Energy correlations for a random matrix model of disordered bosons

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Linearizing the Heisenberg equations of motion around the ground state of an interacting quantum many-body system, one gets a time-evolution generator in the positive cone of a real symplectic Lie algebra. The presence of disorder in the physical system determines a probability measure with support on this cone. The present paper analyzes a discrete family of such measures of exponential type, and does so in an attempt to capture, by a simple random matrix model, some generic statistical features of the characteristic frequencies of disordered bosonic quasiparticle systems. The level correlation functions of the said measures are shown to be those of a determinantal process, and the kernel of the process is expressed as a sum of biorthogonal polynomials. While the correlations in the bulk scaling limit are in accord with sine-kernel or Gaussian Unitary Ensemble universality, at the low-frequency end of the spectrum an unusual type of scaling behavior is found. [DOI: 10.1063/1.2356798]

I. INTRODUCTION

Perturbing the ground state of an interacting quantum many-body system and linearizing the Heisenberg equations of motion for the boson Fock operators, one faces the standard problem of small oscillations. Concrete examples are furnished by the vibrational modes of a solid, the spin waves in a magnet, the electromagnetic modes in an optical medium, and the oscillations of the superfluid density of a Bose–Einstein condensate. Common to these excitations is that they second-quantize as bosons or bosonic quasiparticles.

Adding some amount of disorder to the system, one may ask: what are the statistical features of the excitation spectrum and, in particular, which of these features (if any) reflect the bosonic nature of the quasiparticle excitations? Is there some kind of universality akin to the Wigner–Dyson universality known from other disordered systems? If so, what are the universal laws, and what is the role of symmetry in determining these laws?

In the parallel case of fermionic quasiparticles the situation is now fairly well understood. If the system is of metallic type and in the ergodic limit, the statistical behavior at high energies is in accord with the universal laws of Wigner–Dyson statistics. For low excitation energies, however, the canonical anticommutation relations obeyed by the fermion operators make themselves felt: they constrain the form of the Hamiltonian matrix and thus give rise to several new universality classes beyond Dyson's threefold way.[1] Some of these are realized by chiral Dirac fermions in a random gauge field,[2] others by quasiparticles in disordered gapless superconductors.[3,4] A

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complete symmetry classification of quadratic fermion Hamiltonians has been carried out,[5] and the role of Riemannian symmetric spaces and superspaces in providing an effective description has been emphasized.[6,7]

Progress has been slower for bosonic systems, and so for good reason, as these are set apart by several distinctive features from other random problems studied and solved in the past. For one thing, in the case of bosons it makes little sense to choose—as one often does for fermions—the matrix elements of the quasiparticle Hamiltonian as independent and identically distributed random variables. In fact, most of the boson Hamiltonians produced in such a manner would generate runaway dynamics rather than oscillatory motion around a stable ground state. In the case of bosons one therefore has to pay attention to the fact that the matrix elements depend in a complicated way on the ground state of the many-boson system and, hence, on the disorder of the microscopic parent problem.[8,9] As a technical consequence, a direct analog of the so-called Gaussian Ensembles, which were pivotal in initiating the Wigner–Dyson theory and establishing its universal statistics, is unavailable in the context of bosons.

For another complication, low-frequency bosons are usually insensitive to weak disorder. Many of the excitations listed above are Goldstone bosons associated with a spontaneously broken symmetry, and for such excitations low frequency is tantamount to low wave number, or large wavelength, which causes the scattering by disorder to be suppressed, as the disorder is effectively seen only on average over regions of size given by the large wavelength. Thus the disorder averages out and becomes less effective, and hence, the behavior of weakly disordered Goldstone bosons tends to be system specific. (Of course this still leaves it possible for weakly disordered bosons of *non*-Goldstone type to exhibit universal statistics.[8,9]) In order for any universality to set in, the disorder strength often has to be so large that standard calculational tools such as the impurity diagram technique fail to apply.

In the present paper we are going to introduce and completely solve a simple random matrix model of disordered bosonic quasiparticles, which we believe to be most closely analogous to the Wigner–Dyson Gaussian Ensembles while retaining the crucial features of bosonic statistics and stability of the motion. In a follow-up paper we will investigate the question whether this simple model might be representative of a whole universality class of related problems.

To formulate the model, let q_j, p_j $(j=1,\ldots,N)$ be a canonical set of position and momentum operators, and consider their linearized Heisenberg equations of motion in the most general form

$$\dot{q}_j = \sum_{i=1}^{N} (q_i A_{ij} + p_i C_{ij}), \qquad -\dot{p}_j = \sum_{i=1}^{N} (q_i B_{ij} + p_i A_{ji}),$$

where $B_{ij}=B_{ji}$, $C_{ij}=C_{ji}$, and A_{ij} are real numbers. If the system was invariant under time reversal $(q_j \mapsto q_j, p_j \mapsto -p_j)$, the coefficients A_{ij} would have to be zero, but we here consider the generic case without symmetries. The criterion for stability of the dynamics is that the stability matrix be positive

$$h := \begin{pmatrix} B & A \\ A^t & C \end{pmatrix} > 0.$$

Assuming $h^t = h > 0$, the generator of the Heisenberg time evolution

$$X := \begin{pmatrix} A & -B \\ C & -A^t \end{pmatrix},$$

has eigenvalues that come as imaginary pairs $\pm \mathrm{i}\omega_j$ where $\omega_j > 0$ $(j=1,\ldots,N)$ are the characteristic frequencies (or single-boson energies) of the small-amplitude motion. In a classical setting one would introduce the generator X as the symplectic gradient of the Hamiltonian function linearized at a stable equilibrium point of the classical flow.

The natural transformation group of the problem at hand is the real symplectic group in $2N$ dimensions, $\mathrm{Sp}_{2N}(\mathbb{R})$, acting by linear canonical transformations on the operators q_j, p_j and by conjugation on the generator X. We can now explain one of the distinctive features of the present problem: when formulating the Gaussian Ensembles of the Wigner–Dyson theory one makes the postulate that the transformation group of the problem (O_N, U_N, or USp_{2N}, as the case may be) also be the symmetry group of the chosen probability measure, whereas in our case no such simplification is possible. Indeed, $Sp_{2N}(\mathbb{R})$ is noncompact, and a probability measure $d\mu$ cannot be invariant under a noncompact group action and at the same time have total mass $\int d\mu = 1$.

One is therefore looking for some construction principle other than symmetry. Our key here is the positivity of the real symmetric stability matrix h: a natural way of building positive real symmetric matrices h is by adding a sufficient number of rank-one projectors with positive weights. Equivalently, we may put

$$h_{ij} = \sum_{\alpha=1}^{M} v_{i\alpha} v_{j\alpha} \quad (i,j = 1, \ldots, 2N) \tag{1.1}$$

for some set of real numbers $v_{i\alpha}$. We now consider the $v_{i\alpha}$ as the fundamental variables, and choose them to be independent and normal (or Gaussian) distributed random variables with zero mean and variance τ^{-1}. Then we use Eq. (1.1) to push forward the normal distribution for the $v_{i\alpha}$ to a probability distribution $d\mu(h)$ for h (and, hence, for X). If $M \geqslant 2N$, the result is

$$d\mu(h) \propto e^{-(1/2)\tau \mathrm{Tr} h} \mathrm{Det}(h)^{(1/2)(l-1)} \prod_{i \leqslant j} dh_{ij}, \quad l = M - 2N \geqslant 0, \tag{1.2}$$

with the domain for h still defined by $h > 0$. The probability distribution (1.2) is the object of study of this paper.

We now give a summary of the contents and the results of the paper. After collecting some basic facts from symplectic linear algebra in Sec. II, we reduce $d\mu(h)$ in Sec. III to a probability distribution on the space of characteristic frequencies $\omega_1, \ldots, \omega_N$ (the positive eigenvalues of $-iX$), and find this to be

$$d\mu_{N,l}(\omega_1, \ldots, \omega_N) = c_{N,l}(\tau) \prod_{i<j} (\omega_i - \omega_j)(\omega_i^2 - \omega_j^2) \prod_{k=1}^{N} \omega_k^l e^{-r\omega_k} d\omega_k. \tag{1.3}$$

Using the method of biorthogonal polynomials we show in Sec. V C that the n-level correlation functions of this probability distribution are of determinant type and are completely determined in the usual way—see Eq. (5.19)—by a certain kernel $K_N(\omega, \tilde{\omega})$ given as a sum over biorthogonal polynomials. We compute the large-N asymptotics of this kernel in the bulk of the spectrum (in Sec. V D) and at the "hard" edge $\omega = 0$ (Sec. V E), using a contour integral representation of the biorthogonal polynomials (Sec. V C). In the former case we establish the scaling limit

$$\tau \lim_{N\to\infty} K_N(Nx/\tau + \omega, Nx/\tau + \tilde{\omega}) = \frac{\sin[\pi \rho_\infty(x)(\omega - \tilde{\omega})]}{\pi(\omega - \tilde{\omega})} e^{-r(x)(\omega - \tilde{\omega})}, \tag{1.4}$$

which is independent of l. The function $\rho_\infty(x)$ of the scaling variable $x = \omega\tau/N$ is the large-N limit of the level density. Viewing $\pi\rho_\infty(x)$ as the imaginary part of a Green's function $\lim_{\epsilon\to 0+} g(x + i\epsilon)$, the function $r(x)$ is the real part. We compute $\rho_\infty(x)$ by two independent methods (from a variational calculation in Sec. IV, and from biorthogonal polynomials in Sec. V D), with the result being

$$\rho_\infty(x) = \frac{\tau}{2\pi}(x/b)^{-1/3}[(1 + \sqrt{1 - x^2/b^2})^{1/3} - (1 - \sqrt{1 - x^2/b^2})^{1/3}] \quad (0 < x \leqslant b = 3\sqrt{3}). \tag{1.5}$$

Apart from the last factor, which is irrelevant since it cancels on passing to the level correlation functions, the right-hand side of Eq. (1.4) is the famous sine kernel known from systems with unitary symmetry. Thus we recover Wigner–Dyson universality of the class of the Gaussian Unitary Ensemble (GUE) at bulk frequencies.

At low frequencies $\omega \sim N^{-1/2}$ we find convergence to an unusual kind of scaling limit

$$\lim_{N\to\infty} N^{-1/2} K_N(N^{-1/2}y_1/\tau, N^{-1/2}y_2/\tau) = \frac{\tau^2}{2\pi^2}\int_{i\mathbb{R}+\epsilon} du \oint_{\mathrm{U}_1} \frac{dv}{v} e^{-y_1/u+y_2/v}(v/u)^l \frac{e^{u^2-v^2}-1}{u^2-v^2}, \tag{1.6}$$

where U_1 denotes the unit circle in $\mathbb{C}$, and $i\mathbb{R}+\epsilon$ is any axis in the right half plane parallel to the imaginary axis. The result (1.6) is reminiscent of formulas obtained by Efetov's supersymmetry method, with u and v playing the role of radial polar coordinates of a Riemannian symmetric superspace. We intend to elucidate this connection in a future publication.

