that is *exact*:  $\beta[f] = d(\alpha[f])$ .

The existence of an anomaly in the transformation law for  $D(x, \xi)$  leads one to consider a larger class of objects, namely,  $\Lambda^p(M) \otimes_{\mathcal{A}} \mathcal{D}$ , the sheaf of linear differential operators on  $\mathcal{A}$  with values in the  $p$ -forms on  $M$ . [ $\Lambda^p(M) \otimes_{\mathcal{A}} \mathcal{D}$  naturally is a right  $\mathcal{A}$ -module.] To rescue the simple transformation law (1) one usually passes from  $\Lambda^p(M) \otimes_{\mathcal{A}} \mathcal{D}$  to its quotient by the anomalies.<sup>30</sup> In order for the integral to be well-defined over the quotient, one must take the functions one integrates to be compactly supported.

Sadly, this last option is *not available to us*. The functions that will be encountered in the applications worked out below, do not ever have compact support but are *analytic* functions instead. When integrating such functions, we need to work with the full transformation law (2), which includes the anomaly.

Another way of avoiding the anomaly is to arrange for the transition functions never to shift the even coordinates by nilpotents, by constructing a restricted subatlas.<sup>31</sup> However, because the concept of a restricted subatlas is somewhat contrived, this approach has been found to be of limited use in the type of problem that is of interest here.

To arrive at a definition of superintegration that is useful in practice, we proceed as follows. The supermanifold is covered by a set of charts with domains  $U_i$  and coordinates  $(x_{(i)}, \xi_{(i)})$  ( $i = 1, \dots, n$ ). On chart  $i$  let  $\omega_i := D(x_{(i)}, \xi_{(i)}) \circ \bar{\omega}_i$  with  $\bar{\omega}_i$  a local section of  $\mathcal{A}$ , and let  $\alpha_i \in \Lambda^{p-1}(M) \otimes_{\mathcal{A}} \mathcal{D}|_{U_i}$ . Partition  $M$  into a number of consistently oriented  $p$ -cells  $D_1, \dots, D_n$ , with  $D_i$  contained in  $U_i$ . For  $i < j$  put  $D_{ij} := \partial D_i \cap \partial D_j$  and, if  $D_{ij}$  is nonempty and is a  $(p-1)$ -cell, fix its orientation by  $\partial D_i = +D_{ij} + \dots$ .

*Definition 2.1:* A collection  $\{\omega_i, \alpha_i\}_{i=1, \dots, n}$  is called a *Berezin measure*  $\omega$  if the conditions

$$\bar{\omega}_i / \bar{\omega}_j = \text{Ber}(i/j), \tag{3}$$

$$\omega_i + d\alpha_i = \omega_j + d\alpha_j, \tag{4}$$

are satisfied on overlapping domains. The Berezin integral  $f \mapsto \int_M \omega[f]$  is defined as

$$\int_M \omega[f] = \sum_{i=1}^n \int_{D_i} \omega_i[f] + \sum_{i < j} \int_{D_{ij}} \alpha_{ij}[f], \tag{5}$$

where  $\alpha_{ij} = \alpha_i - \alpha_j$ . The quantities  $\omega_i$  and  $\alpha_i$  are called the principal term and the anomaly of the Berezin measure on chart  $i$ .

*Remark 2.2:* The conditions (3) and (4) ensure the existence of a global section  $\omega \in \Lambda^p(M) \otimes_{\mathcal{A}} \mathcal{D}$ , whose local expression in chart  $i$  is  $\omega_i + d\alpha_i$ . The existence of  $\omega$  means that the distribution (5) is independent of the coordinate systems and the cell partition chosen. Because (5) depends only on the differences  $\alpha_i - \alpha_j$ , one can gauge the anomaly to zero on one of the charts without changing the Berezin integral.

*Example 2.3:* Consider the real supersphere  $S^{p|2}$ , a  $(p, 2)$ -dimensional supermanifold with support  $S^p$ , which is the space of solutions in  $(p+1, 2)$  dimensions of the quadratic equation

$$\tilde{x}_0^2 + \tilde{x}_1^2 + \dots + \tilde{x}_p^2 + 2\tilde{\xi}_1 \tilde{\xi}_2 = 1.$$

Cover  $S^p$  by two domains 1 and 2 obtained by removing the South ( $\tilde{x}_0 = -1$ ) or North Pole ( $\tilde{x}_0 = +1$ ). Introduce stereographic coordinates  $(x_1, \dots, x_p; \xi_1, \xi_2)$  and  $(y_1, \dots, y_p; \eta_1, \eta_2)$  for  $S^{p|2}$  on these domains with transition functions

$$y_1 = -\frac{x_1}{R^2}, \quad y_i = \frac{x_i}{R^2} \quad (i=2, \dots, p), \quad \eta_j = \frac{\xi_j}{R^2} \quad (j=1, 2),$$

where  $R^2 = \sum_{i=1}^p x_i^2 + 2\xi_1\xi_2$ . (The minus sign preserves the orientation.) Consider

$$\begin{aligned} \omega_1 &= D(x, \xi) \circ \left( 1 + \sum x_i^2 + 2\xi_1\xi_2 \right)^{-p+2}, \\ \omega_2 &= D(y, \eta) \circ \left( 1 + \sum y_i^2 + 2\eta_1\eta_2 \right)^{-p+2}, \\ \alpha_{12} &= -\Omega \frac{(\sum x_i^2)^{(p-2)/2}}{(1 + \sum x_i^2)^{p-2}} \otimes 2\partial_{\xi_1}\partial_{\xi_2} \circ \xi_1\xi_2, \end{aligned}$$

where  $\Omega = (\sum_j x_j^2)^{-p/2} \sum_{i=1}^p (-1)^i dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_p$  is the solid-angle  $(p-1)$  form in  $p$  dimensions. It is not difficult to check by direct calculation that  $\omega_1, \omega_2$  and  $\alpha_1 = \alpha_{12}, \alpha_2 \equiv 0$  obey the relations (3) and (4). Hence, they express a globally defined Berezin measure  $\omega$  in the sense of Definition 2.1. (The geometric meaning of  $\omega$  will be specified in Sec. II C.) For  $p \geq 3$ , the anomaly  $\alpha_{12}$  scales to zero when  $\sum x_i^2 \rightarrow \infty$ , so we may shrink cell 2 to a single point (a set of measure zero) and compute the Berezin integral simply from

$$\int_{S^p} \omega[f] = \int_{\mathbb{R}^p} D(x, \xi) \left( 1 + \sum x_i^2 + 2\xi_1\xi_2 \right)^{-p+2} f(x; \xi).$$

In these cases we can get away with using only a single chart. The situation is different for  $p=2$  and  $p=1$ . In the first case the anomaly is scale-invariant (the solid angle is) and by again shrinking cell 2 to one point [the South Pole  $(y_1, y_2) = (0, 0)$  on  $S^2$ ] we get

$$\int_{S^2} \omega[f] = \int_{\mathbb{R}^2} D(x, \xi) f(x; \xi) + 4\pi f \Big|_{\text{South Pole}}.$$

In particular,  $\int_{S^2} \omega[1] = 4\pi$ . For  $p=1$  the anomaly diverges at  $x=0$  and  $x=\infty$ . In this case the general formula (5) must be used, and one finds  $\int_{S^1} \omega[1] = 0$ .

**B. Grassmann-analytic continuation**

In the formulation of BKL, the vector fields of a supermanifold do not constitute a module over  $\mathcal{A}$  but are constrained to be *even* derivations of  $\mathcal{A}$ , which is to say that their coordinate expression is of the form

$$\hat{X} = f^i(x; \xi) \frac{\partial}{\partial x^i} + \varphi^j(x; \xi) \frac{\partial}{\partial \xi^j},$$

where  $f^i$  and  $\varphi^j$  are even and odd superfunctions, respectively. Unfortunately, this formulation is too narrow for most purposes. The reason is that in applications one typically deals not with a single supermanifold but with many copies thereof (one per lattice site of a lattice-regularized field theory, for example). So, in addition to the anticommuting coordinates of the one supermanifold that is singled out for special consideration, there exist many more anticommuting variables associated with the other copies of the supermanifold. When the focus is on one supermanifold, these can be considered as ‘‘parameters.’’ Often one wants to make parameter-dependent coordinate transformations, leading to coefficients  $f_{i_1 \dots i_n}(x)$  in the expansion  $f(x; \xi) = \sum f_{i_1 \dots i_n}(x) \xi^{i_1} \dots \xi^{i_n}$  that depend on extraneous Grassmann parameters. (For example, when the

supermanifold is a Lie supergroup, it is natural to consider making left and right translations  $g \mapsto g_L g g_R$ .) The upshot is that one wants to take  $\mathcal{A}$  as a sheaf of graded commutative algebras not over  $\mathbb{R}$  but over some (large) parameter Grassmann algebra  $\Lambda$  (the Grassmann algebra generated by the anticommuting coordinates of the ‘‘other’’ supermanifolds). Making this extension, which is called ‘‘Grassmann-analytic continuation’’ in Ref. 29, one is led to consider the more general class of vector fields of the form

$$\hat{X} = f^i(x, \xi; \beta) \frac{\partial}{\partial x^i} + \varphi^j(x, \xi; \beta) \frac{\partial}{\partial \xi^j}, \tag{6}$$

where the symbol  $\beta$  stands for the extra Grassmann parameters and the dependences on these are such that  $f^i$  and  $\varphi^j$  continue to be even and odd, respectively (the  $\mathbb{Z}_2$  grading of  $\mathcal{A}$  after Grassmann-analytic continuation is the natural one).

The vector fields (6) still are even derivations of the extended algebra. One can go further by demanding that  $\text{Der } \mathcal{A}$  be free over  $\mathcal{A}$  and including the odd ones, too. When that development is followed to its logical conclusion, one arrives at Rothstein’s axiomatic definition<sup>32</sup> of supermanifolds, superseding an earlier attempt by Rogers.<sup>33,34</sup> Although there is no denying the elegance and consistency of Rothstein’s formulation, we are not going to embrace it here, the main reason being that odd derivations will not really be needed. For the purposes of the present paper we will get away with considering vector fields of the constrained form (6).

### C. The complex Lie supergroups $\text{Gl}(m|n)$ and $\text{Osp}(m|2n)$

The supermanifolds we will encounter all derive from the complex Lie supergroups<sup>29,28</sup>  $\text{Gl}(m|n)$  and  $\text{Osp}(m|2n)$ , by forming cosets. The definition of  $\text{Gl}(m|n)$  rests on the notion of an invertible supermatrix

$$g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix},$$

where  $g_{00}$ ,  $g_{01}$ ,  $g_{10}$ , and  $g_{11}$  are matrices of size  $m \times m$ ,  $m \times n$ ,  $n \times m$ , and  $n \times n$ . The supermanifold structure of  $\text{Gl}(m|n)$  comes from taking the matrix elements of  $g_{00}$  and  $g_{11}$  ( $g_{01}$  and  $g_{10}$ ) for the even (resp., odd) coordinates on suitable domains of the base  $M = \text{Gl}(m, \mathbb{C}) \times \text{Gl}(n, \mathbb{C})$ . The Lie supergroup structure derives from the usual law of matrix multiplication.

For  $m \neq n$ , it is common practice to split off from  $\text{Gl}(m|n)$  the  $\text{Gl}(1)$ -ideal generated by the unit matrix, so as to have an irreducible Lie superalgebra.<sup>35,36</sup> For  $m = n$ , which turns out to be the case of most interest here, one ends up having to remove two  $\text{Gl}(1)$ ’s, one generated by the unit matrix and the other one by the superparity matrix  $\sigma = \text{diag}(1_n, -1_n)$ . And even then the Lie superalgebra is not irreducible in a sense, for the Killing form  $\text{STr ad}(X)\text{ad}(Y)$  vanishes identically. We therefore prefer to take  $\text{Gl}(m|n)$  as it stands (with no ideals removed) and replace the Killing form by the invariant quadratic form  $B(X, Y) = \text{STr } XY$ , which is nondegenerate in all cases (including  $m = n$ ).

The complex orthosymplectic Lie supergroup  $\text{Osp}(m|2n)$  is defined as a connected subgroup of  $\text{Gl}(m|2n)$  fixed by an involutory automorphism  $g \mapsto \hat{\tau}(g) = \tau g^{-1T} \tau^{-1}$ , where  $\tau$  is supersymmetric ( $\tau = \tau^T \sigma = \sigma \tau^T$ ).<sup>37</sup> The support of  $\text{Osp}(m|2n)$  is  $\text{SO}(m, \mathbb{C}) \times \text{Sp}(n, \mathbb{C})$ .

The action of a Lie supergroup on itself by left and right translations gives rise to right- and left-invariant vector fields. A Berezin measure on a Lie supergroup is said to be invariant, and is called a Berezin–Haar measure, if its Lie derivatives<sup>29</sup> with respect to the invariant vector fields vanish.

Given a Lie supergroup  $G$  and a subgroup  $H$ , the coset superspace  $G/H$  is defined by decreeing that the structure sheaf of the coset superspace is a quotient of sheaves. The action of  $G$

on  $G/H$  by left translation gives rise to so-called Killing vector fields. A Berezin measure on  $G/H$  is called invariant if its Lie derivatives with respect to the Killing vector fields are zero.

If  $\text{Osp}_{\mathbb{R}}(m|2n)$  denotes the orthosymplectic supergroup over the reals, the coset space  $\text{Osp}_{\mathbb{R}}(m+1|2n)/\text{Osp}_{\mathbb{R}}(m|2n)$  can be identified with the real supersphere  $S^{m|2n}$ . The Berezin measure discussed in Example 2.3 is invariant with respect to the action of  $\text{Osp}_{\mathbb{R}}(p+1|2)$  on  $S^{p|2}$  and can be viewed as the “volume superform” of  $S^{p|2}$ . Hence we can restate the results of that example as follows:  $\text{vol}(S^{2|2}) := \int_{S^2} \omega[1] = 4\pi$  and  $\text{vol}(S^{1|2}) := \int_{S^1} \omega[1] = 0$ .

#### D. Holomorphic Berezin measures on complex-analytic supermanifolds

To go from real-analytic supermanifolds to complex-analytic ones, one replaces the structure sheaf  $\mathcal{A}$  by a sheaf of graded commutative algebras  $\mathcal{H}$  over  $\mathbb{C}$  such that  $M \simeq \mathcal{H}|\mathcal{N}$  is a complex manifold and  $\mathcal{H}$  is locally modeled by  $H(U) \otimes \Lambda(\mathbb{C}^q)$ , where  $H(U)$  is the algebra of holomorphic functions on  $U \subset M$ . The natural objects to consider then are holomorphic superfunctions, i.e., global sections of the bundle  $\mathcal{H} \rightarrow M$ . In local coordinates  $z^1, \dots, z^p; \zeta^1, \dots, \zeta^q$  such sections are written as  $f(z; \zeta)$ . Grassmann-analytic continuation is done as before when needed. A Berezin measure on a complex-analytic  $(p, q)$ -dimensional supermanifold is a linear differential operator  $\omega$  that takes holomorphic superfunctions  $f$  into holomorphic  $p$ -forms  $\omega[f]$  on  $M$ . The statements made in Sec. II A about the anomalous transformation behavior of Berezin measures apply here, too (*mutatis mutandi*).

To define Berezin’s integral in the present context, one more piece of data must be supplied, namely a *real*  $p$ -dimensional submanifold  $M_r \subset M$  over which the holomorphic  $p$ -form  $\omega[f]$  can be integrated to produce a complex number. Thus, given  $\omega$  and  $M_r$ , Berezin’s integral is the distribution

$$f \mapsto \int_{M_r} \omega[f]. \quad (7)$$

Let me digress and mention that this definition, natural and simple as it is, was not “discovered” by the random-matrix and mesoscopic physics community (including myself) until quite recently. With one notable exception,<sup>23</sup> all past superanalytic work on disordered single-particle systems employed some operation of “complex conjugation” of the Grassmann generators—namely an adjoint of the first or second kind<sup>29</sup>—to make the treatment of the ordinary (“bosonic”) and anticommuting (“fermionic”) degrees of freedom look as much alike as possible. Presumably this was done because it was felt that such egalitarian treatment is what is required by the principle of “supersymmetry.” Specifically, a reality constraint was imposed, not just on the underlying space  $M$  (fixing  $M_r$ ) but on the entire structure sheaf to reduce  $\mathcal{H}$  to a sheaf of algebras over  $\mathbb{R}$ . Although this reduction can be done with impunity in some cases (namely, the classic Wigner–Dyson symmetry classes), it has turned out to lead to insurmountable difficulties in others (the chiral and normal-superconducting symmetry classes). A major incentive of the present paper is to demonstrate that the construction (7) is in fact the “good” one to use for the application of supermanifold theory to disordered single-particle systems in general. Although that construction may hurt the physicists’ aesthetic sense by “torturing supersymmetry,” it should be clear that we are not breaking any rules. Recall that according to Berezin, superintegration is a two-step process: first, the Fermi integral (i.e., differentiation with respect to the anticommuting coordinates) is carried out, and it is only *afterward* that the ordinary (Bose) integrals are done. When the sequential nature of the Berezin integral is taken seriously, there is no compelling reason why one should ever want to “complex conjugate” a Grassmann variable. In the present paper, we take the radical step of abandoning complex conjugation of Grassmann variables altogether.

*Example 2.4:* The simplest nontrivial example<sup>23</sup> is given by  $\text{Gl}(1|1)$ , the Lie supergroup of regular complex  $2 \times 2$  supermatrices

$$g = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$$

with support  $M = \text{Gl}(1, \mathbb{C}) \times \text{Gl}(1, \mathbb{C})$ . The Berezin–Haar measure on  $\text{Gl}(1|1)$  is  $\omega = (2\pi i)^{-1} D(ad; \beta\gamma)$ , where  $D(ad; \beta\gamma) = da \wedge dd \otimes \partial_\beta \partial_\gamma$ . Solving the regularity conditions  $a \neq 0$  and  $d \neq 0$  by parametrizing  $\text{Gl}(1|1)$  through its Lie algebra,

$$g = \exp \begin{pmatrix} z_1 & \zeta_1 \\ \zeta_2 & z_2 \end{pmatrix},$$

one finds

$$2\pi i \omega = D(z_1 z_2; \zeta_1 \zeta_2) \circ \frac{(z_1 - z_2)^2}{(1 - e^{z_1 - z_2})(e^{z_2 - z_1} - 1)} - \left( d \ln(e^{z_1} - e^{z_2}) - \frac{(z_1 - z_2)(dz_1 - dz_2)}{(1 - e^{z_1 - z_2})(e^{z_2 - z_1} - 1)} \right) \otimes \partial_{\zeta_1} \partial_{\zeta_2} \circ \zeta_1 \zeta_2. \tag{8}$$

Note that this expression is holomorphic in a neighborhood of the origin  $z_1 = z_2 = 0$ . The first term on the right-hand side is the principal term, and the second one is the anomaly of  $\omega$  in these coordinates. To integrate  $\omega$ , one might be tempted to choose for  $M_r$  the  $U(1) \times U(1)$  subgroup defined by  $\text{Re}(z_1) = 0 = \text{Re}(z_2)$ . However, since the rank-two tensor  $\text{STr} dg dg^{-1} = da da^{-1} - dd dd^{-1} + \text{nilpotents} = -dz_1^2 + dz_2^2 + \dots$  is not Riemann on  $U(1) \times U(1)$ , this will not be the best choice. A Riemannian structure is obtained by taking  $M_r = \mathbb{R}^+ \times S^1$  defined by  $\text{Im}(z_1) = 0 = \text{Re}(z_2)$ . To compute  $\int_{\mathbb{R}^+ \times S^1} \omega[f]$  we may use a single cell,

$$D: -\infty < x < +\infty, \quad -\pi < y < +\pi,$$

where  $x = \text{Re}(z_1)$  and  $y = \text{Im}(z_2)$ . The boundary  $\partial D$  consists of the two lines  $y = -\pi$  and  $y = \pi$  ( $x \in \mathbb{R}$ ). Using (8), paying attention to the orientation of the boundary, and simplifying terms, one finds the following explicit expression for the integral of  $\omega$ :

$$\int_{\mathbb{R}^+ \times S^1} \omega[f] = \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \int_{-\pi}^{\pi} dy \frac{(x - iy)^2}{\cosh(x - iy) - 1} \partial_{\zeta_1} \partial_{\zeta_2} f \left( \exp \begin{pmatrix} x & \zeta_1 \\ \zeta_2 & iy \end{pmatrix} \right) + \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{\cosh x + 1} f \left( \begin{pmatrix} e^x & 0 \\ 0 & -1 \end{pmatrix} \right).$$

By construction, this Berezin integral is invariant under left and right translations  $f(g) \mapsto f(g_L g g_R)$ . Evaluation gives  $\int_{\mathbb{R}^+ \times S^1} \omega[1] = 1 \neq 0$ . The naive guess would have been  $\int \omega[1] = (2\pi i)^{-1} \int da \wedge dd \partial_\beta \partial_\gamma = 0$  due to  $\partial_\beta \partial_\gamma 1 = 0$ . Such reasoning is false because  $\int_{\mathbb{R}} da = \infty$ .

**E. Symmetric spaces: A reminder**

A Riemannian (globally) symmetric space is a Riemannian manifold  $M$ , such that every  $p \in M$  is an isolated fixed point of an involutive isometry. (In normal coordinates  $x^i$  centered around  $p$ , this isometry is given by  $x^i \mapsto -x^i$ .) This definition implies (cf. Ref. 38) that the Riemann curvature tensor is covariantly constant, so that “the geometry is the same everywhere.” The curvature can be positive, negative, or zero, and the symmetric space is said to be of compact, noncompact, or Euclidean type correspondingly.

According to Cartan’s complete classification scheme, there exist ten<sup>39</sup> large classes of symmetric spaces. Apart from some minor modifications these are the entries of Table I. The differ-



TABLE I. The large families of symmetry spaces.

Class	Noncompact type	Compact type
A	$GI(N, \mathbb{C})/U(N)$	$U(N)$
AI	$GI(N, \mathbb{R})/O(N)$	$U(N)/O(N)$
AII	$U^*(2N)/Sp(N)$	$U(2N)/Sp(N)$
AIII	$U(p, q)/U(p) \times U(q)$	$U(p+q)/U(p) \times U(q)$
BDI	$SO(p, q)/SO(p) \times SO(q)$	$SO(p+q)/SO(p) \times SO(q)$
CII	$Sp(p, q)/Sp(p) \times Sp(q)$	$Sp(p+q)/Sp(p) \times Sp(q)$
BD	$SO(N, \mathbb{C})/SO(N)$	$SO(N)$
C	$Sp(N, \mathbb{C})/Sp(N)$	$Sp(N)$
CI	$Sp(N, \mathbb{R})/U(N)$	$Sp(N)/U(N)$
DIII	$SO^*(2N)/U(N)$	$SO(2N)/U(N)$

ence from the standard table<sup>38</sup> is that some of the entries of Table I, namely, the spaces of type A, AI, and AII, are not irreducible. They can be made so by dividing out a factor  $U(1)$  ( $\mathbb{R}^+$ ) in the compact (resp., noncompact) cases. Division by such a factor is analogous to removing the center of mass motion from a mechanical system with translational invariance. It turns out that, with a view to superanalytic extensions (cf. Example 2.4), it is preferable not to insist on irreducibility but to ‘retain the center of mass motion.’

The next section introduces supergeneralizations of Cartan’s symmetric spaces, which have appeared in the theory of mesoscopic and disordered single-particle systems and have come to play an important role in that field.

## F. Riemannian symmetric superspaces (definition)

Let  $G_\Lambda$  be a complex Lie supergroup that is realized as a group of supermatrices

$$g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix},$$

with matrix elements that take values in a (sufficiently large) parameter Grassmann algebra  $\Lambda = \Lambda_0 + \Lambda_1$ . If  $\mathcal{S}_\mathbb{C} = \mathcal{S}_\mathbb{C}^0 + \mathcal{S}_\mathbb{C}^1$  is the Lie superalgebra of  $G_\Lambda$ , the Lie algebra of  $G_\Lambda$  is obtained by taking the even part of the tensor product with  $\Lambda$ :  $\text{Lie}(G_\Lambda) = \Lambda_0 \otimes \mathcal{S}_\mathbb{C}^0 + \Lambda_1 \otimes \mathcal{S}_\mathbb{C}^1 = (\Lambda \otimes \mathcal{S}_\mathbb{C})_0$ . Thus, if  $\{e_i, \epsilon_j\}$  is a homogeneous basis of complex matrices in  $\mathcal{S}_\mathbb{C}$ , an element  $X \in \text{Lie}(G_\Lambda)$  is expressed by  $X = z^i e_i + \zeta^j \epsilon_j$  with  $z^i \in \Lambda_0$  and  $\zeta^j \in \Lambda_1$ .

Let  $\theta: G_\Lambda \rightarrow G_\Lambda$  be an involutory automorphism and let  $H_\Lambda \subset G_\Lambda$  be the subgroup fixed by  $\theta$ . The decomposition into even and odd eigenspaces of  $\theta_*: \text{Lie}(G_\Lambda) \rightarrow \text{Lie}(G_\Lambda)$  is written as  $\text{Lie}(G_\Lambda) = \text{Lie}(H_\Lambda) + \mathcal{N}_\Lambda$ . This decomposition is orthogonal with respect to the  $\text{Ad}(G_\Lambda)$ -invariant quadratic form  $B: \text{Lie}(G_\Lambda) \times \text{Lie}(G_\Lambda) \rightarrow \Lambda_0$ ,  $B(X, Y) := \text{STr } XY$ .

Both  $G_\Lambda$  and  $H_\Lambda$  are supermanifolds with underlying spaces that are Lie groups and are denoted by  $G_\mathbb{C}$  and  $H_\mathbb{C}$ . Passing to the coset spaces one obtains a graded commutative algebra  $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$  of (Grassmann-analytically continued) holomorphic sections of the bundle  $G_\Lambda/H_\Lambda \rightarrow G_\mathbb{C}/H_\mathbb{C}$ . These sections are called (super-)functions (on  $G_\Lambda/H_\Lambda$ ) for short. In local complex coordinates  $z^1, \dots, z^p; \zeta^1, \dots, \zeta^q$  they are written as  $f(z^1, \dots, z^p; \zeta^1, \dots, \zeta^q) = \sum f_{i_1 \dots i_n}(z^1, \dots, z^p) \zeta^{i_1} \dots \zeta^{i_n}$ , where the coefficients  $f_{i_1 \dots i_n}(z^1, \dots, z^p)$  take values in  $\Lambda$  after Grassmann-analytic continuation. For coordinate-independent calculations the alternative notation  $f(gH_\Lambda)$  or  $f(g \cdot o)$  is used. In the following  $G_\mathbb{C}/H_\mathbb{C}$  is assumed to be connected.

Every  $X \in \text{Lie}(G_\Lambda)$  is associated with a vector field (or even derivation)  $\hat{X}: \mathcal{H} \rightarrow \mathcal{H}$  by

$$(\hat{X}f)(g \cdot o) = \left. \frac{d}{ds} \right|_{s=0} f(e^{-sX} g \cdot o). \quad (9)$$

Here  $e^{sX}g$  means the usual product of supermatrices, and the function  $f(e^{sX}g \cdot o)$  is determined from  $f(g \cdot o)$  by Grassmann-analytic continuation. The Lie algebra of even derivations of  $\mathcal{H}$  is a left  $\mathcal{H}_0$ -module denoted by  $\text{Der}_0 \mathcal{H}$ .<sup>40</sup>

A notion of supergeometry on  $G_\Lambda/H_\Lambda$  is introduced via a left-invariant tensor field  $\langle \bullet, \bullet \rangle: \text{Der}_0 \mathcal{H} \times \text{Der}_0 \mathcal{H} \rightarrow \mathcal{H}_0$ . The details are as follows.  $G_\Lambda$  acts on  $G_\Lambda/H_\Lambda$  by left translation,  $T_h^*: f(g \cdot o) \mapsto f(hg \cdot o)$ . The left-translate  $dT_h(\hat{X})$  of a vector field  $\hat{X}$  is defined by the equation  $T_h^*(dT_h(\hat{X})f) = \hat{X}(T_h^*f)$ , and one requires

$$T_h^* \langle dT_h(\hat{X}), dT_h(\hat{Y}) \rangle = \langle \hat{X}, \hat{Y} \rangle.$$

This equation determines  $\langle \bullet, \bullet \rangle$  uniquely within a multiplicative constant. For vector fields of the special form (9), one obtains

$$\langle \hat{X}, \hat{Y} \rangle(g \cdot o) = c_0 \times B((\text{Ad}(g)^{-1}X)_{\mathcal{M}_\Lambda}, (\text{Ad}(g)^{-1}Y)_{\mathcal{M}_\Lambda}),$$

where the subscript  $\mathcal{M}_\Lambda$  means projection on the odd eigenspace of  $\theta_*$ . Note that since  $(\text{Ad}(gh)^{-1}X)_{\mathcal{M}_\Lambda} = \text{Ad}(h)^{-1}(\text{Ad}(g)^{-1}X)_{\mathcal{M}_\Lambda}$  for  $h \in H_\Lambda$ , this is well-defined as a function on  $G_\Lambda/H_\Lambda$ . The normalization is fixed by choosing  $c_0=1$ .

The metric tensor  $\langle \bullet, \bullet \rangle$  induces a geometry on the ordinary manifold  $G_C/H_C$  by restriction (i.e., by setting all anticommuting variables equal to zero). Of course, since the groups  $G_C$  and  $H_C$  are complex, this geometry is never Riemann. However, there exist submanifolds in  $G_C/H_C$  that are Riemannian symmetric spaces and can be constructed by selecting from the tangent space  $T_o(G_C/H_C)$  a Lie-triple subsystem  $\mathcal{M}$  (i.e.,  $[\mathcal{M}, [\mathcal{M}, \mathcal{M}]] \subset \mathcal{M}$ ), such that the quadratic form  $B$  restricted to  $\mathcal{M}$  is of definite sign. It is then not hard to show<sup>38</sup> that the image of  $\mathcal{M}$  under the exponential map  $X \mapsto e^X H_\Lambda$  is Riemann in the geometry given by restriction of  $\langle \bullet, \bullet \rangle$ . Its completion is a symmetric space.

*Definition 2.5:* A Riemannian symmetric superspace is a pair  $(G_\Lambda/H_\Lambda; M)$ , where  $M$  is a maximal Riemannian submanifold of the base  $G_C/H_C$ .

*Remark 2.6:* The merit of this definition is that it avoids any use of an adjoint (or ‘‘complex conjugation’’) of the Grassmann variables. ■

By the complex structure of  $G_C/H_C$ , the tangent space  $\mathcal{M}_C := T_o(G_C/H_C)$  decomposes as  $\mathcal{M}_C = \mathcal{M} + i\mathcal{M}$ , where  $\mathcal{M}$  is taken to be the subspace of  $\mathcal{M}_C$  on which the quadratic form  $B$  is strictly positive. Now observe that, since an element  $g \in G_C$  is of the form  $g = \text{diag}(g_{00}, g_{11})$ , the group  $G_C$  is a Cartesian product of two factors, and the same is true for  $H_C$ . Hence,  $G_C/H_C$  factors as  $G_C/H_C = M_C^0 \times M_C^1$ , and  $\mathcal{M}$  is a sum of two spaces:  $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ , which are orthogonal with respect to the quadratic form  $B$ . (It may happen, of course, that one of these spaces is trivial.) For  $Z \in \mathcal{M}$ , let the corresponding orthogonal decomposition be written as  $Z = X + Y$ . Then  $B$  restricted to  $\mathcal{M}$  is evaluated as

$$B(Z, Z) = \text{Tr}_0 X^2 - \text{Tr}_1 Y^2,$$

where the relative minus sign between traces is due to supersymmetry ( $\text{STr} = \text{Tr}_0 - \text{Tr}_1$ ). The positivity of  $B$  on  $\mathcal{M}$  is seen to imply  $X = X^\dagger$  and  $Y = -Y^\dagger$  (the dagger denotes Hermitian conjugation, i.e. transposition in conjunction with complex conjugation).

Given  $G_\Lambda/H_\Lambda$ , the condition that  $M$  be Riemann and maximal in  $G_C/H_C$ , fixes  $M$  uniquely up to two possibilities: either  $T_o(M) = \mathcal{M}$ , or  $T_o(M) = i\mathcal{M}$ . In either case,  $M$  is a product of two factors,  $M = M_0 \times M_1$ , both of which are Riemannian symmetric spaces. In the first case,  $M_0$  is of the noncompact type and  $M_1$  is of the compact type, while in the second case it is the other way around. We adopt the convention of denoting the compact space by  $M_F$  and the noncompact one by  $M_B$ .

In view of Cartan’s list of symmetric spaces (Table I), we arrive at Table II listing the large families of Riemannian symmetric superspaces. Although the entries  $A|A$ ,  $BD|C$ , and  $C|BD$  look

TABLE II. The large families of Riemannian symmetric superspaces.

Class	$G_\Lambda/H_\Lambda$	$M_B$	$M_F$
A A	$Gl(m n)$	A	A
AI AII	$Gl(m 2n)/Osp(m 2n)$	AI	AII
AII AI	$Gl(m 2n)/Osp(m 2n)$	AII	AI
AIII AIII	$Gl(m_1+m_2 n_1+n_2)/Gl(m_1 n_1)\times Gl(m_2 n_2)$	AIII	AIII
BD C	$Osp(m 2n)$	BD	C
C BD	$Osp(m 2n)$	C	BD
CI DIII	$Osp(2m 2n)/Gl(m n)$	CI	DIII
DIII CI	$Osp(2m 2n)/Gl(m n)$	DIII	CI
BDI CII	$Osp(m_1+m_2 2n_1+2n_2)/Osp(m_1 2n_1)\times Osp(m_2 2n_2)$	BDI	CII
CII BDI	$Osp(m_1+m_2 2n_1+2n_2)/Osp(m_1 2n_1)\times Osp(m_2 2n_2)$	CII	BDI

extraneous because they are groups rather than coset spaces, they fit in the same framework by putting by  $G_\Lambda = G \times G$  and  $\theta(g_1, g_2) = (g_2, g_1)$ , so  $H_\Lambda = \text{diag}(G \times G) \simeq G$  and  $G_\Lambda/H_\Lambda \simeq G$ .