II. THE HAMILTONIANS OF STABLE MOTIONS

Let there be some position variables $q_1, \ldots, q_N$ and canonical momenta $p_1, \ldots, p_N$, and consider Hamiltonians H of the quadratic form

$$H = \frac{1}{2}\sum_{i,j=1}^{N} [C_{ij}p_ip_j + B_{ij}q_iq_j + A_{ij}(q_ip_j + p_jq_i)], \tag{2.1}$$

where A, B, and C are real matrices satisfying $B=B^t$ and $C=C^t$. Rewriting H as

$$H = \frac{1}{2}(q\ p)\begin{pmatrix} A & -B \\ C & -A^t \end{pmatrix}\begin{pmatrix} p \\ -q \end{pmatrix},$$

we see that the matrix, X, of H satisfies the linear condition

$$X^tJ + JX = 0, \quad J = \begin{pmatrix} 0 & -1_N \\ 1_N & 0 \end{pmatrix}, \quad X = \begin{pmatrix} A & -B \\ C & -A^t \end{pmatrix}. \tag{2.2}$$

This is saying that X lies in $\mathfrak{sp}_{2N}(\mathbb{R})$, the Lie algebra of the real symplectic group defined by

$$\mathrm{Sp}_{2N}(\mathbb{R}) = \{g \in \mathrm{GL}_{2N}(\mathbb{R}) | g^tJg = J\}.$$

A matrix $X \in \mathfrak{sp}_{2N}(\mathbb{R})$ need not be diagonalizable (e.g., the generator of free motion, $A=B=0$ and $C=1_N$, is not); and even if it is, the eigenvalues will in general be complex.

We now impose the condition

$$h := \begin{pmatrix} B & A \\ A^t & C \end{pmatrix} > 0, \tag{2.3}$$

i.e., we require all eigenvalues of the real symmetric matrix h to be positive. The corresponding domain in $\mathfrak{sp}_{2N}(\mathbb{R})$ will be denoted by $\mathfrak{E}^0$:

$$\mathfrak{E}^0 := \{X \in \mathfrak{sp}_{2N}(\mathbb{R}) | X = hJ, h = h^t > 0\}. \tag{2.4}$$

Although the eigenvalues of h have no direct relation to the dynamics of the system, positivity of h ensures that the motion generated by the Hamiltonian H is stable, or "elliptic." As a consequence of ellipticity, there exists some linear canonical transformation $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$ which takes the Hamiltonian to a sum of harmonic oscillators

$$H = \frac{1}{2}\sum_{i=1}^{N} (P_i^2 + \omega_i^2 Q_i^2),$$

with $\omega_i^2 > 0$. Put differently, for $X \in \mathfrak{E}^0$ one can always find a symplectic transformation $g \in \mathrm{Sp}_{2N}(\mathbb{R})$ that conjugates X to quasidiagonal form

$$X = g\Omega g^{-1}, \quad \Omega = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}, \quad \omega = \mathrm{diag}(\omega_1, \omega_2, \ldots, \omega_N), \tag{2.5}$$

with real and positive ω_i $(i=1,\ldots,N)$.

All of the discussion below will be based on the elliptic domain $\mathfrak{E}^0$. Let us therefore collect some of its mathematical properties. First of all, if X is in $\mathfrak{E}^0$, then so is its conjugate gXg^{-1} by any element $g \in \mathrm{Sp}_{2N}(\mathbb{R})$. Thus $\mathfrak{E}^0$ is invariant under the action of $\mathrm{Sp}_{2N}(\mathbb{R})$ on $\mathfrak{E}^0$ by conjugation. Second, let $\mathfrak{t}$ denote the Abelian algebra of block-diagonal matrices of the form of Ω in Eq. (2.5) but with diagonal elements ω_i that are any real numbers (not necessarily positive). Let $\mathfrak{t}_+ \subset \mathfrak{t}$ be the subset of block-diagonal Ω with positive ω_i. Then, as we said earlier, every $X \in \mathfrak{E}^0$ is conjugate to a unique $\Omega \in \mathfrak{t}_+$ by some $g \in \mathrm{Sp}_{2N}(\mathbb{R})$. Third, introducing $T := \exp(\mathfrak{t})$, which is an N-dimensional compact torus, $T \cong (S^1)^N$, let G/T be the quotient of $G \equiv \mathrm{Sp}_{2N}(\mathbb{R})$ by the right action of T. Then the mapping

$$(G/T) \times \mathfrak{t}_+ \to \mathfrak{E}^0, \quad (gT, \Omega) \mapsto g\Omega g^{-1} \tag{2.6}$$

(the reverse of the process of quasidiagonalization), is a smooth bijection.

We are stating these facts without proof, as they are standard facts of symplectic linear algebra.

III. PROBABILITY MEASURE

By placing a probability distribution on the elliptic domain $\mathfrak{E}^0$, one gets a random matrix model for disordered bosonic quasiparticles. We are then interested in the statistics of the characteristic frequencies or levels ω_i.

It is well known that in the Wigner–Dyson situation of random Hermitian or random real symmetric matrices, where the symmetry group is compact, the level correlation functions exhibit universal behavior in a suitable scaling limit. One may therefore ask whether a similar scenario—leading to universal laws, possibly of a new kind—might be at work in the case being considered.

To answer this question we need to investigate a class of probability distributions on $\mathfrak{E}^0$ as wide as possible. As a first step, the present paper deals with a family of well motivated distributions which are easy to analyze.

A. Choice of measure

Coming from the standard Wigner–Dyson situation with a compact symmetry group, one might be inclined to try and consider a Gaussian distribution

$$P(X)dX \overset{?}{\propto} e^{-\mathrm{Tr}X^2} dX,$$

where dX is a Lebesgue measure for $\mathfrak{E}^0$:

$$dX := \prod_{i,j} dA_{ij} \prod_{i\leq j} dB_{ij} dC_{ij}. \tag{3.1}$$

However, such a distribution has infinite mass, since it is invariant under the action $X \mapsto gXg^{-1}$ by the *noncompact* group $\mathrm{Sp}_{2N}(\mathbb{R})$, and it therefore cannot be normalized to be a probability measure.

Staying within the class of Gaussian distributions, a better choice of distribution function is

$$P(X = Jh) \propto e^{-\tau \mathrm{Tr} h/2 - \sigma \mathrm{Tr} h^2} = e^{-\tau \mathrm{Tr}(J^{-1}X)/2 - \sigma \mathrm{Tr}(J^{-1}X)^2} \tag{3.2}$$

for some positive parameters σ, τ. Because of the presence of J^{-1} under the trace, this distribution function is invariant under conjugation $X \mapsto gXg^{-1}$ only if $g \in \mathrm{Sp}_{2N}(\mathbb{R})$ satisfies the additional condition $g^{-1}Jg = J$. Combining the two conditions, $g^t Jg = J = g^{-1}Jg$, one sees that the invariance group of the function $P(X)$ in Eq. (3.2) is the intersection of the real symplectic and orthogonal groups in $2N$ dimensions

$$K = \mathrm{Sp}_{2N}(\mathbb{R}) \cap \mathrm{SO}_{2N}(\mathbb{R}). \tag{3.3}$$

This group K is isomorphic to U_N, the group of unitary transformations in N complex dimensions. Indeed, changing from the symplectic basis $\{q_1, \ldots, q_N, p_1, \ldots, p_N\}$ to the oscillator basis

$$\{a_1, \ldots, a_N, a_1^\dagger, \ldots, a_N^\dagger\}, \quad a_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \quad a_j^\dagger = \frac{1}{\sqrt{2}}(q_j - ip_j),$$

one finds that K is the subgroup of canonical transformations that do not mix the lowering operators $\{a_j\}$ with the raising operators $\{a_j^\dagger\}$. Moreover, $\mathrm{U}_N \cong K \subset \mathrm{Sp}_{2N}(\mathbb{R})$ is known to be a maximal compact subgroup. It therefore is the biggest symmetry group possible in our problem.

In the sequel we will consider Eq. (3.2) with $\sigma=0$. Thus we take our probability distribution to be

$$P(X)dX := c_N(\tau) e^{-\tau \mathrm{Tr}(J^{-1}X)/2} dX, \tag{3.4}$$

with the normalization constant $c_N(\tau)$ chosen in such a way that $\int_{\mathfrak{E}^0} P(X)dX = 1$. Further motivation for this choice of distribution was put forth in the introduction (Sec. I).

B. Polar decomposition and reduction

Let now $F(X) = F(gXg^{-1})$ be some function on $\mathfrak{E}^0$ which is *radial*, i.e., invariant under conjugation by every element $g \in \mathrm{Sp}_{2N}(\mathbb{R})$. Given such a function F, which depends only on the eigenfrequencies $\omega_1, \ldots, \omega_N$ of X, we wish to take the expectation of F with respect to the probability measure $P(X)dX$:

$$\langle F \rangle := \int_{\mathfrak{E}^0} F(X) P(X) dX. \tag{3.5}$$

The problem of computing such expectations is best tackled by using the polar decomposition $\mathfrak{E}^0 \cong \mathfrak{t}_+ \times (G/T)$ which is given by quasidiagonalization of X; see Eq. (2.6). Inserting that decomposition into Eq. (3.5) one has

$$\langle F \rangle = \int_{\mathfrak{t}_+} \left[\int_{G/T} P(g\Omega g^{-1}) dg_T \right] F(\Omega) j(\Omega) d\Omega, \quad d\Omega = d\omega_1 d\omega_2 \cdots d\omega_N, \tag{3.6}$$

where g_T is a G-invariant measure for G/T, and $j(\Omega)$ is the Jacobian of the change of variables $X = g\Omega g^{-1}$.

Let us calculate this Jacobian. Differentiating the polar coordinate mapping Eq. (2.6) we get

$$\delta(g\Omega g^{-1}) = g(\delta\Omega + [g^{-1}\delta g, \Omega])g^{-1}.$$

The Jacobian we are seeking is the product of all nonzero eigenvalues of the linear operator $X \mapsto [X, \Omega]$. These eigenvalues are called the roots of the pair $[\mathfrak{sp}_{2N}(\mathbb{R}), \mathfrak{t}]$. They are

$$\pm(\omega_i + \omega_j) \quad (i \leq j), \qquad \pm(\omega_i - \omega_j) \quad (i < j),$$

each with multiplicity one. Thus, by taking the product of all nonvanishing roots

$$j(\Omega) d\Omega = \prod_{i<j} (\omega_i^2 - \omega_j^2)^2 \prod_{k=1}^{N} (2\omega_k)^2 d\omega_k. \tag{3.7}$$

To complete the polar integration formula (3.6) we need $\int_{G/T} P(g\Omega g^{-1}) dg_T$. In the next subsection we are going to show that this integral can be calculated in closed form and depends on $\omega_1, \ldots, \omega_N$ as

$$\int_{G/T} P(g\Omega g^{-1}) dg_T \propto \prod_{i<j} (\omega_i + \omega_j)^{-1} \prod_{k=1}^{N} \omega_k^{-1} e^{-\tau\omega_k}. \tag{3.8}$$

Thus, in total, the expectation of a radial observable $F(X)=F(\Omega)\equiv F(\omega_1,\ldots,\omega_N)$ becomes

$$\langle F\rangle = \tilde{c}_N(\tau)\int_{\mathbb{R}_+^N} F(\omega_1,\ldots,\omega_N)\prod_{i<j}(\omega_i-\omega_j)(\omega_i^2-\omega_j^2)\prod_{k=1}^{N} e^{-\tau\omega_k}\omega_k d\omega_k, \tag{3.9}$$

where $\tilde{c}_N(\tau)$ is another normalization constant. This expectation, for the special choices of F that give the level correlation functions, will be calculated in Sec. V of the paper.