As far as applications to random-matrix theory and disordered single-particle systems are concerned, the most important structure carried by Riemannian symmetric superspaces is their  $G_\Lambda$ -invariant Berezin measure. Such a measure always exists by Definition 2.1 and the existence of local coordinates. To describe it in explicit terms, one introduces a local coordinate system by the exponential map  $\mathcal{M}_\Lambda \rightarrow G_\Lambda/H_\Lambda$ ,  $Z \mapsto \exp(Z)H_\Lambda$ . By straightforward generalization (replace the Jacobian by the Berezinian) of a corresponding calculation (cf. Ref. 38) for ordinary symmetric spaces, one obtains for the principal term of the invariant Berezin measure the expression  $DZ \circ J(Z)$ , where  $DZ = dz^1 \wedge \cdots \wedge dz^p \otimes \partial_{\zeta^1} \cdots \partial_{\zeta^q}$  denotes the flat Berezin measure on  $\mathcal{M}_\Lambda$ , and if  $T_Z: \mathcal{M}_\Lambda \rightarrow \mathcal{M}_\Lambda$  is the linear operator defined by

$$T_Z = \sum_{n=0}^{\infty} \frac{\text{ad}^{2n}(Z)}{(2n+1)!},$$

the function  $J(Z) = \text{SDet} T_Z$ . [Note  $\sum_{n=0}^{\infty} x^{2n}/(2n+1)! = x^{-1} \sinh x$ ]. A universally valid expression for the anomaly in these coordinates is not available at present.

### III. SUPERSYMMETRY APPLIED TO THE GAUSSIAN RANDOM-MATRIX ENSEMBLE OF CLASS C

The goal of the remainder of this paper will be to demonstrate that Riemannian symmetric superspaces, as defined in Sec. II F, arise in a compelling way when Gaussian ensemble averages of ratios of spectral determinants for random matrices are considered in the large- $N$  limit. The example to be discussed in detail will be the Gaussian ensemble defined over the symplectic Lie algebra  $\text{sp}(N)$ , which has recently been identified<sup>24</sup> as a model for the ergodic limit of normal-superconducting mesoscopic systems with broken time-reversal symmetry.

#### A. The supersymmetry method: A simple example

The pedagogical purpose of this section is to illustrate our strategy at a simple example.<sup>41</sup> If  $\mathfrak{u}(N)$  is the Lie algebra of the unitary group in  $N$  dimensions, consider on  $iu(N)$  (the Hermitian  $N \times N$  matrices) the Gaussian probability measure with width  $v/\sqrt{N}$ . Denoting by  $H$  the elements of  $iu(N)$  and by  $dH$  a Euclidean measure, we write the Gaussian probability measure in the form  $d\mu(H) = \exp(-N \text{Tr} H^2/2v^2) dH$ ,  $\int d\mu(H) = 1$ . This measure is called the Gaussian unitary ensemble (GUE) in random-matrix theory. The object of illustration will be the average ratio of spectral determinants,

$$Z(\alpha, \beta) = \int_{iu(N)} \text{Det} \left( \frac{H - \beta}{H - \alpha} \right) d\mu(H),$$

where  $\alpha, \beta$  are complex numbers and  $\alpha$  is not in the spectrum of  $H$ . Given the generating function  $Z$ , the GUE average resolvent is obtained by

$$\int_{iu(N)} \text{Tr}(H - z)^{-1} d\mu(H) = \left. \frac{\partial}{\partial \alpha} Z(\alpha, \beta) \right|_{\alpha = \beta = z}.$$

We will now show how to compute  $Z$  using a formalism that readily generalizes to more complicated situations.

To avoid the introduction of indices and have a basis-independent formulation, we choose to interpret  $H$  as a self-adjoint endomorphism  $H \in \text{End}(V)$  of  $N$ -dimensional complex space  $V := \mathbb{C}^N$  with a Hermitian quadratic form  $(x, y) \mapsto \langle \bar{x}, y \rangle_V$ .

The supersymmetry method starts by introducing ‘‘bosonic space’’  $W_B = W_0 = \mathbb{C}$  and ‘‘fermionic space’’  $W_F = W_1 = \mathbb{C}$ . Auxiliary space is the  $\mathbb{Z}_2$ -graded sum  $W = W_B \oplus W_F = \mathbb{C}^{1|1}$ . The Cartesian basis of  $W$  is denoted by  $e_B = (1, 0)$  and  $e_F = (0, 1)$ . Let  $\text{Hom}_\lambda(W, V) := \lambda_0 \otimes \text{Hom}(W_B, V) + \lambda_1 \otimes \text{Hom}(W_F, V)$ , where  $\lambda = \lambda_0 + \lambda_1$  is the Grassmann algebra with  $\dim_{\mathbb{C}} \text{Hom}(W_F, V) = N$  generators. (Grassmann-analytic continuation will not be needed here.)  $\text{Hom}_{\tilde{\lambda}}(V, W)$  is defined similarly, with another Grassmann algebra  $\tilde{\lambda}$ . The key idea is to utilize the Gaussian Berezin integral over the complex-analytic superspace  $\text{Hom}_\lambda(W, V) \times \text{Hom}_{\tilde{\lambda}}(V, W)$ . Let  $D(\psi, \tilde{\psi})$  [with  $\psi \in \text{Hom}_\lambda(W, V)$  and  $\tilde{\psi} \in \text{Hom}_{\tilde{\lambda}}(V, W)$ ] denote a translation-invariant holomorphic Berezin measure on this linear space. If  $\psi_B(\tilde{\psi}_B)$  is the restriction of  $\psi(\tilde{\psi})$  to a map  $W_B \rightarrow V$  (resp.,  $V \rightarrow W_B$ ), fix a Berezin integral  $f \mapsto \int D(\psi, \psi) f(\psi, \tilde{\psi})$  by choosing for the domain of integration the subspace  $M_f$  selected by the linear condition  $\tilde{\psi}_B = \psi_B$  (the adjoint  $\psi_B^\dagger: \mathbb{C}^N \rightarrow \mathbb{C}$  being defined by  $\tilde{\psi}_B^\dagger z = \langle \bar{z}, \psi_B \cdot 1 \rangle_V$ ). Because  $\text{Hom}_\lambda(W, V) \times \text{Hom}_{\tilde{\lambda}}(V, W)$  has complex dimension  $(2N, 2N)$ , the integral  $\int D(\psi, \tilde{\psi}) f(\psi, \tilde{\psi})$  does not change its value when  $f$  is replaced by the rescaled function  $f^s(\psi, \tilde{\psi}) = f(s\psi, s\tilde{\psi})$  ( $s \in \mathbb{R}$ ). Now with  $\text{End}_0(W) = \text{End}(W_B) \oplus \text{End}(W_F)$  and  $\text{End}_1(W) = \text{Hom}(W_B, W_F) \oplus \text{Hom}(W_F, W_B)$ , let

$$\text{End}_\Lambda(W) := \Lambda_0 \otimes \text{End}_0(W) + \Lambda_1 \otimes \text{End}_1(W),$$

where  $\Lambda = \Lambda_0 + \Lambda_1$  is the Grassmann algebra with  $\dim_{\mathbb{C}} \text{End}_1(W) = 2$  generators, and pick  $A \in \text{End}(V)$ ,  $B \in \text{End}_\Lambda(W)$ .  $B$  corresponds to what is called a  $2 \times 2$  supermatrix in physics. An elementary but useful result is that, if we normalize  $D(\psi, \tilde{\psi})$  by  $\int D(\psi, \tilde{\psi}) \exp(-s^2 \text{Tr} \psi \tilde{\psi}) = 1$ , the identity

$$\int D(\psi, \tilde{\psi}) \exp(i \text{Tr}_V A \psi \tilde{\psi} - i \text{STr}_W B \tilde{\psi} \psi) = \text{SDet}_{V \otimes W}(A \otimes 1 - 1 \otimes B)^{-c}, \tag{10}$$

holds with  $c = 1$  provided that the integral exists. (The parameter  $c$  is introduced for later convenience.) When  $A$  and  $B$  are represented by diagonal matrices, verification of (10) is a simple matter of doing one-dimensional Gaussian integrals. The general case follows by the invariance of  $D(\psi, \tilde{\psi})$  under unitary transformations of  $V$  and ‘‘super-rotations’’ in  $W$ .

Now introduce elements  $E_{BB}$  and  $E_{FF}$  of  $\text{End}_0(W)$  by  $E_{BB}e_B = e_B$ ,  $E_{FF}e_F = e_F$ , and  $E_{BB}e_F = E_{FF}e_B = 0$ . By setting  $A := H$  and  $B := \alpha E_{BB} + \beta E_{FF} =: \omega$ , and using

$$\text{SDet}_{V \otimes W}(H \otimes 1 - 1 \otimes \omega) = \text{Det}(H - \alpha) / \text{Det}(H - \beta),$$

we get a Gaussian integral representation of  $Z$ :

$$\begin{aligned}
Z(\omega) &:= Z(\alpha, \beta) = \int \text{SDet}_{V \otimes W}(H \otimes 1 - 1 \otimes \omega)^{-c} d\mu(H) \\
&= \int D(\psi, \tilde{\psi}) \int \exp(i \text{Tr}_V H \psi \tilde{\psi} - i \text{STr}_W \omega \tilde{\psi} \psi) d\mu(H).
\end{aligned} \tag{11}$$

In the next step, the GUE ensemble average is subjected to the following manipulations:

$$\begin{aligned}
\int \exp(i \text{Tr} H \psi \tilde{\psi}) d\mu(H) &= \int_{i\mathfrak{u}(N)} \exp\left(i \text{Tr} H \psi \tilde{\psi} - \frac{N \text{Tr} H^2}{2v^2}\right) dH \\
&= \exp - \frac{v^2}{2N} \text{Tr}_V(\psi \tilde{\psi})^2 \\
&= \exp - \frac{v^2}{2N} \text{STr}_W(\tilde{\psi} \psi)^2 \\
&= \int_{\mathbb{R} \times i\mathbb{R}} DQ \exp\left(i \text{STr} Q \tilde{\psi} \psi - \frac{N \text{STr} Q^2}{2v^2}\right) \\
&=: \int D\mu(Q) \exp(i \text{STr} Q \tilde{\psi} \psi).
\end{aligned} \tag{12}$$

The fourth equality sign decouples the quartic term  $\text{STr}_W(\tilde{\psi} \psi)^2$  by introducing an auxiliary integration over  $Q \in \text{End}_\Lambda(W)$ . In order for this Gaussian integral to converge, the integration domain for the BB part  $Q_{\text{BB}}: W_{\text{B}} \rightarrow W_{\text{B}}$  (FF part,  $Q_{\text{FF}}: W_{\text{F}} \rightarrow W_{\text{F}}$ ), is taken to be the real (resp., imaginary) numbers. By using the relations (10)–(12), we obtain

$$\begin{aligned}
Z(\omega) &= \int D(\psi, \tilde{\psi}) \left( \int \exp(i \text{Tr}_V H \psi \tilde{\psi}) d\mu(H) \right) \exp - i \text{STr}_W \omega \tilde{\psi} \psi \\
&= \int D\mu(Q) \int D(\psi, \tilde{\psi}) \exp i \text{Tr}_V \psi(Q - \omega) \tilde{\psi} \\
&= \int D\mu(Q) \text{SDet}_{V \otimes W}(1_N \otimes (Q - \omega))^{-c} \\
&= \int D\mu(Q) \text{SDet}_W(Q - \omega)^{-N} \\
&= \int_{\mathbb{R} \times i\mathbb{R}} DQ \exp - N \text{STr} \left( \frac{Q^2}{2v^2} + \ln(Q - \omega) \right).
\end{aligned} \tag{13}$$

These steps reduce an integral over the  $N \times N$  matrix  $H$  to an integral over the  $2 \times 2$  supermatrix  $Q$ . The large parameter  $N$  now appears in the exponent of the integrand, so that the  $Q$  integral can be evaluated by a saddle-point approximation that becomes exact in the limit  $N \rightarrow \infty$ . By solving the saddle-point equation  $-Q/v^2 = (Q - \omega)^{-1}$  and doing an elementary calculation, one obtains Wigner's semicircle law for the GUE density of states:<sup>42</sup>

$$\int \text{Tr} \delta(E - H) d\mu(H) = \frac{N}{\pi v} \sqrt{1 - \left(\frac{E}{2v}\right)^2}, \tag{14}$$

which will be of use later.

**B. Definition of the Gaussian ensemble of type C**

Having run through a simple and well-known example, we now treat in detail a less trivial case where the reduction to a  $Q$ -integral representation requires more care.

The ‘‘physical space’’ of our model is  $V = \mathbb{C}^2 \otimes \mathbb{C}^N$ . As before, let  $x \mapsto \bar{x}$  denote complex conjugation, and fix a symmetric quadratic form  $\langle \bullet, \bullet \rangle_V : V \times V \rightarrow \mathbb{C}$ , such that the corresponding Hermitian quadratic form  $\langle \bar{x}, y \rangle_V = \overline{\langle y, x \rangle_V}$  is strictly positive. The transpose and the adjoint of a linear transformation  $L \in \text{End}(V)$  are defined by  $\langle x, L^T y \rangle_V = \langle Lx, y \rangle_V$  and  $\langle \bar{x}, L^\dagger y \rangle_V = \overline{\langle Lx, y \rangle_V}$ , as usual.

Consider now the space,  $P$ , of self-adjoint Hamiltonians  $H \in \text{End}(V)$  subject to the linear condition

$$H = -\mathcal{E}H^T\mathcal{E}^{-1}, \tag{15}$$

where  $\mathcal{E}$  is skew and  $\mathcal{E}^2 = -1$ . Clearly,  $iP$  is isomorphic to  $\text{sp}(N) = C_N$  (the symplectic Lie algebra in  $2N$  dimensions). Introducing an orthonormal real basis of  $V$  we can represent  $H$  by a  $2N \times 2N$  matrix. The explicit form of such a matrix is

$$H = \begin{pmatrix} a & b \\ b^\dagger & -a^T \end{pmatrix}, \quad \text{if } \mathcal{E} = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix},$$

where  $a(b)$  is a complex Hermitian (resp., symmetric)  $N \times N$  matrix. The Gaussian ensemble to be studied is defined by the probability measure  $d\mu(H) = \exp(-N \text{Tr } H^2/2v^2)dH$ ,  $\int d\mu(H) = 1$ . For any two  $A, B \in \text{End}(V)$ ,

$$\int_{i \times \text{sp}(N)} \text{Tr}(AH)\text{Tr}(BH)d\mu(H) = \frac{v^2}{2N} \text{Tr}(AB - A\mathcal{E}B^T\mathcal{E}^{-1}). \tag{16}$$

The joint probability density for the eigenvalues of  $H$  has been given in Ref. 24.

The physical motivation for considering a Gaussian random-matrix ensemble of the above type (type  $C$ ) comes from the fact<sup>24</sup> that it describes the ergodic limit of mesoscopic normal-superconducting hybrid systems with time-reversal symmetry broken by the presence of a weak magnetic field. To deal with such systems, the Bogoliubov–deGennes (BdG) independent-quasiparticle formalism is used. The first factor in the tensor product  $V = \mathbb{C}^2 \otimes \mathbb{C}^N$  accounts for the BdG particle-hole degree of freedom, which is introduced for the purpose of treating the pairing field of the superconductor within the formalism of first quantization. The second factor represents the orbital degrees of freedom of the electron.  $H$  is the Hamiltonian that enters into the BdG equations, and the relation (15) expresses the particle-hole symmetry of the BdG formalism.

Our goal is to compute the following ensemble average:

$$Z_n(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n) = \int_{i \text{ sp}(N)} \prod_{i=1}^n \text{Det} \begin{pmatrix} H - \beta_i \\ H - \alpha_i \end{pmatrix} d\mu(H). \tag{17}$$

By the particle-hole symmetry of  $H$ ,  $Z_n$  is invariant under a reversal of sign for any pair  $(\alpha_i, \beta_j)$ , so no information is lost by restricting all  $\alpha_i$  to one-half of the complex plane. For definiteness, we require

$$\text{Im } \alpha_i < 0 \quad (i = 1, \dots, n). \tag{18}$$

All information about the statistical correlations between the eigenvalues of  $H$  can be extracted from  $Z_n$ . For example, the probability that, given there is an eigenvalue at  $E_1$ , there exist  $n-1$  eigenvalues at  $E_2, \dots, E_n$  (regardless of the positions of all other eigenvalues) is equal to

$$R_n(E_1, \dots, E_n) = \lim_{\epsilon \rightarrow 0} \left( \frac{-\epsilon}{\pi} \right)^n \prod_{l=1}^n \left. \frac{\partial}{\partial \alpha_{2l}} \right|_{\alpha_{2l} = E_l - i\epsilon} \prod_{l=1}^n \left. \frac{\partial}{\partial \alpha_{2l-1}} \right|_{\alpha_{2l-1} = -E_l - i\epsilon} \times Z_{2n}(\alpha_1, \dots, \alpha_{2n}; E_1 - i\epsilon, -E_1 - i\epsilon, \dots, E_n - i\epsilon, -E_n - i\epsilon). \tag{19}$$

The function  $R_n(E_1, \dots, E_n)$  is called the  $n$ -level correlation function in random-matrix theory.<sup>42</sup>

### C. Symmetries of the auxiliary space

To transcribe the supersymmetry method of Sec. III A to the computation of  $Z_n$  (which involves  $n$  ratios of spectral determinants), a simple and natural procedure would be to enlarge the auxiliary space  $W$  by taking the tensor product with  $\mathbb{C}^n$ . However, on using the formula

$$\int \exp(i \operatorname{Tr} H \psi \tilde{\psi}) d\mu(H) = \exp\left(-\frac{1}{2} \int_{i \operatorname{sp}(N)} (\operatorname{Tr} H \psi \tilde{\psi})^2 d\mu(H)\right),$$

one faces the complication that the second moment  $\int (\operatorname{Tr} H \psi \tilde{\psi})^2 d\mu(H)$  then is a sum of two terms, see the right-hand side of (16). Consequently, one needs *two* decoupling supermatrices  $Q$  (one for each term). Although this presents no difficulty of a principal nature, it does lead to rather complicated notations. An elegant remedy is to modify the definition of  $\psi$  and  $\tilde{\psi}$  so that  $\psi \tilde{\psi}$  shares the symmetry (15) of the BdG Hamiltonian  $H$ . The two terms then combine into a single one:

$$\int (\operatorname{Tr} H \psi \tilde{\psi})^2 d\mu(H) = \frac{v^2}{N} \operatorname{STr}_W(\tilde{\psi} \psi)^2,$$

which can again be decoupled by a single supermatrix  $Q$ . To implement the symmetry (15), we proceed as follows.

We enlarge the auxiliary space  $W = W_B \oplus W_F$  in some way (left unspecified for the moment) and fix a rule of supertransposition  $\operatorname{Hom}_\lambda(W, V) \rightarrow \operatorname{Hom}_\lambda(V, W)$ ,  $\psi \mapsto \psi^T$ , and  $\operatorname{Hom}_\lambda(V, W) \rightarrow \operatorname{Hom}_\lambda(W, V)$ ,  $\tilde{\psi} \mapsto \tilde{\psi}^T$ . Such a rule obeys  $\psi^{TT} = \psi \sigma$  and  $\tilde{\psi}^{TT} = \sigma \tilde{\psi}$ , where  $\sigma \in \operatorname{End}_0(W)$  is the operator for superparity, i.e.  $\sigma(x+y) = x-y$  for  $x+y \in W_B \oplus W_F = W$ . It induces a rule of supertransposition  $\operatorname{End}_\lambda(W) \rightarrow \operatorname{End}_\lambda(W)$ ,  $Q \mapsto Q^T$  (no separate symbol is introduced). Combination with complex conjugation gives a rule of Hermitian conjugation  $\dagger: \operatorname{End}_0(W) \rightarrow \operatorname{End}_0(W)$ . Now impose on  $\psi \in \operatorname{Hom}_\lambda(W, V)$ ,  $\tilde{\psi} \in \operatorname{Hom}_\lambda(V, W)$  the linear conditions

$$\psi = \mathcal{E} \tilde{\psi}^T \gamma^{-1}, \quad \tilde{\psi} = -\gamma \psi^T \mathcal{E}^{-1}, \tag{20}$$

with some invertible even element  $\gamma$  of  $\operatorname{End}_0(W)$ . The mutual consistency of these equations requires

$$\gamma = \gamma^T \sigma. \tag{21}$$

To see that, insert the transpose of the second equation in (20) into the first one. Using  $\psi^{TT} = \psi \sigma$  you obtain  $\psi = -\mathcal{E} \mathcal{E}^{-1T} \psi \sigma \gamma^T \gamma^{-1}$ . Since  $\mathcal{E} \mathcal{E}^{-1T} = -1$  and  $\sigma \gamma^T = \gamma^T \sigma$ , Eq. (21) follows. The consistency condition can be implemented by taking  $W_B = W_F = \mathbb{C}^2 \otimes \mathbb{C}^n$ , see below. By multiplying the equations (20) we obtain

$$\psi \tilde{\psi} = -\mathcal{E} (\psi \tilde{\psi})^T \mathcal{E}^{-1}, \quad \tilde{\psi} \psi = -\gamma (\tilde{\psi} \psi)^T \gamma^{-1}. \tag{22}$$

The first equation is the desired symmetry relation allowing us to combine terms. To appreciate the consequences of the second equation, note that by the fourth step in (12) the symmetries of  $\tilde{\psi}\psi$  get transferred onto  $Q$ , so that the latter is subject to

$$Q = -\gamma Q^T \gamma^{-1}. \tag{23}$$

This symmetry reflects that of the BdG Hamiltonian  $H$ , see (15). The linear space  $\text{End}_\Lambda(W)$ , when given a Lie bracket by the commutator, can be identified with  $\mathfrak{gl}(2n|2n) = \text{Lie}(\text{Gl}(2n|2n))$ . As  $\gamma$  is supersymmetric ( $\gamma = \gamma^T \sigma$ ), (23) fixes an  $\mathfrak{osp}(2n|2n)$  subalgebra.

$\gamma$  is not unique. For definiteness we choose it as follows. Let  $\{E_{ij}\}_{i,j=1,\dots,M}$  be a canonical basis of  $\text{End}(\mathbb{C}^M)$  satisfying  $E_{ij}E_{kl} = \delta_{jk}E_{il}$  (here  $M=2$  or  $M=n$ ). For  $M=2$  define the Pauli spin operators  $\sigma_x = E_{12} + E_{21}$ ,  $\sigma_y = -iE_{12} + iE_{21}$ , and  $\sigma_z = E_{11} - E_{22}$ . The usual rule of supertransposition on  $\text{End}_\Lambda(W)$  is given by  $(\mu, \nu=1,2$  and  $i,j=1,\dots,n)$

$$\begin{aligned} (E_{\text{BB}} \otimes E_{\mu\nu} \otimes E_{ij})^T &= E_{\text{BB}} \otimes E_{\nu\mu} \otimes E_{ji}, & (E_{\text{BF}} \otimes E_{\mu\nu} \otimes E_{ij})^T &= -E_{\text{FB}} \otimes E_{\nu\mu} \otimes E_{ji}, \\ (E_{\text{FB}} \otimes E_{\mu\nu} \otimes E_{ij})^T &= E_{\text{BF}} \otimes E_{\nu\mu} \otimes E_{ji}, & (E_{\text{FF}} \otimes E_{\mu\nu} \otimes E_{ij})^T &= E_{\text{FF}} \otimes E_{\nu\mu} \otimes E_{ji}. \end{aligned}$$

With these conventions, one possible choice for  $\gamma$  is

$$\gamma = E_{\text{BB}} \otimes \gamma_B + E_{\text{FF}} \otimes \gamma_F, \quad \text{where } \gamma_B = \sigma_x \otimes 1_n, \quad \gamma_F = i\sigma_y \otimes 1_n. \tag{24}$$

This is the choice we make.

#### D. Gaussian Berezin integral

To repeat the steps of Sec. III A and derive a  $Q$ -integral representation for the generating function  $Z_n$ , we must first generalize the basic identity (10), whose left-hand side is

$$\int D(\psi, \tilde{\psi}) \exp(i \text{Tr}_V A \psi \tilde{\psi} - i \text{STr}_W B \tilde{\psi} \psi). \tag{25}$$

By (22) we have

$$\begin{aligned} \text{Tr } A \psi \tilde{\psi} &= \text{Tr}(\psi \tilde{\psi})^T A^T = \frac{1}{2} \text{Tr}(A - \mathcal{C} A^T \mathcal{C}^{-1}) \psi \tilde{\psi}, \\ \text{STr } B \tilde{\psi} \psi &= \text{STr}(\tilde{\psi} \psi)^T B^T = \frac{1}{2} \text{Tr}(B - \gamma B^T \gamma^{-1}) \tilde{\psi} \psi. \end{aligned}$$

In view of this we demand that  $A$  and  $B$  satisfy

$$A = -\mathcal{C} A^T \mathcal{C}^{-1}, \quad B = -\gamma B^T \gamma^{-1}. \tag{26}$$

When carrying out the calculation (11)–(13) we need to apply the identity (10) twice, the first time with  $A = H$ ,  $B = \omega$ , and the second time with  $A = 0$ ,  $B = \omega - Q$ . In order for (26) to be satisfied with these identifications, we choose to set

$$\omega = E_{\text{BB}} \otimes \sigma_z \otimes \sum_{i=1}^n \alpha_i E_{ii} + E_{\text{FF}} \otimes \sigma_z \otimes \sum_{j=1}^n \beta_j E_{jj}.$$

The presence of the factor  $\sigma_z = \text{diag}(+1, -1)$  reverses the sign of the  $\alpha_i$  and  $\beta_j$  on that subspace where  $\sigma_z$  acts by multiplication with  $-1$ . As the imaginary parts of the  $\alpha_i$  control the convergence of the integral, this sign reversal affects the correct choice of integration domain for  $\psi_B$  and  $\tilde{\psi}_B$ . To ensure convergence of the integral (25), we require  $\text{Im STr } \omega \tilde{\psi} \psi \leq 0$ . This inequality is achieved by imposing the condition  $\psi_B = (\sigma_z \otimes 1_n) \psi_B^\dagger$ , which is compatible with  $\mathcal{C} = i\sigma_y \otimes 1_n$ ,  $\psi_B = \mathcal{C} \tilde{\psi}_B^T \gamma_B^{-1}$ , and  $\gamma_B = \sigma_x \otimes 1_n$ .



*Lemma 3.1:* Let  $D(\psi, \tilde{\psi})$  denote a translation-invariant holomorphic Berezin measure on the subspace of  $\text{Hom}_\lambda(W, V) \times \text{Hom}_\lambda(V, W)$  defined by (20). Fix the integration domain by  $\tilde{\psi}_B = (\sigma_z \otimes 1_n) \psi_B^\dagger$ , and normalize  $D(\psi, \tilde{\psi})$  so that  $\int D(\psi, \tilde{\psi}) \exp(-s^2 \text{Tr } \psi \tilde{\psi}) = 1$  ( $s \in \mathbb{R}$ ). Then if  $A \in \text{End}(V)$  and  $B \in \text{End}_\lambda(W)$  are diagonalizable and satisfy the linear conditions (26), the identity (10) holds with  $c=1/2$  provided that the integral exists.

*Proof:* Assume that  $A$  and  $B$  are represented by diagonal matrices,

$$A = \sigma_z \otimes \sum_{i=1}^N x_i E_{ii}, \quad B = E_{BB} \otimes \sigma_z \otimes \sum_{j=1}^n z_j E_{jj} + E_{FF} \otimes \sigma_z \otimes \sum_{j=1}^n y_j E_{jj},$$

which conforms with (26). The right-hand side of (10) then reduces to

$$\text{SDet}_{V \otimes W}(A \otimes 1 - 1 \otimes B)^{-1/2} = \prod_{i=1}^N \prod_{j=1}^n \frac{(x_i - y_j)(x_i + y_j)}{(x_i - z_j)(x_i + z_j)}. \tag{27}$$

To evaluate the left-hand side, write

$$\psi_B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \psi_F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where  $a, b, c, d, \alpha, \beta, \gamma, \delta$  are complex  $N \times n$  matrices with commuting (resp., anticommuting) matrix elements. The constraint  $\tilde{\psi} = -\gamma \psi^T \mathcal{E}^{-1}$  results in

$$\tilde{\psi}_B = \begin{pmatrix} -d^T & b^T \\ -c^T & a^T \end{pmatrix}, \quad \tilde{\psi}_F = \begin{pmatrix} -\delta^T & \beta^T \\ \gamma^T & -\alpha^T \end{pmatrix},$$

and the reality condition  $\tilde{\psi}_B = (\sigma_z \otimes 1_n) \psi_B^\dagger$  means  $d = -\bar{a}$  and  $c = \bar{b}$ . The exponent of the integrand is expressed by

$$\begin{aligned} \frac{1}{2} \text{Tr } A \psi \tilde{\psi} - \frac{1}{2} \text{Tr } B \tilde{\psi} \psi &= \sum_{i=1}^N \sum_{j=1}^n ((x_i - z_j) a_{ij} \bar{a}_{ij} - (x_i + z_j) b_{ij} \bar{b}_{ij} \\ &\quad + (x_i + y_j) \alpha_{ij} \delta_{ij} - (x_i - y_j) \beta_{ij} \gamma_{ij}). \end{aligned}$$

Doing the Gaussian integrals one gets a result that is identical to (27), which proves the Lemma for diagonal  $A$  and  $B$ . The general case follows by the invariance properties of  $D(\psi, \tilde{\psi})$ .

*Remark:* The condition of diagonalizability can of course be weakened but we will not need that here. ■

To apply Lemma 3.1 to our problem, note

$$\text{SDet}_{V \otimes W}(H \otimes 1 - 1 \otimes \omega)^{1/2} = \prod_{i=1}^n \text{Det}_V \left( \frac{(H - \alpha_i)(H + \alpha_i)}{(H - \beta_i)(H + \beta_i)} \right)^{1/2} = \prod_{i=1}^n \text{Det} \left( \frac{H - \alpha_i}{H - \beta_i} \right),$$

where in the second step we used the invariance of the ratio of determinants under  $H \mapsto -H$ , which is due to the particle-hole symmetry  $H = -\mathcal{E} H^T \mathcal{E}^{-1}$ . Moreover, note

$$\text{SDet}_{V \otimes W}(1 \otimes (Q - \omega))^{-1/2} = \text{SDet}_W(Q - \omega)^{-N}.$$

The previous calculation (11)–(13) thus *formally* goes through with  $c=1/2$ , and  $\text{isp}(N)$  for  $\text{iu}(N)$ , and we arrive at the following representation of the generating function:

$$Z_n(\omega) = \int DQ \exp -N \text{STr} \left( \frac{Q^2}{2v^2} + \ln(Q - \omega) \right), \tag{28}$$

where the supermatrix

$$Q = \begin{pmatrix} Q_{BB} & Q_{BF} \\ Q_{FB} & Q_{FF} \end{pmatrix}$$

is subject to (23). To make this rigorous, we have to specify the integration domain for  $Q$  and show that the interchange of the  $(\psi, \tilde{\psi})$ - and  $Q$ -integrations is permitted.

**E. Choice of integration domain**

If the steps (11)–(13) are to be valid, we must arrange for all integrals to be convergent, at least. This is easily achieved for  $Q_{FF}$ , the FF component of  $Q$ , but requires substantial labor for  $\psi_B$ ,  $\tilde{\psi}_B$ , and  $Q_{BB}$ . Consider  $Q_{FF}$  first. Since  $-\text{STr} Q^2 = -\text{Tr} Q_{BB}^2 + \text{Tr} Q_{FF}^2 + \text{nilpotents}$ , we want  $\text{Tr} Q_{FF} Q_{FF} \leq 0$ , which leads us to require that  $Q_{FF}$  be anti-Hermitian. Combining this with (23) we get

$$Q_{FF} = -\gamma_F Q_{FF}^T \gamma_F^{-1} = -Q_{FF}^\dagger,$$

where  $\gamma_F = i\sigma_y \otimes 1_n$ ; see (24). The solution space of these equations is  $\mathfrak{sp}(n)$ , the symplectic Lie algebra in  $2n$  dimensions. Thus we choose  $\mathcal{U} := \mathfrak{sp}(n)$  for the integration domain of  $Q_{FF}$ , and, of course, the integration measure is taken to be the flat one.

The choice of integration domain for  $Q_{BB}$  is a much more delicate matter and will occupy us for the remainder of this section. Recall, first of all, that the convergence of

$$\int D(\psi, \tilde{\psi}) \exp(i \text{Tr} H \psi \tilde{\psi} - i \text{STr} \omega \tilde{\psi} \psi)$$

requires taking  $\tilde{\psi}_B = \beta \psi_B^\dagger$ , where  $\beta := \sigma_z \otimes 1_n$  cancels the minus signs that multiply the imaginary parts of the parameters  $\alpha_i$  in  $\omega$ . To ensure the convergence of

$$\int D(\psi, \tilde{\psi}) \exp i \text{Tr} \psi(Q - \omega) \tilde{\psi},$$

one is tempted to choose  $Q_{BB}$  in such a way that  $\text{Re Tr} \psi Q \tilde{\psi} = 0$ . Unfortunately, when this condition is adopted one gets  $Q_{BB} = \beta Q_{BB}^\dagger \beta$ , which causes  $\text{Tr} Q_{BB}^2 = \text{Tr} Q_{BB} \beta Q_{BB}^\dagger \beta$ , to be of *indefinite sign*, so that the integral over  $Q$  does not exist.

A way out of this difficulty was first described by Schäfer and Wegner<sup>2</sup> in a related context. We are now going to formulate their prescription in a language that anticipates the geometric structure emerging in the large- $N$  limit. To simplify the notation, we put  $Q_{BB} = iZ$ . What we need to do is investigate the expression

$$\exp(-N \text{Tr} Q_{BB}^2 / 2v^2 + i \text{Tr} Q_{BB} \tilde{\psi}_B \psi_B) = \exp(N \text{Tr} Z^2 / 2v^2 - \text{Tr} Z \tilde{\psi}_B \psi_B). \tag{29}$$

The conditions on  $Q_{BB}$  translate into

$$Z = -\gamma_B Z^T \gamma_B^{-1} = -\beta Z^\dagger \beta^{-1}.$$

Because  $\gamma_B = \sigma_x \otimes 1_n$  is symmetric, the solution space of the first equation is a complex Lie algebra  $\mathcal{S}_\mathbb{C} \simeq \mathfrak{so}(2n, \mathbb{C})$ . The matrix representation of an element  $Z \in \mathcal{S}_\mathbb{C}$  is of the form

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where  $B$  and  $C$  are skew. The second equation ( $Z = -\beta Z^\dagger \beta^{-1}$ ) means  $A = -A^\dagger$  and  $C = B^\dagger$ , which fixes a real form  $\mathcal{G} = \mathfrak{so}^*(2n)$  of  $\mathcal{G}_\mathbb{C} = \mathfrak{so}(2n, \mathbb{C})$ . This real form is *noncompact* [i.e.,  $\mathcal{G} = \text{Lie}(G)$  with  $G$  a noncompact Lie group], which is what causes all the trouble and is forcing us to work hard. Its maximal compact subalgebra  $\mathcal{K}$  is the set of solutions of  $X = \beta X \beta^{-1}$  in  $\mathcal{G}$ . From

$$X = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}$$

and  $A = -A^\dagger$  we see that  $\mathcal{K} \cong \mathfrak{u}(n)$ .