C. Computation of the integral (3.8)

We now establish Eq. (3.8). Omitting a normalization constant, we denote the integral on the left-hand side of Eq. (3.8) by

$$I(\Omega) := \int_{G/T} e^{-\tau \mathrm{Tr}(J^{-1} g\Omega g^{-1})/2} dg_T. \tag{3.10}$$

What makes this integral computable in closed form is that J lies in $\mathfrak{sp}_{2N}(\mathbb{R})$ and $\Omega \mapsto g\Omega g^{-1}$ is the adjoint action of $G=\mathrm{Sp}_{2N}(\mathbb{R})$ on its Lie algebra. These circumstances place the integral in the class of integrals of Harish–Chandra–Itzykson–Zuber type, which are covered by the Duistermaat–Heckman theorem and its generalizations. In the present case, the integral can be computed in a particularly simple manner, as follows.

Let dg and dt be Haar measures for G and T, respectively, with $dg=dg_T dt$ and $\int_T dt=\mathrm{vol}(T)$. Our first step is to switch from G/T to integrating over the full symplectic group G:

$$I(\Omega) = \frac{1}{\mathrm{vol}(T)}\int_G e^{-\tau \mathrm{Tr}(J^{-1} g\Omega g^{-1})/2} dg.$$

Next we use that dg is invariant under inversion, $g\mapsto g^{-1}$. After this transformation the integrand is expressed in terms of the combination $gJ^{-1}g^{-1}=-gJg^{-1}$. Since $kJk^{-1}=J$ for $k\in K\cong \mathrm{U}_N$, we can push down the resulting integral over G to an integral over the quotient space G/K. Let dg_K and dk be invariant resp. Haar measures for G/K and K so that $dg=dg_K dk$. Then

$$I(\Omega) = \frac{\mathrm{vol}(K)}{\mathrm{vol}(T)}\int_{G/K} e^{\tau \mathrm{Tr}(\Omega g J g^{-1})/2} dg_K, \quad \mathrm{vol}(K) = \int_K dk. \tag{3.11}$$

The homogeneous space $G/K \cong \mathrm{Sp}_{2N}(\mathbb{R})/\mathrm{U}_N$ has the salient feature of being a noncompact symmetric space of Hermitian type. Such spaces carry the structure of a Kähler manifold, which means that G/K comes with a non-degenerate, closed, and G-invariant two-form (the Kähler form of G/K). Writing $gJg^{-1}=:Q$ this is the form

$$\beta = \mathrm{Tr}(Q\mathrm{d}Q \wedge \mathrm{d}Q). \tag{3.12}$$

Notice that $\dim_{\mathbb{R}} G/K=N(2N+1)-N^2=N(N+1)$. Raising β to its $(1/2)N(N+1)$th exterior power one obtains a top-dimensional form, $\beta^{(1/2)N(N+1)}$, which is still G-invariant and nonzero. Since G/K is homogeneous, there can be at most one such form up to multiplication by scalars. Therefore, there exists some (nonzero) constant such that

$$dg_K = \mathrm{const}\ \beta^{(1/2)N(N+1)}. \tag{3.13}$$

By Darboux's theorem one can find local symplectic coordinates for G/K that bring β into canonical form. While this fact by itself would not be of much practical help, in the present case such coordinates exist *globally* and, moreover, they can be chosen in such a way that $\mathrm{Tr}(\Omega g J g^{-1})$

depends on them *quadratically*.

To describe these perfect coordinates, consider the space of complex symmetric $N\times N$ matrices, $\mathrm{Sym}(\mathbb{C}^N)$, which has dimension $(1/2)N(N+1)$ over $\mathbb{C}$ and thus shares with G/K the dimension $N(N+1)$ over $\mathbb{R}$. With every $Z\in\mathrm{Sym}(\mathbb{C}^N)$ associate a positive Hermitian $2N\times 2N$ matrix $\tilde{g}$ by

$$\tilde{g}=\tilde{g}(Z,Z^\dagger)=\begin{pmatrix}(1+ZZ^\dagger)^{1/2} & Z\\ Z^\dagger & (1+Z^\dagger Z)^{1/2}\end{pmatrix}. \tag{3.14}$$

Now if S is the matrix of the unitary transformation from the real symplectic basis $\{p_j,q_j\}$ of $\mathbb{R}^{2N}$ to the oscillator basis $\{a_j,a_j^\dagger\}$:

$$S:=\frac{1}{\sqrt{2}}\begin{pmatrix}1_N & i1_N\\ -1_N & i1_N\end{pmatrix},$$

then $g:=S^{-1}\tilde{g}S$ is immediately seen to be a real matrix and, using the relation

$$SJS^{-1}=i\Sigma_3,\quad \Sigma_3=\begin{pmatrix}1_N & 0\\ 0 & -1_N\end{pmatrix},$$

one finds that $g=S^{-1}\tilde{g}S$ satisfies $g^\dagger Jg=g^tJg=J$ and hence lies in $\mathrm{Sp}_{2N}(\mathbb{R})$. Moreover, the reverse correspondence $k\mapsto\tilde{k}=SkS^{-1}$ is the isomorphism between K and U_N discussed in the paragraph after Eq. (3.3); it takes $k\in K$ to the block-diagonal form

$$\tilde{k}=\begin{pmatrix}U & 0\\ 0 & \bar{U}\end{pmatrix},\quad U\in\mathrm{U}_N.$$

It is now clear that the mapping $\mathrm{Sym}(\mathbb{C}^N)\to G/K$ by $Z\mapsto S^{-1}\tilde{g}(Z,Z^\dagger)SK\equiv gK$ is a bijection. Using it to express the Kähler form β in terms of the complex symmetric matrix Z, one obtains

$$\beta=\mathrm{Tr}(Q\mathrm{d}Q\wedge\mathrm{d}Q)=-4i\mathrm{Tr}(\Sigma_3\mathrm{d}\tilde{g}^{-1}\wedge\mathrm{d}\tilde{g})=8i\mathrm{Tr}(\mathrm{d}Z\wedge\mathrm{d}Z^\dagger). \tag{3.15}$$

Thus, the top-dimensional form $\beta^{(1/2)N(N+1)}$ is constant in Z:

$$\frac{\beta^{(1/2)N(N+1)}}{\left[\frac{1}{2}N(N+1)\right]!}=(8i)^{(1/2)N(N+1)}2^{(1/2)N(N-1)}\prod_{i\leqslant j}dZ_{ij}\wedge d\bar{Z}_{ij}, \tag{3.16}$$

and from Eq. (3.13) the invariant measure dg_K is a constant multiple of the Lebesgue measure for $\mathrm{Sym}(\mathbb{C}^N)$.

Finally, from $g=S^{-1}\tilde{g}S$, $SJS^{-1}=\mathrm{i}\Sigma_3$, and Eq. (3.14) one has

$$-\mathrm{Tr}(\Omega gJg^{-1})=\mathrm{Tr}\,\omega(1+2ZZ^\dagger)+\mathrm{Tr}\,\omega(1+2Z^\dagger Z),\quad \omega=\mathrm{diag}(\omega_1,\ldots,\omega_N).$$

Our integral (3.11) now becomes a Gaussian integral

$$I(\Omega)=\mathrm{const}\;e^{-\tau\Sigma_k\omega_k}\int e^{-2\tau\mathrm{Tr}Z^\dagger(Z\omega+\omega Z)}\prod_{i\leqslant j}dZ_{ij}d\bar{Z}_{ij}.$$

Doing this integral one immediately obtains the result for $I(\Omega)$ stated in Eq. (3.8).

D. Generalization

A slight generalization of Eq. (3.4) is afforded by the observation that the determinant of X in Eq. (2.2) is always positive:

$$\mathrm{Det}(X) = \mathrm{Det}(\Omega) = \prod_{k=1}^{N} \omega_k^2.$$

Thus, by multiplying the probability measure $P(X)dX$ by some power $l-1>-1$ of the positive square root $\mathrm{Det}(X)^{1/2}$ and adjusting the normalization constant, we get another probability measure

$$P_l(X)dX = \text{const}\ \mathrm{Det}(X)^{(1/2)(l-1)} e^{-(1/2)\tau \mathrm{Tr}(J^{-1}X)} dX. \tag{3.17}$$

This measure is still U_N invariant. By the process of quasidiagonalization and drawing on our results above, we push it forward to a measure for the eigenfrequencies. The result is

$$d\mu_{N,l}(\omega_1, \ldots, \omega_N) = c_{N,l}(\tau) \prod_{i<j} (\omega_i - \omega_j)^2 (\omega_i + \omega_j) \prod_{k=1}^{N} \omega_k^l e^{-\tau \omega_k} d\omega_k. \tag{3.18}$$

This, for any non-negative power $l \in \mathbb{Z}$, is the family of probability distributions to be studied in the present paper.

IV. LARGE-*N* LIMIT OF THE ONE-POINT DENSITY IN THE BULK

The one-point density $\rho(\omega)d\omega$ is defined as the probability density for any one of the eigenfrequencies ω_i to have the value of ω, irrespective of what the values of the other eigenfrequencies are; thus $\rho(\omega)$ is the function

$$\rho(\omega) := \int \sum_{i=1}^{N} \delta(\omega - \omega_i) d\mu_{N,l}(\omega_1, \ldots, \omega_N), \tag{4.1}$$

which has the properties $\rho(\omega) \geqslant 0$ and

$$\int_0^\infty \rho(\omega) d\omega = N. \tag{4.2}$$

We are now interested in the behavior of the density function $\rho(\omega)$ in the limit of $N \to \infty$. From the expression (3.18) and experience with similar problems (see, e.g., Ref. 11), we expect that this limit can be obtained by maximizing the functional

$$F = \frac{1}{2} \int_0^\infty \int_0^\infty \ln[(\omega - \omega')^2 (\omega + \omega')] \rho(\omega) \rho(\omega') d\omega' d\omega + \int_0^\infty \ln(\omega^l e^{-\omega \tau}) \rho(\omega) d\omega \tag{4.3}$$

subject to the constraint (4.2) and the condition $\rho(\omega) \geqslant 0$. More precisely, the limit is expected to exist in the scaled variable $x := \omega\tau/N$; i.e., there should exist a certain non-negative function $\rho_\infty(x)$ with $\int \rho_\infty(x) dx = 1$ such that $\rho(\omega)$ is asymptotic to $\tau \rho_\infty(\omega\tau/N)$.

Varying F with respect to $\rho(\omega)$ we get

$$\frac{\delta F}{\delta \rho(\omega)} = \int_0^\infty [2 \ln|\omega - \omega'| + \ln(\omega + \omega')] \rho(\omega') d\omega' + l \ln \omega - \omega\tau.$$

We now insert the asymptotic equality $\rho(Nx/\tau) \approx \tau \rho_\infty(x)$ and pass to the limit $N \to \infty$ in the scaling variable x. Let $\mathrm{supp}(\rho_\infty) = [0, b]$ be the region of support of ρ_∞. Then the condition $\delta F / \delta \rho(\omega) = N\lambda$, where λ is a Lagrange multiplier for the constraint (4.2), yields the equation

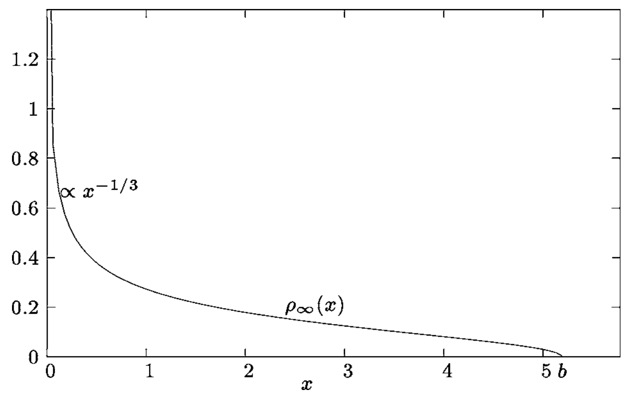

FIG. 1. The graph of the function ρ_∞.

$$\int_0^b [2\ln|x-x'| + \ln(x+x')]\rho_\infty(x')dx' - x = \lambda \quad (0 < x < b), \tag{4.4}$$

which no longer depends on the parameter l. It can be shown that our functional F is convex; as a result, the solution ρ_∞ of Eq. (4.4) exists and is unique when supplemented by the normalization condition

$$\int_0^b \rho_\infty(x)dx = 1. \tag{4.5}$$

In the following subsections, we are going to construct the solution to the mathematical problem posed by Eqs. (4.4) and (4.5). It will turn out to be

$$\rho_\infty(x) = \frac{1}{2\pi}(x/b)^{-1/3}[(1+\sqrt{1-x^2/b^2})^{1/3} - (1-\sqrt{1-x^2/b^2})^{1/3}] \quad (0 < x \leqslant b = 3\sqrt{3}). \tag{4.6}$$

The graph of this function is plotted in Fig. 1. From the expression (4.6) the behavior near the lower edge $x=0$ is

$$\rho_\infty(x) \simeq \frac{1}{2\pi}(2b/x)^{1/3} \quad (0 < x \ll b),$$

while close to the upper edge $x=b$ one gets

$$\rho_\infty(x) \simeq \frac{1}{3\pi}(1-x^2/b^2)^{1/2} \quad (x < b,\ x \to b).$$

In the vicinity of the upper and lower edges there exists crossover to a fine-scale behavior that cannot be found by the present method of maximization of the functional F. The crossover at the upper edge involves Airy functions on a scale $N^{1/3}$, which is small compared to the bulk scale N. At the lower edge, the crossover occurs on a very fine scale, $N^{-1/2}$, which is small even in comparison with the bulk mean level spacing (which is of order N^0).

A. Method of solution (idea)

We do not know how to solve Eq. (4.4) for the unknown function $\rho_\infty(x)$ directly. Therefore, to simplify the problem we differentiate once with respect to x to obtain the equation

$$2\mathcal{P}\int_0^b \frac{\rho_\infty(x')dx'}{x-x'} + \int_0^b \frac{\rho_\infty(x')dx'}{x+x'} = 1, \tag{4.7}$$

where $\mathcal{P}$ means the principal value of the integral. At this stage, the value of b is unknown but assumed to be finite.

Introducing the Green's function (or Stieltjes transform)

$$g(z) := \int_0^b \frac{\rho(x)dx}{x-z}, \quad z \in \mathbb{C}\setminus[0,b], \tag{4.8}$$

and the related functions

$$g_\pm(x) := \lim_{\varepsilon\to 0+} g(x \pm i\varepsilon), \quad g_0(x) := -g(-x), \tag{4.9}$$

we bring Eq. (4.7) into the form

$$g_+(x) + g_-(x) + g_0(x) = -1 \quad (0 < x < b). \tag{4.10}$$

To solve this equation, we are led to do an exercise in complex analysis which is motivated as follows.

Let $w \mapsto f(w)$ be some meromorphic function of a complex variable w, and let the equation $z=f(w)$ have r simple roots $w_1(z), w_2(z), \ldots, w_r(z)$, i.e., $z=f[w_1(z)]=\ldots=f[w_r(z)]$. If the function f is analytic in $1/w$ at $w=\infty$, these roots add up to a constant

$$\sum_{i=1}^r w_i(z) = \text{const} =: c \quad (\text{independent of } z). \tag{4.11}$$

Indeed, if γ is a closed contour encircling all of the roots in the counterclockwise sense, then

$$\sum w_i'(z) = \sum \frac{1}{f'[w_i(z)]} = \frac{1}{2\pi i}\oint_\gamma \frac{dw}{f(w)-z} = 0,$$

where the second equality is by the residue theorem, and the last equality follows by contracting γ to the point at infinity. Thus $\Sigma w_i'(z)=0$ and, hence, $\Sigma w_i(z)=\text{const}$.

Equation (4.11) for $r=3$ looks similar to Eq. (4.10) and can, in fact, be made to look identical to it by the following observation. Notice that the function $z \mapsto g(z)$ defined by Eq. (4.8) is holomorphic in the interior of the left half of the complex plane. Suppose, therefore, that we have found a root $g(z)$ of $z=f[g(z)]$ which is holomorphic in the left half plane, and that $g_\pm(x) = \lim_{\varepsilon\to 0+} g(x\pm i\varepsilon)$ are its two analytic continuations to positive real $x \in (0,b)$. Moreover, suppose that the function f has a reflection symmetry

$$f(w) = -f(2a-w) \quad (a \in \mathbb{C}). \tag{4.12}$$

Then $z \mapsto 2a - g(-z)$ is a root of $z=f(w)$ holomorphic in the right half plane, and from Eq. (4.11) we infer that

$$g_+(x) + g_-(x) + [2a - g(-x)] = c.$$

Setting $g_0(x) \equiv -g(-x)$ this becomes the same as Eq. (4.10) if

$$c - 2a = -1. \tag{4.13}$$

Thus we are inspired to interpret g_+, g_-, and $2a+g_0$ as the three roots of an equation $z=f(w)$. Given this interpretation, solving Eq. (4.10) amounts to finding the function f.

B. The good function *f* to consider

We are looking for a certain meromorphic function f on $\mathbb{C}$. By adding a point at infinity we can view such a function f as a mapping of the Riemann sphere $S^2=\mathbb{C}\cup\{\infty\}$ to itself. We want this mapping to have degree $r=3$; i.e., every regular point z of f is to have three distinct preimages

$$f^{-1}(z)=\{w_1(z),w_2(z),w_3(z)\}.$$

Such a mapping can be presented in the general form

$$f(\omega)=f_\infty+\sum_{i=1}^{3}\frac{b_i}{w-a_i} \tag{4.14}$$

with some complex numbers a_i, b_i, and f_∞.

Let us narrow down the choice of parameters. From the normalization condition (4.5) and the definition of $g(z)$ in Eq. (4.8), we have the limit $zg(z)\to-1$ for $z\to\infty$. Therefore, since $f[g(z)]=z$ by construction, we need $f(w)$ to have a pole at $w=g(\infty)=0$ with residue -1. So we choose $a_1=0$ and $b_1=-1$. The reflection symmetry (4.12) is then implemented by setting $f_\infty=0$, $b_3=b_1$, and $a_i=(i-1)\,a$ for $i=1,2,3$ and some $a\in\mathbb{C}$. Thus

$$f(w)=-\frac{1}{w}+\frac{b_2}{w-a}-\frac{1}{w-2a},$$

where the parameters a and b_2 are still unknown.

Next observe that for a degree-r holomorphic mapping f: $S^2\to S^2$, the number of singular points, where $f'(w)=0$, is $2r-2$. Indeed, writing f as $f(w)=p(w)/q(w)$ where p and q are polynomials of degree r, one has

$$f'(w)=\frac{p'(w)q(w)-p(w)q'(w)}{q(w)^2},$$

the numerator of which is a polynomial of degree $2r-2$ and so has $2r-2$ zeros.

Thus we should expect our function (4.14) to have $2\times3-2=4$ singular points. The reflection symmetry (4.12) makes for their images $\{f(w)\in\mathbb{C}\,|\,f'(w)=0\}$ to be arranged symmetrically around $z=0$. Now notice that our Green's function $g(z)$, being the Stieltjes transform of $\rho_\infty(x)$ with support $[0,b]$, must have singularities at $z=0$ and $z=b$. The image of the singular set had better contain these values, and thus is determined to be $\{-b,0,+b\}$ by reflection symmetry. Actually, since our situation calls for f to have four singular points, the singularity at $z=0$ (corresponding to $w=\infty$) must have multiplicity two. This is achieved by choosing $b_2=-b_1-b_3=+2$, so that

$$f(w)=-\frac{1}{w}+\frac{2}{w-a}-\frac{1}{w-2a}=\frac{-2a^2}{w(w-a)(w-2a)},$$

resulting in the behavior $f(w)\sim w^{-3}$ for $w\to\infty$. The singular points of f now are $w=a\pm a/\sqrt{3}$, and ∞. These correspond to $z=f(w)=\pm3\sqrt{3}/a$, and 0, respectively, so we infer

$$b=3\sqrt{3}/|a|. \tag{4.15}$$

It remains to pin down the last unknown parameter a. For that purpose, recall that the sum of the roots $f^{-1}(z)=\{w_1(z),w_2(z),w_3(z)\}$ is a constant, c, independent of z. To determine this constant, look at $\Sigma w_i(\infty)$ and use that the poles of f are at $w=0$, a, $2a$ to obtain

$$c=\Sigma w_i(z)=\Sigma w_i(\infty)=3a. \tag{4.16}$$

We then conclude $a=-1$ from Eq. (4.13), and $b=3\sqrt{3}$ from Eq. (4.15). In summary, the good meromorphic function f for us to consider is

$$w \mapsto f(w) = \frac{-2}{w(w+1)(w+2)}. \tag{4.17}$$

Let us mention in passing that the idea to consider the equation $z = f(w)$ or, equivalently

$$w(w+1)(w+2) + 2/z = 0,$$

first came to one of us (H.-J.S.) from previous work[10] on the Green's function of the Bures measure, whose large-N limit leads to a similar equation.