To display clearly the general nature of the following construction, we introduce a symmetric quadratic form  $B: \mathcal{G}_\mathbb{C} \times \mathcal{G}_\mathbb{C} \rightarrow \mathbb{C}$  by  $B(X, Y) = \text{Tr} XY$ . The Cartan (orthogonal) decomposition of  $\mathcal{G}$  with respect to this quadratic form is written as  $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$ . An element  $Y$  of  $\mathcal{M}$  satisfies  $Y = -\beta Y \beta^{-1}$ . From this, in conjunction with the equation fixing  $\mathcal{K}$  ( $X = +\beta X \beta^{-1}$ ), one deduces the commutation relations

$$[\mathcal{M}, \mathcal{M}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{K}, \mathcal{K}] \subset \mathcal{K}. \tag{30}$$

Note that the elements of  $\mathcal{M}$  are Hermitian while those of  $\mathcal{K}$  are anti-Hermitian. We will also encounter the complexified spaces  $\mathcal{K}_\mathbb{C} = \mathcal{K} + i\mathcal{K}$  and  $\mathcal{M}_\mathbb{C} = \mathcal{M} + i\mathcal{M}$ . They, too, are orthogonal with respect to  $B$  and satisfy the commutation relations (30). The element  $\beta = \sigma_z \otimes 1_n$  satisfies  $\beta = -\gamma_B \beta^T \gamma_B^{-1}$  and can therefore be regarded as an element of  $\mathcal{G}_\mathbb{C}$ . Moreover,  $\beta \in i\mathcal{K} \subset \mathcal{G}_\mathbb{C}$ .

Now we embed  $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$  into  $\mathcal{G}_\mathbb{C}$  by a map  $\phi_b$ ,

$$\phi_b: \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{G}_\mathbb{C},$$

$$(X, Y) \mapsto \phi_b(X, Y) = b \times (X + e^Y \beta e^{-Y}),$$

where  $b \neq 0$  is some constant that will be specified later.

*Lemma 3.2:*  $\phi_b(\mathcal{K} \times \mathcal{M})$  is an analytic manifold without boundary, and is diffeomorphic to  $\mathcal{G}$ .

*Proof:* Analyticity is clear. To prove the other properties, we first establish that  $\phi_b$  is injective. For that purpose, we write  $e^Y \beta e^{-Y} = e^{\text{ad}(Y)} \beta$ , where  $\text{ad}(Y)\beta = [Y, \beta]$  is the adjoint action on  $\mathcal{G}_\mathbb{C}$ . Decomposing the exponential function according to  $\exp = \cosh + \sinh$ , we write  $\phi_b = \phi_+ + \phi_-$ , where

$$\phi_+(X, Y) = b \times (X + \cosh \text{ad}(Y)\beta),$$

$$\phi_-(X, Y) = b \times \sinh \text{ad}(Y)\beta.$$

From the commutation relations (30) and  $\beta \in i\mathcal{K}$ , we see that  $\phi_\pm$  takes values  $\phi_+(X, Y) \in \mathcal{K}_\mathbb{C}$  and  $\phi_-(X, Y) \in \mathcal{M}_\mathbb{C}$ . Since  $\mathcal{G}_\mathbb{C} = \mathcal{K}_\mathbb{C} \oplus \mathcal{M}_\mathbb{C}$  (direct sum), injectivity is equivalent to the regularity of the maps  $X \mapsto \phi_+(X, Y)$  (with  $Y$  viewed as a parameter) and  $Y \mapsto \phi_-(X, Y)$ . The function  $\phi_+(X, \cdot) = X + \text{const}$  is obviously regular. By  $Y = Y^\dagger$  the element  $Y$  is diagonalizable with real eigenvalues. The regularity of  $\phi_-$  then follows from  $\sinh: \mathbb{R} \rightarrow \mathbb{R}$  being monotonic and  $Y \mapsto \text{ad}(Y)\beta$  being regular. This completes the proof that  $\phi_b$  is injective. The injectivity of  $\phi_b$  means that  $\phi_b(\mathcal{K} \times \mathcal{M})$  is diffeomorphic to  $\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$ . This in turn means that, since  $\mathcal{G}$  has no boundary,  $\phi_b(\mathcal{K} \times \mathcal{M})$  has no boundary either. ■

We are now going to demonstrate that  $\phi_b(\mathcal{H} \times \mathcal{M})$  for any  $b > 0$  may serve as a mathematically satisfactory domain of integration for the variable  $Z$  in (29). We begin by investigating the quadratic form  $\text{Tr } Z^2 = \text{B}(Z, Z)$  on  $\phi_b(\mathcal{H} \times \mathcal{M})$ . For this we set  $Z = Z_+ + Z_-$  with  $Z_{\pm} = \phi_{\pm}(X, Y)$ . Using  $\text{B}(Z_+, Z_-) = 0$  (recall  $\mathcal{H} \perp \mathcal{M}$ ),  $\text{B}(\text{ad}(Y)A, B) = -\text{B}(A, \text{ad}(Y)B)$  and  $\cosh^2 - \sinh^2 = 1$ , we obtain

$$\text{B}(Z, Z)/b^2 = \text{B}(X, X) + 2\text{B}(X, \cosh \text{ad}(Y)\beta) + \text{B}(\beta, \beta).$$

The antihermiticity of  $X \in \mathcal{H}$  gives  $\text{B}(X, X) \leq 0$ . In contrast,  $\cosh \text{ad}(Y)\beta \in i\mathcal{H}$  is Hermitian, so  $\text{B}(X, \cosh \text{ad}(Y)\beta) \in i\mathbb{R}$ . It follows that  $\exp(N \text{Tr } Z^2/2v^2) = \exp(N \text{Tr } \phi_b(X, Y)^2/2v^2)$  is decaying with respect to  $X$  and oscillatory with respect to  $Y$ .

We have not yet made any use of  $b > 0$  yet. This inequality comes into play when the coupling term

$$-\text{Tr } Z \tilde{\psi}_B \psi_B = -\text{B}(Z, \tilde{\psi}_B \psi_B) = -b\text{B}(X, \tilde{\psi}_B \psi_B) - b\text{B}(e^Y \beta e^{-Y}, \tilde{\psi}_B \psi_B)$$

is considered. From (22) and  $\tilde{\psi}_B = \beta \psi_B^\dagger$  we see that  $\tilde{\psi}_B \psi_B$  satisfies

$$\tilde{\psi}_B \psi_B = -\gamma_B (\tilde{\psi}_B \psi_B)^T \gamma_B^{-1} = +\beta (\tilde{\psi}_B \psi_B)^\dagger \beta^{-1},$$

so  $\tilde{\psi}_B \psi_B \in i\mathcal{S}$ . Since  $\text{B}$  is real valued on  $\mathcal{S} \times \mathcal{S}$ , the term  $\text{B}(X, \tilde{\psi}_B \psi_B)$  is purely imaginary. The other term,

$$-b\text{B}(e^Y \beta e^{-Y}, \tilde{\psi}_B \psi_B) = -b \text{Tr}(\psi_B e^{2Y} \psi_B^\dagger) \leq 0,$$

is never positive if  $b > 0$ . Hence the real part of the exponential in (29) is negative semidefinite for  $Q = iZ \in i\phi_b(\mathcal{H} \times \mathcal{M})$  and  $b > 0$ . As a result, the integrals over  $Q$  and  $\psi, \tilde{\psi}$  converge if the integration domain for  $Q$  is taken to be  $i\phi_b(\mathcal{H} \times \mathcal{M}) \times \mathcal{U}$  ( $b > 0$ ). Because  $i\phi_b(\mathcal{H} \times \mathcal{M}) \times \mathcal{U}$  is an analytic manifold without boundary and Cauchy's theorem applies, we may perform the shift of integration variables that is implied by the fourth equality sign in (12). Moreover, the presence of the nonvanishing imaginary parts of the parameters  $\alpha_i$  in  $\omega$  ensures *uniform convergence* of the  $(\psi, \tilde{\psi})$  integral with respect to  $Q$ , so that we may interchange the order of integration [the second equality sign in (13)]. And finally, any breakdown of diagonalizability of  $Q - \omega$  occurs on a set of measure zero, so that the identity (10) (Lemma 3.1) may be used, and all steps leading to (28) are rigorous. In summary, we have proved the following result.

**Theorem 3.3:** For  $V = \mathbb{C}^2 \otimes \mathbb{C}^N$  and  $W = \mathbb{C}^{11} \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$  define the generating function

$$Z_{n,N}(\omega) = \int_{i \times \text{sp}(N)} \text{SDet}_{V \otimes W}(H \otimes 1 - 1 \otimes \omega)^{-1/2} \exp\left(-\frac{N \text{Tr } H^2}{2v^2}\right) dH,$$

$$\omega = E_{\text{BB}} \otimes \sigma_z \otimes \sum_{i=1}^n \alpha_i E_{ii} + E_{\text{FF}} \otimes \sigma_z \otimes \sum_{j=1}^n \beta_j E_{jj} \quad (\text{Im } \alpha_i < 0).$$

Let  $DQ$  denote a translation-invariant holomorphic Berezin measure of the complex-analytic superspace  $\text{osp}(2n|2n)$ . Then for all  $N \in \mathbb{N}$ ,  $n \in \mathbb{N}$  and  $b > 0$ ,  $DQ$  can be normalized so that

$$Z_{n,N}(\omega) = \int_{i\phi_b(\mathcal{H} \times \mathcal{M}) \times \mathcal{U}} DQ \exp -N \text{STr} \left( \frac{Q^2}{2v^2} + \ln(Q - \omega) \right), \tag{31}$$

where  $\mathcal{U} = \text{sp}(n)$ ,  $\mathcal{H} \simeq \mathfrak{u}(n)$ ,  $\mathcal{M}$  is determined by  $\mathcal{H} \oplus \mathcal{M} = \mathfrak{so}^*(2n)$ , and  $\phi_b(X, Y) = b(X + \text{Ad}(e^Y)(\sigma_z \otimes 1_n))$ . ■

We conclude this section with a comment. In the literature a parametrization of the form  $Q = TPT^{-1}$  (cf. Ref. 43) has been very popular. In our language, this factorization amounts to choosing for the integration domain of  $Q_{\text{BB}}$  the image of  $\varphi: \mathcal{S} = \mathcal{H} \oplus \mathcal{M} \rightarrow \mathcal{S}, X + Y \mapsto e^Y X e^{-Y}$ . This is *not* a valid choice as  $\varphi(\mathcal{S})$  *does have a boundary*, namely the light cone  $\{Z | B(Z, Z) = 0\}$  in  $\mathcal{S}$ , so that shifting of integration variables is not permitted. (However, it turns out that the error made becomes negligible in the limit  $N \rightarrow \infty$ , so that the final results remain valid if that limit is assumed.)

**F. Saddle-point supermanifold**

The result (31) holds for all  $N \in \mathbb{N}$ . We are now going to use the method of steepest descent to show that in the limit  $N \rightarrow \infty$ , the integral on the right-hand side reduces to an integral over a Riemannian symmetric superspace of type  $DIII|CI$ .

With our choice of normalization, the mean spacing between the eigenvalues of  $H$  scales as  $N^{-1}$  for  $N \rightarrow \infty$ ; see (14). We are most interested in the eigenvalues close to zero, as their statistical properties describe those of the low-lying Bogoliubov independent-quasiparticle energy levels of mesoscopic normal-superconducting systems.<sup>24</sup> To probe their statistical behavior, what we need to do is keep  $\hat{\omega} = N\omega/\pi v$  (i.e.,  $\omega$  scaled by the mean level spacing) fixed as  $N$  goes to infinity. In this limit  $\omega \sim \mathcal{O}(1/N)$  can be treated as a small perturbation and we may expand  $N \text{STr} \ln(Q - \omega) = N \text{STr} \ln Q - \pi v \text{STr} Q^{-1} \hat{\omega} + \mathcal{O}(1/N)$  if  $Q^{-1}$  exists.

To evaluate the integral (31) by the method of steepest descent, we first look for the critical points of the function  $NF(Q) = N \text{STr}(Q^2/2v^2 + \ln Q)$ . These are the solutions of

$$F'(Q) = Q/v^2 + Q^{-1} = 0,$$

or  $Q^2 = -v^2$ . The solution spaces, the so-called ‘‘saddle-point supermanifolds,’’ are nonlinear subspaces of  $\text{osp}(2n|2n)$ , which can be distinguished by the eigenvalues of  $Q$ . Of these supermanifolds, which are the ones to select for the steepest-descent evaluation of the integral (31)?

To tackle this question, we start out by setting all Grassmann variables to zero. The BB part of the saddle-point manifold(s) is uniquely determined by the forced choice of integration domain  $i\phi_b(\mathcal{H} \times \mathcal{M})$  and by analyticity. This is because the saddle-point manifold must be deformable (using Cauchy’s theorem) into the integration domain without crossing any of the singularities of  $\text{SDet}(Q - \omega)^{-N}$ ; and by inspection one finds that this condition rules out all saddle-point manifolds except for one, which is  $i\phi_v(0 \times \mathcal{M})$ , the subspace of the integration domain  $i\phi_b(\mathcal{H} \times \mathcal{M})|_{b=v}$  obtained by dropping from  $\mathcal{S} = \mathcal{H} \oplus \mathcal{M}$  the  $\mathcal{H}$  degrees of freedom (these are the directions of steepest descent). By an argument given in the proof of Lemma 3.2 we know that  $i\phi_v(0 \times \mathcal{M})$  is diffeomorphic to  $\mathcal{M}$ . On general grounds the latter is diffeomorphic to a coset space  $G/K$  by the exponential map  $\mathcal{M} \rightarrow G/K, Y \mapsto e^Y K$ ; where in the present case  $G = \{g \in \text{Gl}(2n, \mathbb{C}) | g = \gamma_B g^{-1T} \gamma_B^{-1} = \beta g^{-1\ddagger} \beta^{-1}\}$ , and  $K = \{k \in G | k = \beta k \beta^{-1}\}$  (on setting  $g = \exp Z, k = \exp X$  and linearizing, we recover the conditions  $Z = -\gamma_B Z^T \gamma_B^{-1} = -\beta Z^\ddagger \beta^{-1}$  defining  $\mathcal{S}$  and the condition  $X = \beta X \beta^{-1}$  fixing the subalgebra  $\mathcal{H}$ ). We already know  $\mathcal{S} = \text{so}^*(2n)$  and  $\mathcal{H} = \mathfrak{u}(n)$ , so  $G = \exp \mathcal{S} = \text{SO}^*(2n)$  and  $K = \exp \mathcal{H} = \text{U}(n)$ . Because  $K$  is a maximal compact subgroup, the coset space  $G/K$  is a Riemannian symmetric space of the noncompact type. In Cartan’s notation,  $G/K = \text{SO}^*(2n)/\text{U}(n)$  is called type  $DIII$ . For better distinction from its FF analog, we will henceforth denote  $G/K$  by  $G/K_B$ .

We turn to the FF sector. Since  $\text{SDet}(Q - \omega)^{-N}$  does not have poles but only has *zeros* as a function of  $Q_{\text{FF}}$ , analyticity provides *no* criterion for selecting any specific solution space of the saddle-point equation  $Q_{\text{FF}}^2 = -v^2$ . Instead, the determining agent now is the limit  $N \rightarrow \infty$ . From (31) it is seen that integration over the Gaussian fluctuations around the saddle-point manifold produces one factor of  $N^{-1}(N^{+1})$  for every commuting (resp., anticommuting) direction of steepest descent. Therefore, the limit  $N \rightarrow \infty$  is dominated by the saddle-point manifold that has the minimal transverse (super-)dimension  $d_B^\perp - d_F^\perp$ . A little thought shows that the transverse dimen-

sion is minimized by choosing  $Q_{FF}$  to possess  $n$  eigenvalues  $+iv$  and  $n$  eigenvalues  $-iv$ . Thus, the dominant saddle-point manifold is unique and contains the special point  $q_0 := iv\beta$  ( $\beta = \sigma_z \otimes 1_n$  now acts in the fermionic subspace).

Recall that the integration domain for  $Q_{FF}$  is a compact Lie algebra  $\mathcal{U} = \mathfrak{sp}(n)$ . The corresponding Lie group  $U = \text{Sp}(n)$  operates on  $\mathcal{U}$  by the adjoint action  $\text{Ad}(u): \mathcal{U} \rightarrow \mathcal{U}, X \mapsto uXu^{-1}$ . Because the saddle-point equation  $Q_{FF} = -v^2 Q_{FF}^{-1}$  is invariant under this action, the FF part of the (dominant) saddle-point manifold can be viewed as the orbit of the action of  $\text{Ad}(U)$  on the special point  $q_0 \in \mathcal{U}$ . Let  $K_F$  be the stability group of  $q_0$ , i.e.,  $K_F = \{k \in U | kq_0k^{-1} = q_0\}$ . By  $\text{Ad}(K_F)q_0 = q_0$  the orbit  $\text{Ad}(U)q_0$  is diffeomorphic to the coset space  $U/K_F$ . Arguing in the same way as for the BB sector, one shows that  $K_F \simeq K_B \simeq U(n)$ . Hence  $U/K_F = \text{Sp}(n)/U(n)$ , which in Cartan's notation is a compact Riemannian symmetric space of type CI.