C. Solution of the problem

The situation can now be succinctly described like this: thinking of

$$W := \mathbb{C} \setminus \{-1 + 1/\sqrt{3}, -1 - 1/\sqrt{3}\}, \quad Z := (\mathbb{C} \setminus \{b, 0, -b\}) \cup \{\infty\} \quad (b = 3\sqrt{3}),$$

as two Riemann surfaces W and Z, the function f of Eq. (4.17) gives us a holomorphic cover

$$f: W \to Z, \quad f^{-1}(z) = \{w_1(z), w_2(z), w_3(z)\}.$$

What is the monodromy of this cover, i.e., what happens when the locally defined functions $z \mapsto w_i(z)$ are analytically continued around one of the singular points $z = b, 0, -b$? At the point $z = 0$ (or $w = \infty$) we have a cubic singularity $z \sim w^{-3}$. Consequently, the monodromy at $z = 0$ cyclically permutes the roots $w_i(z)$. Turning to $z = \pm b$, we see that linearization $z = \pm b + \delta z$ and $w = f^{-1}(\pm b) + \delta w$ gives

$$\delta z \sim (\delta w)^2.$$

In the latter two cases the monodromy must exchange two of the $w_i(z)$ while leaving the third one invariant.

Now focus on the situation near the singular point $z = -b$ and denote by $w(z) \equiv g(z)$ the root which, there, is trivial under monodromy and, hence, exists as a holomorphic function in some neighborhood of $z = -b$. With the remaining two singularities being at $z = 0$ and $z = b$, the function $g(z)$ actually extends to a holomorphic function on the Riemann sphere $\mathbb{C} \cup \{\infty\}$ cut along, say, $[0, b] \subset \mathbb{R}$. Let us verify that this holomorphic function $g: (\mathbb{C} \setminus [0, b]) \cup \{\infty\} \to W$ coincides with the Green's function (4.8) solving our problem (4.10).

By the holomorphic nature of g and Cauchy's theorem, we have that

$$g(z) = \frac{1}{2\pi i} \oint_\gamma \frac{g(z')dz'}{z' - z},$$

where γ is a small loop running around z in the counterclockwise sense. Since g is holomorphic at infinity, the loop γ can be deformed (through infinity) to a loop encircling the cut $[0, b]$, but now with the orientation reversed. Collapse the deformed loop to the two line segments connecting 0 with b. Then, setting $g_\pm(x) = \lim_{\varepsilon \to 0+} g(x \pm i\varepsilon)$ and

$$\rho_\infty(x) := \frac{g_+(x) - g_-(x)}{2\pi i} \quad (0 < x < b), \tag{4.18}$$

$g(z)$ is obviously given by the integral in Eq. (4.8).

Because $g_+(x)$ and $g_-(x)$ arise by analytic continuation from $g(z) \in f^{-1}(z)$, these are two of the three elements in the set $f^{-1}(x)$. How is the third element of $f^{-1}(x)$ related to $g(z)$? To see that, recall $a = -1$ and from Eq. (4.12) the invariance of the equation $z = f(w)$ under $(z, w) \mapsto (-z, 2a - w)$. Thus, if $g(-z)$ is a root over $-z$, then $-2 - g(-z)$ is a root over z, and it follows that $g_0(x) - 2$ with $g_0(x) := -g(-x)$ (for $0 < x < b$) is a root over x. The roots $g_+(x)$, $g_-(x)$, and $g_0(x) - 2$ all are different as functions. In fact, $\mathfrak{Im}\, g_+(x) > 0 = \mathfrak{Im}\, g_0(x) > \mathfrak{Im}\, g_-(x)$ for $0 < x < b$. So,

$$f^{-1}(x) = \{g_+(x), g_-(x), g_0(x) - 2\},$$

and from Eq. (4.16) we deduce that

$$g_+(x) + g_-(x) + g_0(x) - 2 = \sum_{w \in f^{-1}(x)} w = c = 3a = -3 \quad (0 < x < b),$$

which agrees with Eq. (4.10). Recall that in order for our analysis to work out we must choose

$$b = 3\sqrt{3}. \tag{4.19}$$

With a full understanding of the situation in hand, it is now an easy exercise to obtain $\rho_\infty(x)$ in explicit form. Solving the equation $z=f(w)$ one finds the holomorphic function $g(z)$ in the interval $-b<x<0$ to be

$$g(x) = (-x)^{-1/3}(1 + \sqrt{1 - x^2/b^2})^{1/3} + (-x)^{-1/3}(1 - \sqrt{1 - x^2/b^2})^{1/3} - 1,$$

where all square roots and cubic roots are understood to be positive. This function indeed extends holomorphically to a neighborhood of $x=-b$, as the Taylor expansion at $x=-b$ contains only even powers of $\sqrt{1-x^2/b^2}$. Analytic continuation around the singularity at $z=0$ gives

$$g_\pm(x) = x^{-1/3}e^{\pm\pi i/3}(1 + \sqrt{1 - x^2/b^2})^{1/3} + x^{-1/3}e^{\mp\pi i/3}(1 - \sqrt{1 - x^2/b^2})^{1/3} - 1 \quad (0 < x \le b).$$

Computing the difference (4.18) we then get the result for $\rho_\infty(x)$ claimed in Eq. (4.6), with the value for b given by Eq. (4.19).

As a final remark, let us note that the good form of $g(z)$ to use near infinity is

$$g(z) = -1 + e^{i\pi/6}\left(\sqrt{\frac{1}{b^2} - \frac{1}{z^2}} + \frac{i}{z}\right)^{1/3} + e^{-i\pi/6}\left(\sqrt{\frac{1}{b^2} - \frac{1}{z^2}} - \frac{i}{z}\right)^{1/3}. \tag{4.20}$$

From this, all moments of $\rho_\infty(x)dx$ can be found by expanding $g(z)$ in powers of $1/z$.

V. EXACT SOLUTION USING BIORTHOGONAL POLYNOMIALS

We now express the probability measure (3.18) as

$$d\mu_{N,l}(\omega_1, \ldots, \omega_N) = c_{N,l}(\tau)\prod_{i<j}(\omega_i - \omega_j)(\omega_i^2 - \omega_j^2)\prod_{k=1}^{N} e^{-\tau\omega_k}\omega_k^l d\omega_k, \tag{5.1}$$

and embark on another approach to handling it.

To get started, recall the formula for the Vandermonde determinant

$$\prod_{i>j}(\omega_i - \omega_j) = \mathrm{Det}(\omega_j^{i-1})_{i,j=1,\ldots,N} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \omega_1 & \omega_2 & \cdots & \omega_N \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^{N-1} & \omega_2^{N-1} & \cdots & \omega_N^{N-1} \end{vmatrix}.$$

Using it we reorganize the probability measure (5.1) as

$$d\mu_{N,l}(\omega_1, \ldots, \omega_N) = c_{N,l}(\tau)\mathrm{Det}(\omega_j^{i-1})\mathrm{Det}(\omega_j^{2i-2})\prod_{k=1}^{N} e^{-\tau\omega_k}\omega_k^l d\omega_k. \tag{5.2}$$

We also simplify our notation by setting $\tau=1$.

By standard properties of the determinant, $\mathrm{Det}(\omega_j^{i-1})$ changes only by a multiplicative constant when the monomials ω_j^{i-1} are replaced by any polynomials in ω_j of degree $i-1$. We have two

Vandermonde determinants, $\Pi_{i<j}(\omega_i-\omega_j)$ and $\Pi_{i<j}(\omega_i^2-\omega_j^2)$, so we introduce two sets of polynomials, denoting those of the first set by $P_{i-1}(\omega_j)$ and those of the second one by $Q_{i-1}(\omega_j^2)$. Our measure then becomes

$$d\mu_{N,l}(\omega_1,\ldots,\omega_N)=\tilde{c}_{N,l}\mathrm{Det}[P_{i-1}(\omega_j)]\mathrm{Det}[Q_{i-1}(\omega_j^2)]\prod_{k=1}^{N}e^{-\omega_k}\omega_k^l d\omega_k. \tag{5.3}$$

In order for the introduction of the polynomials $P_n(\omega)$ and $Q_n(\omega^2)$ to be useful we require them to be orthogonal with respect to the integration measure $e^{-\omega}\omega^l d\omega$:

$$I_{m,n}\equiv\int_0^\infty P_m(\omega)Q_n(\omega^2)e^{-\omega}\omega^l d\omega=h_n\delta_{m,n}, \tag{5.4}$$

where the numbers $h_n=I_{n,n}$ depend on the choice of normalization for $P_n(\omega)$ and $Q_n(\omega^2)$. Such polynomials are constructed by a variant of the Gram–Schmidt algorithm, as follows.

A. Biorthogonal polynomials

We review the construction in the general setting of two real vector spaces V, W with a pairing (or nondegenerate bilinear form)

$$\gamma: V\times W\to\mathbb{R}.$$

Given some basis $v_0,v_1,v_2,\ldots$ of V, and a basis $w_0,w_1,w_2,\ldots$ of W, let the entries of the corresponding pairing matrix be denoted by

$$\gamma_{m,n}:=\gamma(v_m,w_n)\quad(m,n=0,1,2,\ldots).$$

The goal now is to construct a new basis $e_0,e_1,e_2,\ldots$ of V, and a new basis $f_0,f_1,f_2,\ldots$ of W such that

$$e_n=v_n+\sum_{n'=0}^{n-1}A_{nn'}v_{n'},\quad f_n=w_n+\sum_{n'=0}^{n-1}B_{nn'}w_{n'},$$

(with real coefficients $A_{nn'}$ and $B_{nn'}$), and the transformed basis vectors form a biorthogonal system

$$\gamma(e_m,f_n)=0\quad m\neq n.$$

This problem has a unique solution by the process of Gram–Schmidt orthogonalization. A nice way of presenting the solution is by means of the following determinants (where, by a slight abuse of notation, the matrix entries in the last column resp. last row are vectors, whereas all of the other matrix entries are numbers)

$$e_n=D_{n-1}^{-1}\begin{vmatrix}\gamma_{0,0} & \cdots & \gamma_{0,n-1} & v_0\\ \gamma_{1,0} & \cdots & \gamma_{1,n-1} & v_1\\ \vdots & \ddots & \vdots & \vdots\\ \gamma_{n,0} & \cdots & \gamma_{n,n-1} & v_n\end{vmatrix},\quad f_n=D_{n-1}^{-1}\begin{vmatrix}\gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n}\\ \vdots & \vdots & \ddots & \vdots\\ \gamma_{n-1,0} & \gamma_{n-1,1} & \cdots & \gamma_{n-1,n}\\ w_0 & w_1 & \cdots & w_n\end{vmatrix}\quad(n=1,2,\ldots), \tag{5.5}$$

with normalization factor

$$D_n=\begin{vmatrix}\gamma_{0,0} & \cdots & \gamma_{0,n}\\ \vdots & \ddots & \vdots\\ \gamma_{n,0} & \cdots & \gamma_{n,n}\end{vmatrix}.$$

These formulas are easily verified. Indeed, pairing e_m with w_n for $m>n$ one gets

$$\gamma(e_m,w_n) = D_{m-1}^{-1}\begin{vmatrix} \gamma_{0,0} & \cdots & \gamma_{0,m-1} & \gamma_{0,n} \\ \gamma_{1,0} & \cdots & \gamma_{1,m-1} & \gamma_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ \gamma_{m,0} & \cdots & \gamma_{m,m-1} & \gamma_{m,n} \end{vmatrix} = 0,$$

which vanishes because the last column coincides with one of the other columns. Since f_n is a linear combination of the vectors $w_{n'}$ with $n' \leqslant n$, it follows that $\gamma(e_m,f_n)=0$ for $m>n$. The same conclusion for $m<n$ follows by reversing the roles of V and W. Notice that $e_n = 1v_n + \ldots$ and $f_n = 1w_n + \ldots$ by insertion of the factor D_{n-1}^{-1}. The nonvanishing pairing matrix elements for $n \geqslant 1$ are $\gamma_{n,n} = \gamma(e_n,f_n) = \gamma(e_n,w_n) = D_n/D_{n-1}$.