We are finally in a position to construct the full saddle-point supermanifold. Recall, first of all, that  $Q$  is subject to the condition  $Q = -\gamma Q^T \gamma^{-1}$ , which defines an orthosymplectic complex Lie algebra  $\mathcal{S}_\Lambda := \mathfrak{osp}(2n|2n)$  in  $\text{End}_\Lambda(W)$ . The solution spaces in  $\mathcal{S}_\Lambda$  of the equation  $Q/v^2 + Q^{-1} = 0$  are complex-analytic supermanifolds that are invariant under the adjoint action of the complex Lie supergroup  $G_\Lambda := \text{Osp}(2n|2n)$ . They can be regarded as  $\text{Ad}(G_\Lambda)$  orbits of elements  $Q_0 \in \text{Lie}(G_\Lambda)$  that are solutions of  $(Q_0)^2 = -v^2$ . From the above analysis of the BB and FF sectors, we know that the saddle-point supermanifold that dominates in the large- $N$  limit is obtained by setting  $Q_0 = iv\Sigma_z$  where  $\Sigma_z = 1_{\mathbb{B}|F} \otimes \beta = (E_{BB} + E_{FF}) \otimes \sigma_z \otimes 1_n$ . If  $H_\Lambda$  is the stability group of  $Q_0$ , the orbit  $\text{Ad}(G_\Lambda)Q_0$  is diffeomorphic to the coset space  $G_\Lambda/H_\Lambda$ . From  $\gamma\Sigma_z + \Sigma_z\gamma = 0$  and the equation  $h\Sigma_z h^{-1} = \Sigma_z$  (or, equivalently,  $h = \Sigma_z h \Sigma_z$ ) for  $h \in H_\Lambda$  one infers  $H_\Lambda \simeq \text{Gl}(n|n)$ . Hence the unique complex-analytic saddle-point supermanifold that dominates the large- $N$  limit is  $G_\Lambda/H_\Lambda \simeq \text{Osp}(2n|2n)/\text{Gl}(n|n)$ .

Turning to the integral (31) we note the relations  $\text{STr } Q_0^2 = -v^2 \text{STr } 1 = 0$  and  $\ln \text{SDet } Q_0 = \ln 1 = 0$ . These imply that the function  $F(Q) = \text{STr}(Q^2/2v^2 + \ln Q)$  vanishes identically on  $\text{Ad}(G_\Lambda)Q_0$ . Hence the exponent of the integrand in (31) restricted to  $G_\Lambda/H_\Lambda$  is

$$\pi v \text{STr } Q^{-1} \hat{\omega}|_{G_\Lambda/H_\Lambda} + \mathcal{O}(1/N) = -i\pi B(\hat{\omega}, \text{Ad}(g)\Sigma_z) + \mathcal{O}(1/N).$$

To complete the steepest-descent evaluation of (31) we need to Taylor expand the exponent of the integrand up to second order and do a Gaussian integral. By the  $\text{Ad}(G_\Lambda)$  invariance of the function  $NF(Q)$  it is sufficient to do this calculation for one element of the saddle-point supermanifold, say  $Q = Q_0$ . Putting  $Q = Q_0 + Z$  ( $Z \in \mathcal{S}_\Lambda$ ) we get

$$NF(Q_0 + Z) = \frac{N}{2v^2} \text{STr}(Z^2 + Z\Sigma_z Z\Sigma_z) + \mathcal{O}(Z^3).$$

Now we make the orthogonal decomposition  $\mathcal{S}_\Lambda = \text{Lie}(H_\Lambda) + \mathcal{M}_\Lambda$ ,  $Z = X + Y$ , where  $Y = -\Sigma_z Y \Sigma_z$  are the degrees of freedom tangent to the saddle-point supermanifold, and  $X = +\Sigma_z X \Sigma_z$  are the degrees of freedom transverse to it. The translation-invariant Berezin measure  $DZ$  of  $\mathcal{S}_\Lambda$  factors as  $DZ = DY DX$ . We thus obtain the transverse Gaussian integral

$$\int DX \exp\left(-\frac{N}{v^2} \text{STr } X^2 + \mathcal{O}(N^0)\right).$$

The integration domain for  $X$  is  $i\mathcal{K}_B \times \mathcal{K}_F \simeq i\mathfrak{u}(n) \times \mathfrak{u}(n)$ . By  $\dim \text{Lie}(H_\Lambda) = (p, q)$  and  $p = q$ , this integral reduces to a constant independent of  $N$  in the limit  $N \rightarrow \infty$ .

What remains is an integral over the saddle-point supermanifold itself. Since  $DY$  is the local expression of the invariant Berezin measure of  $G_\Lambda/H_\Lambda$  at  $\text{Ad}(e^Y)Q_0|_{Y=0} = Q_0$  we arrive at the following result.

**Theorem 3.4:** If  $Dg_H$  is a suitably normalized invariant holomorphic Berezin measure of the complex-analytic supermanifold  $G_\Lambda/H_\Lambda \simeq \text{Osp}(2n|2n)/\text{Gl}(n|n)$ ,

$$\lim_{N \rightarrow \infty} Z_{n,N} \left( \frac{\pi \nu \hat{\omega}}{N} \right) = \int_{M_B \times M_F} Dg_H \exp -i \pi B(\hat{\omega}, \text{Ad}(g) \Sigma_z), \tag{32}$$

where  $\Sigma_z = 1_{\mathbb{B}|\mathbb{F}} \otimes \sigma_z \otimes 1_n$ ,  $M_B \simeq \text{SO}^*(2n)/\text{U}(n)$ , and  $M_F \simeq \text{Sp}(n)/\text{U}(n)$ .

*Remark 3.5:* This result expresses the generating function for  $N \rightarrow \infty$  as an integral over a Riemannian symmetric superspace of type  $DIII|CI$  (see Tables I and II) with  $m = n$ .

In Ref. 44 the  $n$ -level correlation function  $R_n$  is calculated exactly from (32) for all  $n$ .

#### IV. OTHER SYMMETRY CLASSES

There exist ten known universality classes of ergodic disordered single-particle systems. These are the three classic Wigner-Dyson classes (GOE, GUE, GSE), the three ‘‘chiral’’ ones describing a Dirac particle in a random gauge field (chGUE, chGOE, chGSE), and the four classes that can be realized in mesoscopic normal-superconducting (NS) hybrid systems. In Ref. 25 it was noted that there exists a one-to-one correspondence between these universality classes and the large families of symmetric spaces (with the exception of the orthogonal group in odd dimensions). Specifically, the Gaussian random-matrix ensemble over the tangent space of the symmetric space describes the corresponding universality class, in the limit  $N \rightarrow \infty$ . In the notation of Table I the correspondences are  $A \leftrightarrow \text{GUE}$ ,  $AI \leftrightarrow \text{GOE}$ ,  $AII \leftrightarrow \text{GSE}$ ,  $AIII \leftrightarrow \text{chGUE}$ ,  $B DI \leftrightarrow \text{chGOE}$ ,  $CII \leftrightarrow \text{chGSE}$ , and the four NS classes correspond to  $C$ ,  $D$ ,  $CI$ , and  $DIII$ .

We have shown in detail how to use the supersymmetry method for the Gaussian ensemble over  $C_N = \text{sp}(N)$ , the tangent space of the symplectic Lie group. There are nine more ensembles to study. We will now briefly run through all these cases, giving only a summary of the essential changes.

##### A. Class $D$

Recall the definitions given at the beginning of Sec. III B and replace the symplectic unit by  $\mathcal{E} = \sigma_x \otimes 1_N$ . What you get is a Gaussian random-matrix ensemble over  $D_N = \text{so}(2N)$ , the orthogonal Lie algebra in  $2N$  dimensions. The explicit form of the Hamiltonian is

$$H = \begin{pmatrix} a & b \\ b^\dagger & -a^T \end{pmatrix},$$

where  $a(b)$  is complex Hermitian (resp., skew). The treatment of this ensemble closely parallels that of type  $C$ . A change first occurs in the consistency condition for  $\gamma$ , which now reads as  $\gamma = -\gamma^T \sigma$  (instead of  $\gamma = +\gamma^T \sigma$ ) by  $\mathcal{E} \mathcal{E}^{-1^T} = +1$ . The extra minus sign can be accommodated by simply exchanging the BB and FF sectors ( $\gamma_B \leftrightarrow \gamma_F$ ). The linear constraint  $Q = -\gamma Q^T \gamma^{-1}$  again defines an  $\text{osp}(2n|2n)$  Lie algebra, the only difference being that the BB sector is now ‘‘symplectic’’ while the FF sector has turned ‘‘orthogonal.’’ Everything else goes through as before and we arrive at the statement of Theorem 3.3 with  $\mathcal{U} = \text{so}(2n)$ ,  $\mathcal{H} = \mathfrak{u}(n)$ , and  $\mathcal{H} \oplus \mathcal{M} = \text{sp}(n, \mathbb{R})$ .

A novel feature arises in the large- $N$  limit, where instead of one dominant saddle-point supermanifold there now emerge *two*. One of them is the orbit with respect to the adjoint action of  $\text{Osp}(2n|2n)$  on  $Q_0 = i\nu 1_{\mathbb{B}|\mathbb{F}} \otimes \sigma_z \otimes 1_n$  as before, and the other one is the orbit of

$$Q_1 = i\nu E_{\text{BB}} \otimes \sigma_z \otimes 1_n + i\nu E_{\text{FF}} \otimes \sigma_z \otimes \left( E_{11} - \sum_{i=2}^n E_{ii} \right).$$

[The orbits of  $Q_0$  and  $Q_1$  are disconnected because the Weyl group of  $\text{so}(2n)$  is ‘‘too small.’’] Consequently, the right-hand side of Theorem 3.4 is replaced by a sum of two terms, one for each of the two saddle-point supermanifolds. The integral is over a Riemannian symmetric superspace of type  $CI|DIII$  ( $m = n$ ) in both cases.

**B. Class CI**

Let  $V = \mathbb{C}^2 \otimes \mathbb{C}^N$  carry a Hermitian inner product (as always), and consider the space,  $P$ , of self-adjoint Hamiltonians  $H \in \text{End}(V)$  of the form

$$H = H^T = -\mathcal{E}H^T\mathcal{E}^{-1} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad \text{where } \mathcal{E} = i\sigma_y \otimes 1_N = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}.$$

The  $N \times N$  matrices  $a$  and  $b$  are real symmetric. It is easy to see<sup>25</sup> that  $P$  is isomorphic to the tangent space of the symmetric space  $\text{Sp}(N)/\text{U}(N)$  (type CI). A Gaussian measure  $d\mu(H)$  on  $P$  is completely specified by its first two moments,  $\int_P \text{Tr}(AH)d\mu(H) = 0$  and

$$\int_P \text{Tr}(AH)\text{Tr}(BH)d\mu(H) = \frac{v^2}{4N} \text{Tr}(A(B+B^T) - A\mathcal{E}(B+B^T)\mathcal{E}^{-1}).$$

To deal with the random-matrix ensemble defined by this measure, we take  $W = \mathbb{C}^{1|1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ . Recall  $\psi \in \text{Hom}_\lambda(W, V)$  and  $\tilde{\psi} \in \text{Hom}_{\tilde{\lambda}}(V, W)$ . The symmetries of  $H$  are copied to  $\psi\tilde{\psi}$  by imposing the linear conditions

$$\psi = \mathcal{E}\tilde{\psi}^T\gamma^{-1}, \quad \tilde{\psi} = -\gamma\psi^T\mathcal{E}^{-1}; \quad \psi = \tilde{\psi}^T\tau^{-1}, \quad \tilde{\psi} = \tau\psi^T.$$

In order for these conditions to be mutually consistent,  $\tau, \gamma \in \text{End}_0(W)$  must satisfy

$$\gamma = \gamma^T\sigma, \quad \tau = \tau^T\sigma, \quad \gamma\tau^{-1} = -\tau\gamma^{-1}.$$

Without loss, we take  $\gamma$  and  $\tau$  to be orthogonal. The consistency conditions can then be written in the form

$$\gamma^2 = \sigma = \tau^2, \quad \gamma\tau + \tau\gamma = 0.$$

If  $\text{Gl}(W) \simeq \text{Gl}(4n|4n)$  is the Lie supergroup of regular elements in  $\text{End}_\Lambda(W)$ , the equation  $\gamma^2 = \sigma$  in conjunction with  $g^{\text{TT}} = \sigma g \sigma$  means that the automorphism  $\hat{\gamma}: \text{Gl}(W) \rightarrow \text{Gl}(W)$  defined by  $\hat{\gamma}(g) = \gamma g^{-1T} \gamma^{-1}$  is involutory. The same is true for  $\hat{\tau}$  defined by  $\hat{\tau}(g) = \tau g^{-1T} \tau^{-1}$  and, moreover,  $\hat{\gamma}$  and  $\hat{\tau}$  commute by  $\gamma\tau + \tau\gamma = 0$ . For definiteness we take

$$\begin{aligned} \gamma &= E_{\text{BB}} \otimes \gamma_B + E_{\text{FF}} \otimes \gamma_F, & \gamma_B &= \sigma_x \otimes \sigma_z \otimes 1_n, & \gamma_F &= i\sigma_y \otimes 1_2 \otimes 1_n, \\ \tau &= E_{\text{BB}} \otimes \tau_B + E_{\text{FF}} \otimes \tau_F, & \tau_B &= 1_2 \otimes \sigma_x \otimes 1_n, & \tau_F &= \sigma_z \otimes i\sigma_y \otimes 1_n. \end{aligned}$$

(This choice is consistent with  $\tilde{\psi}_B = \beta\psi_B^\dagger, \beta = \sigma_z \otimes 1_2 \otimes 1_n$ .) Let

$$\mathcal{Q} = \{Q \in \text{End}_\Lambda(W) \mid Q = -\gamma Q^T \gamma^{-1} = +\tau Q^T \tau^{-1}\}$$

be the subspace distinguished by the symmetry properties of  $\tilde{\psi}\psi$ . The group  $\text{Gl}(W)$  acts on  $\mathcal{Q}$  by  $Q \mapsto gQg^{-1}$ . We now ask what is the subgroup  $G_\Lambda$  of  $\text{Gl}(W)$  that leaves the symmetries of  $\mathcal{Q}$  invariant [the normalizer of  $\mathcal{Q}$  in  $\text{Gl}(W)$ ].

*Lemma 4.1:*  $G_\Lambda$  is isomorphic to  $\text{Osp}(2n|2n) \times \text{Osp}(2n|2n)$ .

*Proof:* The conditions on  $g \in G_\Lambda$  can be phrased as follows:

$$\gamma = g\gamma g^T, \quad \tau = g\tau g^T.$$

Equivalently,  $G_\Lambda$  can be described as the simultaneous ‘‘fixed point set’’<sup>45</sup> of the involutory automorphisms  $\hat{\gamma}$  and  $\hat{\tau}$ . We first describe the fixed point set of  $\hat{\gamma} \circ \hat{\tau}$ , which acts by  $(\hat{\gamma} \circ \hat{\tau})(g) = \epsilon g \epsilon^{-1}$ , where  $\epsilon = -i\gamma\tau^{-1}$ . From the explicit expression  $\epsilon = 1_{\text{BF}} \otimes \sigma_x \otimes \sigma_y \otimes 1_n$  we see that  $\epsilon$  has  $4n$  eigenvalues equal to  $+1$ ,  $4n$  eigenvalues equal to  $-1$ , and these are equally



distributed over the bosonic and fermionic subspaces. Hence the subgroup of  $\text{Gl}(W)$  fixed by  $\hat{\gamma} \circ \hat{\tau}$  is isomorphic to  $G_+ \times G_-$ , where  $G_+ \simeq \text{Gl}(2n|2n) \simeq G_-$ . Denote the embedding  $G_+ \times G_- \rightarrow \text{Gl}(W)$  by  $\varphi(g_+, g_-) = g$ . The group  $G_\Lambda$  is the fixed point set of  $\hat{\tau}$  (or, equivalently, of  $\hat{\gamma}$ ) in  $\varphi(G_+ \times G_-)$  [ $\hat{\tau}$  commutes with  $\hat{\gamma} \circ \hat{\tau}$  and therefore takes  $\varphi(G_+ \times G_-)$  into itself]. Note  $\epsilon \tau = -\tau \epsilon$ ,  $\epsilon^{-1 \text{T}} = -\epsilon$ , and for  $g \in \varphi(G_+ \times G_-)$  do the following little calculation:

$$\epsilon \hat{\tau}(g) = \epsilon \tau g^{-1 \text{T}} \tau^{-1} = -\tau \epsilon g^{-1 \text{T}} \tau^{-1} = \tau(\epsilon g)^{-1 \text{T}} \tau^{-1} = \hat{\tau}(\epsilon g).$$

Combining this with  $\epsilon \varphi(g_+, g_-) = \varphi(g_+, -g_-)$  one infers that  $\hat{\tau}$  acting on  $\varphi(G_+ \times G_-)$  is of the form  $\hat{\tau} \varphi(g_+, g_-) = \varphi(\hat{\tau}_+(g_+), \hat{\tau}_-(g_-))$ . By a short calculation (work in an eigenbasis of  $\epsilon$ ) one sees that the involutory automorphisms  $\hat{\tau}_i: \text{Gl}(2n|2n) \rightarrow \text{Gl}(2n|2n)$  ( $i = \pm$ ) are expressed by  $\hat{\tau}_i(g) = \tau_i g^{-1 \text{T}} \tau_i^{-1}$  with supersymmetric  $\tau_i$  ( $\tau_i = \tau_i^{\text{T}} \sigma$ ). It follows that  $\hat{\tau}_i$  fixes an orthosymplectic subgroup of  $G_i \simeq \text{Gl}(2n|2n)$ , so  $G_\Lambda \simeq \text{Osp}(2n|2n) \times \text{Osp}(2n|2n)$ , as claimed.