To apply these general formulas to the case under consideration, we choose the vectors v_m and w_n to be the functions $\omega \mapsto \omega^m$ resp. $\omega \mapsto \omega^{2n}$, and take the pairing to be given by integration with our measure $e^{-\omega}\omega^l d\omega$:

$$\gamma_{m,n} = \int_0^\infty \omega^{m+2n} e^{-\omega}\omega^l d\omega = \Gamma(m+2n+l+1). \tag{5.6}$$

Making the identification $e_n \equiv P_n(\omega)$, the general formula for e_n in Eq. (5.5) then gives $P_0(\omega) = 1$ and

$$P_n(\omega) = D_{n-1}^{-1}\begin{vmatrix} \Gamma(l+1) & \cdots & \Gamma(l+2n-1) & \omega^0 \\ \Gamma(l+2) & \cdots & \Gamma(l+2n) & \omega^1 \\ \vdots & \ddots & \vdots & \vdots \\ \Gamma(l+n+1) & \cdots & \Gamma(l+3n-1) & \omega^n \end{vmatrix} \quad (n \geqslant 1). \tag{5.7}$$

Similarly, identifying $f_n \equiv Q_n(\omega^2)$ we obtain $Q_0(\omega^2) = 1$ and

$$Q_n(\omega^2) = D_{n-1}^{-1}\begin{vmatrix} \Gamma(l+1) & \Gamma(l+3) & \cdots & \Gamma(l+2n+1) \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma(l+n) & \Gamma(l+n+2) & \cdots & \Gamma(l+3n) \\ \omega^0 & \omega^2 & \cdots & \omega^{2n} \end{vmatrix} \quad (n \geqslant 1). \tag{5.8}$$

Using the relation $\Gamma(z+1) = z\Gamma(z)$ an easy Gauss elimination process gives the normalization constant as

$$D_n = \begin{vmatrix} \Gamma(l+1) & \cdots & \Gamma(l+2n+1) \\ \vdots & \ddots & \vdots \\ \Gamma(l+n+1) & \cdots & \Gamma(l+3n+1) \end{vmatrix} = \prod_{k=0}^{n} 2^k k!(2k+1)!. \tag{5.9}$$

From this, note the diagonal pairing matrix elements $h_0 = l!$ and

$$\int_0^\infty P_n(\omega)Q_n(\omega^2)e^{-\omega}\omega^l d\omega = h_n = D_n/D_{n-1} = 2^n n!(2n+l)! \quad (n \geqslant 1). \tag{5.10}$$

B. *n*-level correlation functions

The n-level correlation function $R_n(\omega_1,\ldots,\omega_n)$ in the present context is defined as

$$R_n(\omega_1,\omega_2,\ldots,\omega_n) = n!\int_{\mathbb{R}_+^N}\sum_{i_1<i_2<\cdots<i_n}\delta(\omega_1-\tilde{\omega}_{i_1})\delta(\omega_2-\tilde{\omega}_{i_2})\cdots\delta(\omega_n-\tilde{\omega}_{i_n})d\mu_{N,l}(\tilde{\omega}_1,\ldots,\tilde{\omega}_N). \tag{5.11}$$

A closed-form expression for it can be given from the biorthogonal polynomials $P_{n'}(\omega)$ and $Q_{n'}(\omega^2)$ for $0\leq n'\leq N$. The result will take its most succinct form when expressed in terms of the modified functions

$$\tilde{P}_n(\omega) := (-2)^{-n}n!^{-1}e^{-\omega}P_n(\omega), \tag{5.12}$$

$$\tilde{Q}_n(\omega) := (-1)^n(2n+l)!^{-1}\omega^l Q_n(\omega^2), \tag{5.13}$$

[the motivation for the sign $(-1)^n$ will become clear later], which from Eqs. (5.4) and (5.10) obey the orthogonality relations

$$\int_0^\infty \tilde{P}_m(\omega)\tilde{Q}_n(\omega)d\omega = \delta_{m,n}. \tag{5.14}$$

The probability measure (5.3) expressed by these functions takes the form

$$d\mu_{N,l}(\omega_1,\ldots,\omega_N) = \frac{1}{N!}\mathrm{Det}[\tilde{P}_{i-1}(\omega_j)]\mathrm{Det}[\tilde{Q}_{i-1}(\omega_j)]\prod_k d\omega_k.$$

Now, by using the multiplicative property of the determinant, we can also write

$$d\mu_{N,l}(\omega_1,\ldots,\omega_N) = \frac{1}{N!}\mathrm{Det}[K_N(\omega_i,\omega_j)]_{i,j=1,\ldots,N}\prod_k d\omega_k, \tag{5.15}$$

where the kernel $K(\omega_i,\omega_j)$ is defined by

$$K_N(\omega_i,\omega_j) = \sum_{n=0}^{N-1}\tilde{P}_n(\omega_i)\tilde{Q}_n(\omega_j). \tag{5.16}$$

From the orthogonality relations (5.14) this kernel has the reproducing property

$$\int_0^\infty K_N(\omega_i,\omega)K_N(\omega,\omega_j)d\omega = K_N(\omega_i,\omega_j), \tag{5.17}$$

and the trace

$$\int_0^\infty K_N(\omega,\omega)d\omega = N. \tag{5.18}$$

To proceed further, take notice of the relation

$$\int_0^\infty \begin{vmatrix} K_N(\omega_1,\omega_1) & \cdots & K_N(\omega_1,\omega_n) \\ \vdots & \ddots & \vdots \\ K_N(\omega_n,\omega_1) & \cdots & K_N(\omega_n,\omega_n)\end{vmatrix} d\omega_n = (N-n+1)\begin{vmatrix} K_N(\omega_1,\omega_1) & \cdots & K_N(\omega_1,\omega_{n-1}) \\ \vdots & \ddots & \vdots \\ K_N(\omega_{n-1},\omega_1) & \cdots & K_N(\omega_{n-1},\omega_{n-1})\end{vmatrix},$$

which is proved by expanding the determinant with respect to the last row or column and exploiting the properties (5.17) and (5.18). Using it, an inductive procedure starting from

$R_N(\omega_1,\ldots,\omega_N)=\mathrm{Det}[K_N(\omega_i,\omega_j)]_{i,j=1,\ldots,N}$ gives the n-level correlation functions as

$$R_n(\omega_1,\ldots,\omega_n)=\mathrm{Det}[K_N(\omega_i,\omega_j)]_{i,j=1,\ldots,n}. \tag{5.19}$$

Thus the correlations are those of a determinantal process and are completely determined by the kernel $K_N(\omega_i,\omega_j)$. The remaining discussion therefore focuses on this kernel, but first we make another preparatory step.

C. Contour integral representation

We are now going to show that the functions $\tilde{P}_n(\omega)$ and $\tilde{Q}_n(\omega)$ have expressions as complex contour integrals

$$\tilde{P}_n(\omega)=\oint_{S_\epsilon(1)} e^{-\omega u}(1-u^{-2})^{-n-1}u^{l-2}du/\pi i, \tag{5.20}$$

$$\tilde{Q}_n(\omega)=\oint_{S_\epsilon(0)} e^{\omega v}(1-v^{-2})^{n}u^{-l-1}dv/2\pi i. \tag{5.21}$$

Both integrals are over circles in the complex plane with radius ϵ and counterclockwise orientation; the first circle is centered at $u=1$ and has radius $\epsilon<2$ (to avoid the singularity at $u=-1$), the second one is centered at $v=0$.

Our proof of these expressions for $\tilde{P}_n(\omega)$ and $\tilde{Q}_n(\omega)$ will be indirect and in two steps. First, we establish some information on power series. In the case of $\tilde{Q}_n(\omega)$ we insert the power series of the exponential function $\mathrm{e}^{\omega v}$, use the binomial expansion of $(1-v^{-2})^n$, and compute a residue to obtain

$$\tilde{Q}_n(\omega)=\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k\omega^{2k+l}}{(2k+l)!}. \tag{5.22}$$

In the case of $\tilde{P}_n(\omega)$, calculating the residue at $u=1$ we have that

$$\tilde{P}_n(\omega)=\left.\frac{2}{n!}\frac{d^n}{du^n}\left[\frac{e^{-\omega u}u^{n+l-1}}{(1+u^{-1})^{n+1}}\right]\right|_{u=1}. \tag{5.23}$$

In both cases, defining $P_n(\omega)$ and $Q_n(\omega^2)$ by the reverse of the relations (5.12) and (5.13), we see from Eqs. (5.22) and (5.23) that these are polynomials of degree n in ω resp. ω^2 and that the highest-degree term (ω^n resp. ω^{2n}) has coefficient one.

Recall now from Sec. V A that, given these properties, the polynomials $P_n(\omega)$ and $Q_n(\omega^2)$ are completely determined by the orthogonality relations (5.4) for $m\neq n$. Via Eqs. (5.12) and (5.13) the latter are in one-to-one correspondence with the orthogonality relations (5.14) (still for $m\neq n$). Therefore, defining

$$\tilde{I}_{m,n}=\int_0^\infty \tilde{P}_m(\omega)\tilde{Q}_n(\omega)d\omega, \tag{5.24}$$

the second and final step of our proof is to show that $\tilde{I}_{m,n}=0$ for $m\neq n$.

To that end, we insert the expressions (5.20) and (5.21) into Eq. (5.24). The ω dependence then is $e^{-\omega(u-v)}$ with $u\in S_\epsilon(1)$ and $v\in S_\epsilon(0)$. Taking the radius ϵ to be very small ($\epsilon\ll 1$), we have that $e^{-\omega(u-v)}$ decreases rapidly as ω goes to $+\infty$. Therefore, the integral over ω exists, and we may interchange the order of integrations. Doing first the ω integral

$$\int_0^\infty e^{-\omega(u-v)}d\omega = \frac{1}{u-v},$$

the remaining contour integrals for $\widetilde{I}_{m,n}$ defined by Eq. (5.24) are

$$\widetilde{I}_{m,n} = \oint_{S_\epsilon(1)} \frac{u^{l-2}}{(1-u^{-2})^{m+1}}\left[\oint_{S_\epsilon(0)} \frac{(1-v^{-2})^n dv}{v^{l+1}(v-u)}\right]\frac{du}{2\pi^2}.$$

To simplify the inner integral over v we use the identity

$$\left(\frac{1-v^{-2}}{1-u^{-2}}\right)^n = 1 - \frac{v^2-u^2}{v^2(1-u^2)}\sum_{k=0}^{n-1}\left(\frac{1-v^{-2}}{1-u^{-2}}\right)^k.$$

Inserting this into the expression for $\widetilde{I}_{m,n}$ we see that the terms in the k sum do not contribute as the residue at $v=0$ vanishes for all of those terms. Doing the v integral for the first term on the right-hand side, we get

$$\oint_{S_\epsilon(0)} v^{-l-1}(v-u)^{-1}dv = -2\pi i u^{-l-1},$$

so the remaining u integral is

$$\widetilde{I}_{m,n} = (\pi i)^{-1}\oint_{S_\epsilon(1)} (1-u^{-2})^{n-m-1}u^{-3}du.$$

This integrand is holomorphic near $u=1$ for $m<n$, and the integral therefore vanishes in that case. For $m>n$ we use the invariance of the integration form under $u\to -u$ to write $I_{m,n}$ as an integral over a sum of two circles

$$\widetilde{I}_{m,n} = \frac{1}{2\pi i}\oint_\gamma \frac{u^{2m-2n-1}du}{(u^2-1)^{m-n+1}}, \quad \gamma = S_\epsilon(1) + S_\epsilon(-1).$$

The integrand in this case is holomorphic near $u=0$. In the punctured plane $\mathbb{C}\setminus\{1,-1\}$ the chain $S_\epsilon(1)+S_\epsilon(-1)$ is homologous to the circle at infinity, where the integrand vanishes. Therefore the integral again is zero. This proves that $\widetilde{I}_{m,n}=0$ for $m\neq n$, which in turn completes our proof that the contour integrals (5.20) and (5.21) are the same as the functions $\widetilde{P}_n(\omega)$, $\widetilde{Q}_n(\omega)$ defined from Eqs. (5.7) and (5.8) by Eqs. (5.12) and (5.13). As a final check, note that

$$\widetilde{I}_{n,n} = (\pi i)^{-1}\oint_{S_\epsilon(1)} \frac{du}{u(u^2-1)} = 1,$$

which is what it ought to be in view of Eq. (5.14).

Now we harvest a major benefit from the contour integral representations (5.20) and (5.21): using these, we can carry out the sum in the definition (5.16) of the kernel K_N as a geometric sum. The result is a double contour integral

$$K_N(\omega_1,\omega_2) = \oint_{S_\epsilon(1)} du \oint_{S_\epsilon(0)} dv F_N(u,v;\omega_1,\omega_2), \tag{5.25}$$

$$F_N(u,v;\omega_1,\omega_2) = \frac{1}{2\pi^2} e^{-\omega_1 u+\omega_2 v} \frac{u^l v^{-l+1}}{u^2-v^2}\left[\left(\frac{1-v^{-2}}{1-u^{-2}}\right)^N - 1\right]. \tag{5.26}$$

This exact expression represents the complete solution of our problem. We will now use it to determine the large-N asymptotics in the bulk and at the hard edge $\omega=0$.

D. Asymptotics in the bulk

The kernel on the diagonal $\omega_1=\omega_2$ is the same as the one-level function, $R_1(\omega)=K_N(\omega,\omega)$; see Eq. (5.19). We already know from Sec. IV the asymptotics of $R_1(\omega)\equiv\rho(\omega)$ in the bulk: introducing the scaling variable $x=\omega/N$ (formerly $x=\omega\tau/N$), this is

$$\lim_{N\to\infty} K_N(Nx,Nx) = \rho_\infty(x),$$

with $\rho_\infty(x)$ given by Eq. (4.6). In the present subsection we are going to demonstrate that the scaling limit of the kernel $K_N(\omega_1,\omega_2)$ off the diagonal leads to sine-kernel universality for all level correlation functions:

$$\lim_{N\to\infty} R_n(Nx+\omega_1,\ldots,Nx+\omega_n) = \mathrm{Det}\left\{\frac{\sin[\pi\rho_\infty(x)(\omega_i-\omega_j)]}{\pi(\omega_i-\omega_j)}\right\}_{i,j=1,\ldots,n}, \tag{5.27}$$

as is expected for systems in the universality class of the Gaussian Unitary Ensemble. As a corollary, we will obtain an independent confirmation of the result (4.6).

Looking at the integral representation (5.25) one might think that the large-N limit could be taken by applying the saddle-point method to that integral. However, as we shall see, the dominant saddle points lie on the line $u=v$ where the integrand has a singularity of type $0/0$ which, albeit removable, complicates the saddle-point evaluation.

Therefore, rather than calculating $K_N(\omega_1,\omega_2)$ directly, we look at the product $(\omega_1-\omega_2)K_N(\omega_1,\omega_2)$. Using the relation $(\omega_2-\omega_1)\mathrm{e}^{\omega_2 v-\omega_1 u}=(\partial_v+\partial_u)\mathrm{e}^{\omega_2 v-\omega_1 u}$ and partially integrating, we rewrite Eq. (5.25) as

$$(\omega_1-\omega_2)K_N(\omega_1,\omega_2) = \oint_{S_\epsilon(1)} du \oint_{S_\epsilon(0)} dv \tilde{F}_N(u,v;\omega_1,\omega_2), \tag{5.28}$$

$$\tilde{F}_N(u,v;\omega_1,\omega_2) = \frac{1}{2\pi^2} e^{-\omega_1 u+\omega_2 v}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial u}\right)\frac{u^l v^{-l+1}}{u^2-v^2}\left[\left(\frac{1-v^{-2}}{1-u^{-2}}\right)^N - 1\right], \tag{5.29}$$

which constitutes the starting point for the following analysis.

In preparation for taking the limit $N\to\infty$, we set $\omega_1=Nx+\omega$ and $\omega_2=Nx+\tilde{\omega}$. The deciding factor in the integrand of $(\omega_1-\omega_2)K_N(\omega_1,\omega_2)$ in the large-N limit will then be

$$\exp[-Nx(u-v)+N\log(1-v^{-2})-N\log(1-u^{-2})],$$

leading to the saddle-point equation

$$\varphi(u) = x = \varphi(v), \quad \varphi(w) = -\partial_w \log(1-w^{-2}) = \frac{-2}{w(w^2-1)}. \tag{5.30}$$

Notice that φ is related to our function (4.17) by $f(w-1)=\varphi(w)$. A comprehensive study of the equation $f(w)=z$ and its solutions for w was made in Sec. IV. From there we know that the saddle-point equation $\varphi(w)=x$ has three solutions in general, and for $0<x\leq b=3\sqrt{3}$ these are

$$w_\alpha(x) = -x^{-1/3}e^{-2\pi i\alpha/3}(1+\sqrt{1-x^2/b^2})^{1/3} - x^{-1/3}e^{2\pi i\alpha/3}(1-\sqrt{1-x^2/b^2})^{1/3} \quad (\alpha = 1,-1,0). \tag{5.31}$$

In the range of interest $(0<x<b)$ the first two solutions, $w_{\pm 1}(x)$, are complex conjugates of each other while the third one, $w_0(x)$, is negative. Expanding the logarithm of $(1-v^{-2})^N/(1-u^{-2})^N$ to second order around a pair of saddle points w_α, w_β we encounter the Gaussian

$$\exp\left\{\frac{1}{2}N\varphi'[w_\alpha(x)](\delta u)^2 - \frac{1}{2}N\varphi'[w_\beta(x)](\delta v)^2\right\}, \quad \varphi'(x) = \frac{6w^2-2}{w^2(w^2-1)^2}.$$

For the negative saddle point one has $\varphi'[w_0(x)]>0$, so its path of steepest descent would be perpendicular to the real axis in the case of u and along the real axis in the case of v. The latter is inconsistent with the original integration contour for v being $S_\epsilon(0)$. In the former case, $w_0(x)<-1$ is inaccessible because of the singularity of $(1-u^{-2})^{-N}$ intervening at $u=-1$. Thus this saddle point is irrelevant for present purposes and may be discarded.

We now make another preparation of the saddle-point evaluation of the integral, by investigating the behavior of the integrand near the two remaining saddle points. We set

$$u = w_\alpha(x) + N^{-1/2}\delta u, \quad v = w_\beta(x) + N^{-1/2}\delta v \quad (\alpha,\beta = \pm 1),$$

and first look at the diagonal case, $\alpha=\beta$. Using the identity

$$\frac{1}{u-v}\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{1-u^{-2}}{1-u^{-2}}\right)^N = 2N\frac{u^2+uv+v^2-1}{u(u^2-1)v(v^2-1)}\left(\frac{1-v^{-2}}{1-u^{-2}}\right)^N, \tag{5.32}$$

we find the scaling limit of the integrand $\widetilde{F}_N$ to be

$$\lim_{N\to\infty} N^{-1}\widetilde{F}_N[w_\alpha(x)+N^{-1/2}\delta u, w_\alpha(x)+N^{-1/2}\delta v; Nx+\omega, Nx+\widetilde{\omega}]$$
$$= (2\pi)^{-2}e^{-w_\alpha(x)(\omega-\widetilde{\omega})}\varphi'[w_\alpha(x)]e^{(1/2)\varphi'[w_\alpha(x)](\delta u^2-\delta v^2)}.$$

The same limit in the off-diagonal case $(\alpha\neq\beta)$ vanishes. Indeed,

$$w_\alpha^2 + w_\alpha w_\beta + w_\beta^2 - 1 = \frac{w_\alpha(w_\alpha^2-1)-w_\beta(w_\beta^2-1)}{w_\alpha - w_\beta} = \frac{-2}{w_\alpha-w_\beta}(x^{-1}-x^{-1}) = 0 \quad (\alpha\neq\beta),$$

and therefore the factor in the numerator on the right-hand side of Eq. (5.32) gives zero.

We now deform the contours of integration as indicated in Fig. 2. The deformed contours pass through the saddle points $w_{\pm 1}$ but miss the saddle point w_0. At $w_{\pm 1}$ the paths of steepest descent for u and v cross at right angles, valleys in one case being mountains in the other case and vice versa.