*Corollary 4.2:* The space  $\mathcal{Q}$  is isomorphic to the complement of  $\text{osp}(2n|2n) \oplus \text{osp}(2n|2n)$  in  $\text{osp}(4n|4n)$ .

*Proof:* The solution space in  $\text{End}_\Lambda(W)$  of  $Q = -\gamma Q^{\text{T}} \gamma^{-1}$  is an  $\text{osp}(4n|4n)$  algebra. Implementing the second condition  $Q = +\tau Q^{\text{T}} \tau^{-1}$  amounts to removing from  $\text{osp}(4n|4n)$  the subalgebra fixed by  $X = -\tau X^{\text{T}} \tau^{-1}$ . By linearization of the conditions  $g = \hat{\gamma}(g) = \hat{\tau}(g)$ , this subalgebra is identified as  $\text{Lie}(G_\Lambda) \simeq \text{osp}(2n|2n) \oplus \text{osp}(2n|2n)$ . ■

The Gaussian integral identity (10) continues to hold, albeit with a different value of  $c = 1/4$ . The proof is essentially the same as before.

Since  $\mathcal{Q}$  is not a Lie algebra, the description of the correct choice of integration domain for the auxiliary variable  $Q$  is more complicated than before. In the FF sector we take  $\mathcal{U} := \{Q_{\text{FF}} \in \mathcal{Q}_{\text{FF}} | Q_{\text{FF}} = -Q_{\text{FF}}^\dagger\}$ . By Corollary 4.2,  $\text{sp}(2n) \simeq (\text{sp}(n) \oplus \text{sp}(n)) \oplus \mathcal{U}$ . To deal with the BB sector we introduce the spaces

$$\begin{aligned} \mathcal{S} &= \{X \in \mathfrak{gl}(2n, \mathbb{C}) | X = -\gamma_B X^{\text{T}} \gamma_B^{-1} = -\tau_B X^{\text{T}} \tau_B^{-1} = -\beta X^\dagger \beta^{-1}\}, \\ \mathcal{M} &= \{Y \in \mathcal{S} | Y = -\beta Y \beta^{-1}\}, \quad \mathcal{P}^\pm = \{X \in \mathcal{Q}_{\text{BB}} | X = -\beta X^\dagger \beta^{-1} = \pm \beta X \beta^{-1}\}, \end{aligned}$$

where  $\beta = \sigma_z \otimes 1_2 \otimes 1_n$ . The Lie algebra  $\mathcal{S}$  is a noncompact real form of the BB part of  $\text{Lie}(G_\Lambda)$ . By  $\beta \in i\mathcal{P}^+$  and the commutation relations  $[\mathcal{M}, \mathcal{P}^+] \subset \mathcal{P}^-$  and  $[\mathcal{M}, \mathcal{P}^-] \subset \mathcal{P}^+$ , we have an embedding,

$$\begin{aligned} \phi_b: \mathcal{P}^+ \times \mathcal{M} &\rightarrow \mathcal{Q}_{\text{BB}} = \mathcal{P}_\mathbb{C}^+ + \mathcal{P}_\mathbb{C}^-, \\ (X, Y) &\mapsto b \times (X + e^{\text{ad}(Y)} \beta). \end{aligned}$$

Similar considerations as in Sec. III E show that all integrals are rendered convergent by the choice of integration domain  $\phi_b(\mathcal{P}^+ \times \mathcal{M}) \times \mathcal{U}$  ( $b > 0$ ) for  $Q$ . With this choice we again arrive at Theorem 3.3.

The large- $N$  limit is dominated by a single saddle-point supermanifold, which can be taken as the orbit of  $Q_0 = i\nu \Sigma_z$  (here  $\Sigma_z = 1_{\mathbb{B}|F} \otimes \sigma_z \otimes 1_2 \otimes 1_n$ ) under the adjoint action of  $G_\Lambda$ . This orbit is diffeomorphic to  $G_\Lambda/H_\Lambda$ , where  $H_\Lambda = \{h \in G_\Lambda | h \Sigma_z h^{-1} = \Sigma_z\}$ . The stability group  $H_\Lambda$  can equivalently be described as the fixed point set of  $\tilde{\Sigma}_z: G_\Lambda \rightarrow G_\Lambda$ ,  $\tilde{\Sigma}_z(g) = \Sigma_z g \Sigma_z$ . By the relations  $\tilde{\Sigma}_z = \Sigma_z^{\text{T}} = -\gamma \Sigma_z \gamma^{-1} = \tau \Sigma_z \tau^{-1}$  ( $\Sigma_z \in \mathcal{Q}$ ), the element  $\Sigma_z$  anticommutes with  $\epsilon = -i\gamma \tau^{-1}$ , and  $\tilde{\Sigma}_z$  commutes with  $\hat{\gamma} \circ \hat{\tau}$ . These relations are compatible with the existence of an embedding  $\phi: \text{Osp}(2n|2n) \times \text{Osp}(2n|2n) \rightarrow \text{Gl}(W)$ , such that  $(\tilde{\Sigma}_z \circ \phi)(g_+, g_-) = \phi(g_-, g_+)$ . (Such an embedding is easily constructed.) Hence  $H_\Lambda \simeq \text{diag}(\text{Osp}(2n|2n) \times \text{Osp}(2n|2n)) \simeq \text{Osp}(2n|2n)$ . In this way we arrive at Theorem 3.4 with  $G_\Lambda/H_\Lambda \simeq \text{Osp}(2n|2n)$ , and the maximal Riemannian submanifold  $M_B \times M_F$ , where  $M_B \simeq \text{SO}(2n, \mathbb{C})/\text{SO}(2n)$  and  $M_F \simeq \text{Sp}(n)$  (type  $D|C$ ).<sup>47</sup>

**C. Class DIII**

Consider for  $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$  the linear space

$$P = \{H \in \text{End}(V) \mid H = H^\dagger = -\mathcal{E}H^T\mathcal{E}^{-1} = +\mathcal{F}H^T\mathcal{F}^{-1}\},$$

where  $\mathcal{E} = \sigma_x \otimes 1_2 \otimes 1_N$  and  $\mathcal{F} = 1_2 \otimes i\sigma_y \otimes 1_N$ . It has been shown<sup>25</sup> that  $P$  is isomorphic to the tangent space of  $\text{SO}(4N)/\text{U}(2N)$  (a symmetric space of type DIII). Introducing an orthonormal real basis of  $V$ , we can represent  $H$  by a  $4N \times 4N$  matrix. If  $\mathcal{E}$  and  $\mathcal{F}$  are given by

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 1_N & 0 \\ 0 & 0 & 0 & 1_N \\ 1_N & 0 & 0 & 0 \\ 0 & 1_N & 0 & 0 \end{pmatrix}, \quad \mathcal{F} = \begin{pmatrix} 0 & 1_N & 0 & 0 \\ -1_N & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_N \\ 0 & 0 & -1_N & 0 \end{pmatrix},$$

the explicit form of such a matrix is

$$H = \begin{pmatrix} a & b & c & d \\ b^\dagger & a^T & -d^T & -c^\dagger \\ c^\dagger & -d^T & -a^T & b^\dagger \\ d & -c & b & -a \end{pmatrix},$$

where all entries are complex  $N \times N$  matrices and  $a, d(b, c)$  are Hermitian (skew).

The Gaussian random-matrix ensemble on  $P$  is defined by the Gaussian measure  $d\mu(H)$  with vanishing first moment, and second moment,

$$\int_P \text{Tr}(AH)\text{Tr}(BH)d\mu(H) = \frac{v^2}{4N} \text{Tr}(AB - A\mathcal{E}B^T\mathcal{E}^{-1} + A\mathcal{F}B^T\mathcal{F}^{-1} - A(\mathcal{E}\mathcal{F})B(\mathcal{E}\mathcal{F})^{-1}).$$

Given the auxiliary space  $W := \mathbb{C}^{1|1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ , we impose on  $\psi \in \text{Hom}_\lambda(W, V)$ ,  $\tilde{\psi} \in \text{Hom}_{\tilde{\lambda}}(V, W)$  the linear conditions

$$\begin{aligned} \psi &= \gamma\tilde{\psi}^T\mathcal{E}^{-1}, & \tilde{\psi} &= -\mathcal{E}\psi^T\gamma^{-1}, \\ \psi &= \tau\tilde{\psi}^T\mathcal{F}^{-1}, & \tilde{\psi} &= \mathcal{F}\psi^T\tau^{-1}, \end{aligned}$$

with some invertible orthogonal elements  $\gamma, \tau$  of  $\text{End}_0(W)$ . Consistency requires  $\gamma^2 = -\sigma = \tau^2$  and  $\gamma\tau + \tau\gamma = 0$ . A possible choice is

$$\gamma = (E_{\text{BB}} \otimes i\sigma_y \otimes 1_2 + E_{\text{FF}} \otimes \sigma_x \otimes \sigma_z) \otimes 1_n,$$

$$\tau = (E_{\text{BB}} \otimes \sigma_z \otimes i\sigma_y + E_{\text{FF}} \otimes 1_2 \otimes \sigma_x) \otimes 1_n.$$

Because this differs from class CI only by the exchange of the bosonic and fermionic subspaces, the following development closely parallels that for CI, and we arrive at another variant of Theorem 3.3.

The large- $N$  limit is dominated by a pair of complex-analytic saddle-point supermanifolds, each being isomorphic to  $\text{Osp}(2n|2n)$ . [The reason why there are two is that  $\text{O}(2n, \mathbb{C})$  has two connected components.] The first one is the orbit under  $\text{Ad}(G_\Lambda)$  of  $Q_0 = i\nu 1_{\text{BF}} \otimes \sigma_z \otimes 1_2 \otimes 1_n$ , and the second one is the orbit of

$$Q_1 = iv \left( E_{BB} \otimes \sigma_z \otimes 1_2 \otimes 1_n + E_{FF} \otimes \left( 1_2 \otimes \sigma_x \otimes E_{11} + \sigma_z \otimes 1_2 \otimes \sum_{i=2}^n E_{ii} \right) \right).$$

Both saddle-point supermanifolds are Riemannian symmetric superspaces of type  $C|D$  with dimensionality  $m = 2n$  (Table II).<sup>47</sup>

**D. Class AIII**

The tangent space at the origin of  $U(p, q)/U(p) \times U(q)$  consists of the matrices of the form

$$H = \begin{pmatrix} 0 & Z \\ Z^\dagger & 0 \end{pmatrix},$$

where  $Z$  is complex and has dimension  $p \times q$ . Such matrices are equivalently described by  $H^\dagger = H = -\mathcal{P}H\mathcal{P}^{-1}$ , where  $\mathcal{P} = \text{diag}(1_p, -1_q)$ . For simplicity, we will consider only the case  $p = q$  (the general case has not yet been analyzed in the present formalism). The Gaussian ensemble of random matrices  $H$  is taken to have second moment

$$\int \text{Tr}(AH)\text{Tr}(BH)d\mu(H) = \frac{v^2}{2N} \text{Tr}(AB - A\mathcal{P}B\mathcal{P}^{-1}).$$

The physical space is  $V = \mathbb{C}^2 \otimes \mathbb{C}^p$ , and the auxiliary space is  $W = \mathbb{C}^{11} \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ . The definition of  $\omega$  is unchanged from class  $C$ . To implement the symmetry condition  $\psi\psi = -\mathcal{P}\psi\tilde{\psi}\mathcal{P}^{-1}$  we set

$$\psi = i\mathcal{P}\psi\pi^{-1}, \quad \tilde{\psi} = i\pi\tilde{\psi}\mathcal{P}^{-1},$$

where  $\pi = 1_{\mathbb{B}|F} \otimes i\sigma_y \otimes 1_n$ . This choice is consistent with the relation  $\tilde{\psi}_B = \beta\psi_B^\dagger$  which ensures convergence of the  $(\psi, \tilde{\psi})$  integration. The auxiliary variable  $Q$  ranges over the complex-analytic superspace

$$\mathcal{Q} = \{Q \in \text{End}_\Lambda(W) | Q = -\pi Q \pi^{-1}\},$$

and the normalizer of  $\mathcal{Q}$  in  $\text{Gl}(W)$  is

$$G_\Lambda = \{g \in \text{Gl}(W) | g = \pi g \pi^{-1}\} \simeq \text{Gl}(n|n) \times \text{Gl}(n|n).$$

For the integration domain  $\mathcal{U}$  in the FF sector we again take the anti-Hermitian matrices in  $\mathcal{Q}_{FF}$ . In the BB sector we set

$$\mathcal{M} = \{Y \in \text{End}_{\mathbb{C}}(W_B) | Y = \pi Y \pi^{-1} = -\beta Y \beta^{-1} = Y^\dagger\},$$

$$\mathcal{P}^\pm = \{X \in \text{End}_{\mathbb{C}}(W_B) | X = -\pi X \pi^{-1} = \pm \beta X \beta^{-1} = \mp X^\dagger\}.$$

The treatment of Sec. III E then goes through as before, leading again to Theorem 3.3.

There is a single dominant saddle-point supermanifold, which is the  $\text{Ad}(G_\Lambda)$  orbit of  $Q_0 = iv 1_{\mathbb{B}|F} \otimes \sigma_z \otimes 1_n$  and is diffeomorphic to  $G_\Lambda/H_\Lambda \simeq \text{Gl}(n|n)$ . The integration domain  $M_B \times M_F$  is given by  $M_B \simeq \text{Gl}(n, \mathbb{C})/U(n)$  and  $M_F = U(n)$ . The invariant Berezin measure of this Riemannian symmetric superspace of type  $A|A$  was discussed for  $n = 1$  in Example 2.4.

**E. Class BDI**

The form of the random-matrix Hamiltonian  $H$  for class  $BDI$  can be obtained from the preceding case by taking the  $p \times q$  matrix  $Z$  to be real. Put in formulas,  $H$  is subject to  $H^\dagger = H = H^T = -\mathcal{P}H\mathcal{P}^{-1}$ . We again make the restriction to  $p = q$ . The basic correlation law of the Gaussian ensemble is

$$\int \text{Tr}(AH)\text{Tr}(BH)d\mu(H) = \frac{v^2}{4N} \text{Tr}(A(B+B^T) - A\mathcal{P}(B+B^T)\mathcal{P}^{-1}).$$

To accommodate the extra symmetry  $H = H^T$ , auxiliary space is extended to  $W = \mathbb{C}^{1|1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ . The symmetry conditions on  $\psi, \tilde{\psi}$  are

$$\psi = i\mathcal{P}\psi\pi^{-1}, \quad \tilde{\psi} = i\pi\tilde{\psi}\mathcal{P}^{-1}; \quad \psi = \tilde{\psi}^T\tau^{-1}, \quad \tilde{\psi} = \tau\psi^T,$$

where  $\pi = 1_{\mathbb{B}|F} \otimes i\sigma_y \otimes 1_2 \otimes 1_n$  and  $\tau = (E_{\mathbb{B}\mathbb{B}} \otimes 1_2 \otimes \sigma_x + E_{\mathbb{F}\mathbb{F}} \otimes 1_2 \otimes i\sigma_y) \otimes 1_n$ . The auxiliary integration space,

$$\mathcal{Q} = \{Q \in \text{End}_\Lambda(W) \mid Q = -\pi Q \pi^{-1} = +\tau Q^T \tau^{-1}\},$$

has the symmetry group (or normalizer)

$$G_\Lambda = \{g \in \text{Gl}(W) \mid g = \pi g \pi^{-1} = \tau g^{-1T} \tau^{-1}\} \simeq \text{Gl}(2n|2n).$$

For the integration domain  $\mathcal{U}$  in the FF sector we once again take the anti-Hermitian matrices in  $\mathcal{Q}_{\mathbb{F}\mathbb{F}}$ . In the BB sector we set

$$\mathcal{M} = \{Y \in \text{End}_{\mathbb{C}}(W_{\mathbb{B}}) \mid Y = \pi Y \pi^{-1} = -\tau Y^T \tau^{-1} = -\beta Y \beta^{-1} = Y^\dagger\},$$

$$\mathcal{P}^\pm = \{X \in \text{End}_{\mathbb{C}}(W_{\mathbb{B}}) \mid X = -\pi X \pi^{-1} = +\tau X^T \tau^{-1} = \pm \beta X \beta^{-1} = \mp X^\dagger\}.$$

The treatment of Sec. III E then goes through with modifications as in Sec. IV B.

There is a single dominant saddle-point supermanifold, which is the  $\text{Ad}(G_\Lambda)$  orbit of  $Q_0 = iv 1_{\mathbb{B}|F} \otimes \sigma_z \otimes 1_2 \otimes 1_n$  and is diffeomorphic to  $G_\Lambda/H_\Lambda \simeq \text{Gl}(2n|2n)/\text{Osp}(2n|2n)$ . The integration domain  $M_{\mathbb{B}} \times M_{\mathbb{F}}$  is given by  $M_{\mathbb{B}} \simeq \text{Gl}(2n, \mathbb{R})/\text{O}(2n)$  and  $M_{\mathbb{F}} = \text{U}(2n)/\text{Sp}(n)$ . This is a Riemannian symmetric superspace of type AI|AII with  $m = 2n$  (Table II).

**F. Class CII**

The tangent space at the origin of  $\text{Sp}(N, N)/\text{Sp}(N) \times \text{Sp}(N)$  (a noncompact symmetric space of type CII) can be described by the equations

$$H^\dagger = H = -\mathcal{P}H\mathcal{P}^{-1} = -\mathcal{T}H^T\mathcal{T}^{-1},$$

where  $\mathcal{P} = \sigma_z \otimes 1_2 \otimes 1_N$  and  $\mathcal{T} = 1_2 \otimes i\sigma_y \otimes 1_N$  (the physical space is  $V = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^N$ ). The explicit form of the matrices is

$$H = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \\ a^\dagger & -b^T & 0 & 0 \\ b^\dagger & a^T & 0 & 0 \end{pmatrix},$$

if

$$\mathcal{P} = \begin{pmatrix} 1_N & 0 & 0 & 0 \\ 0 & 1_N & 0 & 0 \\ 0 & 0 & -1_N & 0 \\ 0 & 0 & 0 & -1_N \end{pmatrix}$$

and

$$\mathcal{T} = \begin{pmatrix} 0 & 1_N & 0 & 0 \\ -1_N & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_N \\ 0 & 0 & -1_N & 0 \end{pmatrix},$$

where  $a$  and  $b$  are complex and have dimension  $N \times N$ . The correlation law of the Gaussian random-matrix ensemble of type CII is

$$\int \text{Tr}(AH)\text{Tr}(BH)d\mu(H) = \frac{v^2}{4N} \text{Tr}(A - \mathcal{P}A\mathcal{P}^{-1})(B - \mathcal{T}B^T\mathcal{T}^{-1}).$$

As before,  $W = \mathbb{C}^{1|1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$ . The symmetry conditions on  $\psi, \tilde{\psi}$  are

$$\psi = i\mathcal{P}\psi\pi^{-1}, \quad \tilde{\psi} = i\pi\tilde{\psi}\mathcal{P}^{-1}; \quad \psi = \mathcal{T}\tilde{\psi}^T\tau^{-1}, \quad \tilde{\psi} = \tau\psi^T\mathcal{T}^{-1},$$

where  $\pi = 1_{\mathbb{B}|\mathbb{F}} \otimes i\sigma_y \otimes 1_2 \otimes 1_n$  and  $\tau = (E_{\mathbb{B}\mathbb{B}} \otimes 1_2 \otimes i\sigma_y + E_{\mathbb{F}\mathbb{F}} \otimes 1_2 \otimes \sigma_x) \otimes 1_n$ . This differs from class BDI only by the exchange of the bosonic and fermionic subspaces. Once more we arrive at another version of Theorem 3.3.

There is only one complex-analytic supermanifold of saddle-points that dominates for  $N \rightarrow \infty$ . It is isomorphic to that for class BDI. The integration domain  $M_{\mathbb{B}} \times M_{\mathbb{F}}$  changes to  $M_{\mathbb{B}} \simeq \text{U}^*(2n)/\text{Sp}(n)$  and  $M_{\mathbb{F}} \simeq \text{U}(2n)/\text{O}(2n)$  [not  $\text{U}(2n)/\text{SO}(2n)$ ]. This is a Riemannian symmetric superspace of type AII|AI with  $m = 2n$  (Table II). The group  $\text{U}^*(2n)$  is defined as the non-compact real subgroup of  $\text{Gl}(2n, \mathbb{C})$  fixed by  $g = \mathcal{E}\bar{g}\mathcal{E}^{-1}$ , where  $\mathcal{E} = i\sigma_y \otimes 1_n$ .

### G. Class A

This class for  $n = 1$  was used to illustrate our general strategy in Sec. III A. Let us now do the case of arbitrary  $n$ ,

$$Z_n(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_n) = \int_{i \in \text{u}(N)} \prod_{i=1}^n \text{Det} \left( \frac{H - \beta_i}{H - \alpha_i} \right) d\mu(H).$$

The classes treated so far (C, D, CI, DIII, AIII, BDI, CII) all share one feature, namely the existence of a particle-hole type of symmetry ( $H = -\mathcal{P}H\mathcal{P}^{-1}$  or  $H = -\mathcal{E}H^T\mathcal{E}^{-1}$ ), which allows us to restrict all  $\alpha_i$  to one-half of the complex plane. Such a symmetry is absent for the Wigner–Dyson symmetry classes A, AI, and AII, which results in a somewhat different scenario, as it now matters how many  $\alpha_i$  lie above or below the real axis. For definiteness let

$$\text{Im } \alpha_i < 0 \quad (i = 1, \dots, n_A), \quad \text{Im } \alpha_j > 0 \quad (j = n_A + 1, \dots, n),$$

and set  $n_R = n - n_A$ .

Auxiliary space is taken to be  $W = \mathbb{C}^{1|1} \otimes \mathbb{C}^n$ . The definition of  $\omega$  changes to

$$\omega = E_{\text{BB}} \otimes \sum_{i=1}^n \alpha_i E_{ii} + E_{\text{FF}} \otimes \sum_{j=1}^n \beta_j E_{jj}.$$

Recall that the imaginary parts of the  $\alpha_i$  steer the convergence of the  $(\psi, \tilde{\psi})$  integration. Since  $\omega$  couples to  $\psi, \tilde{\psi}$  by  $\exp -i \text{STr}_W \omega \tilde{\psi} \psi$ , convergence forces us to take  $\tilde{\psi}_B = \beta \psi_B^\dagger$ , where

$$\beta = \sum_{i=1}^{n_A} E_{ii} - \sum_{j=n_A+1}^n E_{jj}.$$

There are no further constraints on  $\psi, \tilde{\psi}$ , or  $Q$ . Thus the complex-analytic auxiliary integration space is  $\mathcal{Q} = \text{End}_\Lambda(W)$ , and  $G_\Lambda = \text{Gl}(W) \simeq \text{Gl}(n|n)$ .

The integration domain for  $Q$  in the FF sector is taken to be the anti-Hermitian matrices  $\mathcal{U} = \mathfrak{u}(n)$ . In the BB sector we introduce

$$\mathcal{S} = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid X = -\beta X^\dagger \beta^{-1}\}, \quad \mathcal{K} = \{X \in \mathcal{S} \mid X = \beta X \beta^{-1}\}.$$

The Lie algebra  $\mathcal{S}$  is a noncompact real form  $\mathfrak{u}(n_A, n_R)$  of  $\mathfrak{gl}(n, \mathbb{C})$ , and  $\mathcal{K} = \mathfrak{u}(n_A) \oplus \mathfrak{u}(n_R)$  is a maximal compact subalgebra. The space  $\mathcal{M}$  is defined by the Cartan decomposition  $\mathcal{S} = \mathcal{K} \oplus \mathcal{M}$ . The integration domain for  $Q_{\text{BB}}$  is taken to be  $i\phi_b(\mathcal{K} \times \mathcal{M})$ , where  $\phi_b(X, Y) = b(X + e^{\text{ad}(Y)}\beta)$  ( $b > 0$ ). This gives Theorem 3.3.

By simple power counting, the limit  $N \rightarrow \infty$  is again dominated by a single complex-analytic saddle-point supermanifold, which is the  $\text{Ad}(G_\Lambda)$ -orbit of  $Q_0 = i\nu 1_{\text{BIF}} \otimes \beta$ . The stability group  $H_\Lambda$  of  $Q_0$  is  $H_\Lambda = \text{Gl}(n_A|n_A) \times \text{Gl}(n_R|n_R)$ , so

$$\text{Ad}(G_\Lambda)Q_0 \simeq G_\Lambda / H_\Lambda = \text{Gl}(n|n) / \text{Gl}(n_A|n_A) \times \text{Gl}(n_R|n_R).$$

The intersection of  $\text{Ad}(G_\Lambda)Q_0$  with  $i\phi_v(\mathcal{K} \times \mathcal{M}) \times \mathcal{U}$  is  $M_B \times M_F$ , where  $M_B \simeq \text{U}(n_A, n_R) \text{U}(n_A) \times \text{U}(n_R)$  and  $M_F \simeq \text{U}(n_A + n_R) / \text{U}(n_A) \times \text{U}(n_R)$ . This is a Riemannian symmetric superspace of type AIII|AIII with  $m_1 = n_1 = n_A$  and  $m_2 = n_2 = n_R$  (see Table II).

### H. Class AI

The tangent space of  $\text{U}(N)/\text{O}(N)$  is the same as ( $i$  times) the real symmetric matrices  $H^\dagger = H = H^T$ . It differs from the tangent space of  $\text{SU}(N)/\text{SO}(N)$ , a symmetric space of type AI in an inessential way (just remove the multiples of the unit matrix). The Gaussian ensemble over the real symmetric matrices has its second moment given by

$$\int \text{Tr}(AH) \text{Tr}(BH) d\mu(H) = \frac{v^2}{2N} \text{Tr}(AB + AB^T).$$

This ensemble is related to type A in the same way that type CI is related to type C.

To implement the symmetry  $H = H^T$  we set  $W = \mathbb{C}^{1|1} \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$  and require  $\psi = \tilde{\psi}^T \tau^{-1}$ ,  $\tilde{\psi} = \tau \psi^T$ , where  $\tau = (E_{\text{BB}} \otimes \sigma_x + E_{\text{FF}} \otimes i\sigma_y) \otimes 1_n$ . The auxiliary integration space,

$$\mathcal{Q} = \{Q \in \text{End}_\Lambda(W) \mid Q = \tau Q^T \tau^{-1}\},$$

has the symmetry group

$$G_\Lambda = \{g \in \text{Gl}(W) \mid g = \tau g^{-1T} \tau^{-1}\} \simeq \text{Osp}(2n|2n).$$

The intersection  $\mathcal{U}$  of the FF sector  $Q_{\text{FF}}$  with the anti-Hermitian matrices is given by  $\mathfrak{sp}(n) \oplus \mathfrak{u} = \mathfrak{u}(2n)$ . In the BB sector we put

$$\begin{aligned} \mathcal{M} &= \{Y \in \text{End}_{\mathbb{C}}(W_{\text{B}}) \mid Y = -\tau Y^T \tau^{-1} = -\beta Y \beta^{-1} = Y^\dagger\}, \\ \mathcal{F}^\pm &= \{X \in \text{End}_{\mathbb{C}}(W_{\text{B}}) \mid X = +\tau X^T \tau^{-1} = \pm \beta X \beta^{-1} = \mp X^\dagger\}, \end{aligned}$$

which leads to yet another version of Theorem 3.3.

The large- $N$  limit is controlled by a single complex-analytic saddle-point supermanifold  $\text{Ad}(G_\Lambda)Q_0 \simeq G_\Lambda/H_\Lambda$ , where  $H_\Lambda \simeq \text{Osp}(2n_A|2n_A) \times \text{Osp}(2n_R|2n_R)$  is the stability group of  $Q_0 = iv 1_{\text{B|F}} \otimes (\sum_{i=1}^{n_A} E_{ii} - \sum_{j=n_A+1}^n E_{jj})$ . The intersection of  $\text{Ad}(G_\Lambda)Q_0$  with the integration domain  $\phi_v(\mathcal{P}^+ \times \mathcal{M}) \times \mathcal{U}$  is  $M_{\text{B}} \times M_{\text{F}}$ , where  $M_{\text{B}} \simeq \text{SO}(2n_A, 2n_R)/\text{SO}(2n_A) \times \text{SO}(2n_R)$  and  $M_{\text{F}} \simeq \text{Sp}(n_A + n_R)/\text{Sp}(n_A) \times \text{Sp}(n_R)$ . This is a Riemannian symmetric superspace of type  $B\text{DI|CII}$  (Table II) with  $m_1 = 2n_1 = 2n_A$  and  $m_2 = 2n_2 = 2n_R$ .

**I. Class AII**

Finally, the tangent space of  $\text{U}(2N)/\text{Sp}(N)$  [a symmetric space of type  $\text{AII}$ , except for the substitution  $\text{SU}(2N) \rightarrow \text{U}(2N)$ ] can be described as ( $i$  times) the subspace of  $\text{End}(\mathbb{C}^2 \otimes \mathbb{C}^N)$  fixed by the linear equations  $H^\dagger = H = \mathcal{F} H^T \mathcal{F}^{-1}$ ,  $\mathcal{F} = i\sigma_y \otimes 1_N$ . The explicit matrix form of  $H$  is

$$H = \begin{pmatrix} a & b \\ b^\dagger & a^T \end{pmatrix},$$

where  $b$  is skew and  $a$  is Hermitian.

The conditions  $\psi = \mathcal{F} \tilde{\psi}^T \tau^{-1}$  and  $\tilde{\psi} = \tau \psi^T \mathcal{F}^{-1}$  are mutually consistent if, say,  $\tau = (E_{\text{BB}} \otimes i\sigma_y + E_{\text{FF}} \otimes \sigma_x) \otimes 1_n$ . The rest of the manipulations leading up to Theorem 3.3 are the same as for class  $\text{AI}$ , except for the exchange of the bosonic and fermionic subspaces ( $\tau_{\text{B}} \leftrightarrow \tau_{\text{F}}$ ). The large- $N$  limit is controlled by a single saddle-point supermanifold  $(G_\Lambda/H_\Lambda, M_{\text{B}} \times M_{\text{F}})$ , where

$$\begin{aligned} G_\Lambda/H_\Lambda &= \text{Osp}(2n|2n)/\text{Osp}(2n_A|2n_A) \times \text{Osp}(2n_R|2n_R), \\ M_{\text{B}} &= \text{Sp}(n_A, n_R)/\text{Sp}(n_A) \times \text{Sp}(n_R), \\ M_{\text{F}} &= \text{SO}(2n_A + 2n_R)/\text{SO}(2n_A) \times \text{SO}(2n_R), \end{aligned}$$

which is a Riemannian symmetric superspace of type  $\text{CII|BDI}$  (Table II) with  $m_1 = 2n_1 = 2n_A$  and  $m_2 = 2n_2 = 2n_R$ .

**V. SUMMARY**

When Dyson realized<sup>46</sup> that the random-matrix ensembles he had introduced were based on the symmetric spaces of type  $A$ ,  $\text{AI}$ , and  $\text{AII}$ , he wrote: ‘‘The proof of (the) Theorem... is a mere verification. It would be highly desirable to find a more illuminating proof, in which the appearance of the (final result) might be related directly to the structure of the symmetric space...’’. The advent of the supersymmetry method of Efetov and others has improved the situation lamented by Dyson. The present work takes the Gaussian random-matrix ensembles defined over Cartan’s large families of symmetric spaces and, going to the limit of large matrix dimension, expresses their spectral correlation functions as integrals over the corresponding Riemannian symmetric superspaces. These correspondences are summarized in Table III. The Riemannian symmetric superspaces that appear there all have superdimension  $(p, q)$  with  $p = q$ . We say that they are ‘‘perfectly graded’’ or ‘‘supersymmetric.’’ An interesting question for future mathematical research is whether our procedure can be optimized by reducing it to a computation involving no more than the root system of the symmetric space, thereby obviating the space- and time-consuming need to distinguish cases. (Although I have treated all ten cases separately, it is possible, following Efetov,<sup>3</sup> to shorten the derivation by starting from a large ‘‘master ensemble’’ of highest symme-

TABLE III. The symmetric-space based random-matrix theories of the first column map onto the Riemannian symmetric superspaces listed in the third column. The notation for the dimensions is taken from Table II.

RMT	Comments	RSS	Dimensions
A	Wigner–Dyson (GUE)	AIII AIII	$m_1 = n_1 = n_A, m_2 = n_2 = n_R$
AI	Wigner–Dyson (GOE)	BDI CII	$m_1 = 2n_1 = 2n_A, m_2 = 2n_2 = 2n_R$
AII	Wigner–Dyson (GSE)	CII BDI	$m_1 = 2n_1 = 2n_A, m_2 = 2n_2 = 2n_R$
AIII ( $p = q$ )	chiral GUE	A A	$m = n$
BDI ( $p = q$ )	chiral GOE	AI AII	$m = n$
CII ( $p = q$ )	chiral GSE	AII AI	$m = n$
C	NS	DIII CI	$m = n$
CI	NS	D C	$m = 2n$
D	NS	CI DIII	$m = n$
DIII	NS	C D	$m = 2n$

try and then reducing it by the addition of symmetry-breaking terms. I chose not to follow this route as it involves handling large tensor products, which makes the computations less transparent and the identification of the spaces involved more difficult.)

The great strength of the supersymmetry method, as compared to other methods of mesoscopic physics, stems from the fact that it easily extends beyond the universal random-matrix limit to diffusive and localized systems. What one obtains for these more general systems are field theories of the nonlinear  $\sigma$  model type, with fields that take values in a Riemannian symmetric superspace. The method also extends beyond spectral correlations and allows the calculation of wave function statistics and of transport coefficients such as the electrical conductance (see the literature cited in the Introduction).

Let me end on a provocative note. Mathematicians and mathematical physicists working on supermanifold theory have taken much guidance from developments in such esoteric subjects as supergravity and superstring theory. Would it not be just as worthwhile to investigate the beautiful structures outlined in the present paper, whose physical basis is not speculative but firmly established, and which are of direct relevance to experiments that are currently being performed in physics laboratories all over the world?

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- <sup>37</sup>Supertransposition  $T$  is an operation with the properties  $(AB)^T = B^T A^T$  and  $A^{TT} = \sigma A \sigma$ .
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