Next we do the Gaussian integrals. Given the counterclockwise orientations of the original contours $S_\epsilon(1)$ resp. $S_\epsilon(0)$, and taking into account the directions of the paths of steepest descent, we get

$$\int e^{(1/2)\varphi'(w_{+1})\delta u^2}d(\delta u) = \sqrt{2\pi}|\varphi'(w_{+1})|^{-1/2}e^{(1/2)[-\pi i - i\arg\varphi'(w_{+1})]},$$

$$\int e^{-(1/2)\varphi'(w_{+1})\delta v^2}d(\delta v) = \sqrt{2\pi}|\varphi'(w_{+1})|^{-1/2}e^{(1/2)[2\pi i - i\arg\varphi'(w_{+1})]}.$$

The product of these two integrals is $2\pi i/\varphi'(w_{+1})$. The same calculation for the other saddle w_{-1} gives $-2\pi i/\varphi'(w_{-1})$. Thus, putting the factors together and summing over the contributions from diagonal pairs of saddle points $(\alpha=\beta)$ we obtain

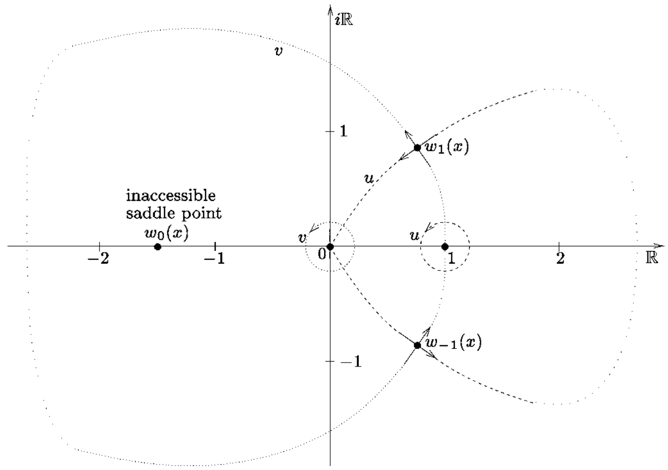

FIG. 2. Sketch of the saddle points for the case of $x=1$. By deforming the original contours, which are small circles around the singular points $u=1$ and $v=0$, one arranges for the contours of integration to pass through the saddle points $w_{\pm 1}(x)$ in the direction of steepest descent. Away from the saddle points the deformed contours are drawn as paths of constant phase, which interpolate between different zeroes of the integrand: they run between 0 and $+\infty$ for u, and between 1 and $-\infty$ for v.

$$(\omega-\tilde{\omega})\lim_{N\to\infty} K_N(Nx+\omega,Nx+\tilde{\omega}) = \frac{1}{2\pi i}\left(e^{-w_{-1}(x)(\omega-\tilde{\omega})} - e^{-w_{+1}(x)(\omega-\tilde{\omega})}\right).$$

Since $\mathfrak{Re}\,w_{+1}=\mathfrak{Re}\,w_{-1}$ and $\mathfrak{Im}\,w_{+1}=-\mathfrak{Im}\,w_{-1}$ this means that, with $\mathfrak{Im}\,w_{+1}(x)=:\pi\rho_\infty(x)$, we have

$$\lim_{N\to\infty} K_N(Nx+\omega,Nx+\tilde{\omega}) = e^{-\mathfrak{Re}\,w_{\pm 1}(x)(\omega-\tilde{\omega})}\frac{\sin[\pi\rho_\infty(x)(\omega-\tilde{\omega})]}{\pi(\omega-\tilde{\omega})}.$$

The exponential factor $\mathrm{e}^{-\mathfrak{Re}\,w_{\pm 1}(x)(\omega-\tilde{\omega})}$ drops out when forming the determinant on the right-hand side of Eq. (5.19). Thus we arrive at the universal sine-kernel (or GUE) correlation functions (5.27).

Setting $\omega=\tilde{\omega}$ notice the special result $\lim_{N\to\infty}K_N(Nx,Nx)=\rho_\infty(x)$. Since the kernel on the diagonal is none other than the one-level function, $K_N(\omega,\omega)=\rho(\omega)$, this gives another determination of the large-N level density ρ_∞. From Eq. (5.31) one sees that $\rho_\infty(x)=\pi^{-1}\mathfrak{Im}\,w_{+1}(x)$ agrees with our earlier result Eq. (4.6).

E. Asymptotics near $\omega=0$

At the lower edge $(\omega=0)$ of the spectrum, a new type of behavior is expected to emerge. This behavior, as we shall see presently, occurs on a scale $\omega\sim N^{-1/2}$.

To exhibit the scaling limit near $\omega=0$, it is best to send the integration variables u, v to their reciprocals, $u\to u^{-1}$ and $v\to v^{-1}$. Then $du\to -u^{-2}du$, $dv\to -v^{-2}dv$, and the integration contour for v has its radius inverted and orientation reversed, $S_\epsilon(0)\to -S_{1/\epsilon}(0)$. However, since the integrand is holomorphic in v on $\mathbb{C}\backslash\{0\}$ we may return to the original radius ϵ (or any other radius, for that matter). In the case of u we take the radius ϵ of $S_\epsilon(1)$ to be very small. Then inversion $u\to u^{-1}$ sends $S_\epsilon(1)$ to itself (or, in any case, to the same homology class on $\mathbb{C}\backslash\{1\}$), with no change of orientation. Altogether, then, carrying out the transformation $(u,v)\to(u^{-1},v^{-1})$ the integral representation (5.25) continues to hold true if we make the replacement

$$F_N(u,v;\omega_1,\omega_2)\to -u^{-2}v^{-2}F_N(u^{-1},v^{-1};\omega_1,\omega_2)=\frac{1}{2\pi^2}e^{-\omega_1/u+\omega_2/v}\frac{u^{-l}v^{l-1}}{u^2-v^2}\left[\left(\frac{1-v^2}{1-u^2}\right)^N-1\right].$$

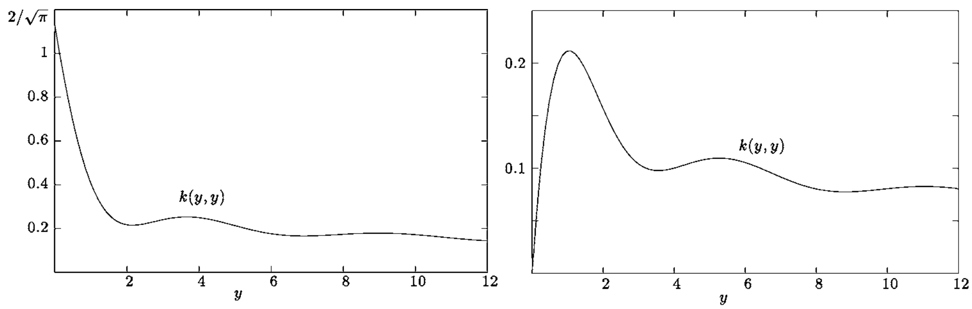

FIG. 3. The graph of the scaling function $k(y,y)$ for the case of $l=0$ (left) and $l=1$ (right).

Next, as another preparation for taking the limit $N\to\infty$, we deform the u-contour $S_\epsilon(1)$ to some axis parallel to the imaginary axis. The deformed contour crosses the real axis between $u=0$ and $u=1$ and is directed from $u=+i\infty$ to $u=-i\infty$. We also reverse the direction of integration for u and change the overall sign of the integral.

Then we set $\omega_j=N^{-1/2}y_j$ and rescale $u\to N^{-1/2}u$ and $v\to N^{-1/2}v$ accordingly. Again, in view of the analytic properties of the integrand we can keep the integration contours fixed while rescaling. Because the u integral converges at infinity we have a good limit

$$\lim_{N\to\infty}(1-u^2/N)^{-N}=\exp(u^2).$$

In total, we thus obtain the following scaling limit for our kernel K_N:

$$k(y_1,y_2) \coloneqq \lim_{N\to\infty} N^{-1/2}K_N(N^{-1/2}y_1,N^{-1/2}y_2)=\frac{1}{2\pi^2}\int_{i\mathbb{R}+\epsilon}du\int_{\mathrm{U}_1}\frac{dv}{v}e^{-y_1/u+y_2/v}(v/u)^l\frac{e^{u^2-v^2}-1}{u^2-v^2}, \tag{5.33}$$

where $\mathrm{U}_1\equiv S_1(0)$ means the unitary numbers, and $i\mathbb{R}+\epsilon$ is the imaginary axis translated by $\epsilon>0$ into the right half of the complex plane. Plots of the scaling function $k(y,y)$ for $l=0,1$ are shown in Fig. 3. Using the method of saddle-point evaluation as in Sec. V C one can show that this function behaves as $k(y,y)\sim y^{-1/3}$ for large y.

Taking the same scaling limit for the functions $\widetilde{P}_N(\omega)$ and $\widetilde{Q}_N(\omega)$ in Eqs. (5.20) and (5.21) one gets

$$p(y)=\lim_{N\to\infty}N^{-(l-1)/2}\widetilde{P}_N(N^{-1/2}y)=\frac{1}{\pi i}\int_{i\mathbb{R}+\epsilon}e^{u^2-y/u}u^{-l}du, \tag{5.34}$$

$$q(y)=\lim_{N\to\infty}N^{l/2}\widetilde{Q}_N(N^{-1/2}y)=\frac{1}{2\pi i}\int_{\mathrm{U}_1}e^{-v^2+y/v}v^{l-1}dv. \tag{5.35}$$

Both functions have convergent series expansions

$$p(y)=\sum_{n=0}^{\infty}\frac{(-y)^n}{n!\Gamma[(l+n+1)/2]},\quad q(y)=y^l\sum_{n=0}^{\infty}\frac{(-y^2)^n}{n!(2n+l)!}. \tag{5.36}$$

The expansion for $q(y)$ can be obtained either directly from Eq. (5.35), or by taking the limit $N\to\infty$ in Eq. (5.22). In the case of $p(y)$, the earlier formula (5.23) is not suitable; rather, in order to verify Eq.(5.36) for $p(y)$ one expands the integrand of Eq. (5.34) in powers of y, makes use of $\mathfrak{Re}\,u=\epsilon>0$ to write

$$u^{-n-l} = (n+l-1)!^{-1} \int_0^\infty e^{-tu} t^{n+l-1} dt \quad (n+l>0),$$

does the Gaussian u integral by completing the square, and uses the duplication formula for the Gamma function.

Note added in proof. After submission of this manuscript, P. Forrester pointed out to us that the joint eigenvalue distribution derived and analyzed here falls in a broad class of models solved by Borodin.[12] Borodin's expression for the kernel $K_N(x,y)$ is equivalent to ours by old work of Konhauser.[13] The mathematical results of Konhauser were first introduced into random matrix physics by Muttalib,[14] who suggested to use them for an approximate treatment of the statistics of transmission eigenvalues of disordered conductors.

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[1] F. J. Dyson, J. Math. Phys. **3**, 1199–1215 (1962).
[2] J. J. M. Verbaarschot, Phys. Rev. Lett. **72**, 2531–2533 (1994).
[3] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142–1161 (1997).
[4] A. Altland, B. D. Simons, and M. R. Zirnbauer, Phys. Rep. **359**, 283–354 (2002).
[5] P. Heinzner, A. Huckleberry, and M. R. Zirnbauer, Commun. Math. Phys. **257**, 725–771 (2005).
[6] M. Caselle and U. Magnea, Phys. Rep. **394**, 41–156 (2004).
[7] M. R. Zirnbauer, J. Math. Phys. **37**, 4986–5018 (1996).
[8] V. Gurarie and J. T. Chalker, Phys. Rev. Lett. **89**, 136801 (2002).
[9] V. Gurarie and J. T. Chalker, Phys. Rev. B **68**, 134207 (2003).
[10] H.-J. Sommers and K. Zyczkowski, J. Phys. A **37**, 8457–8466 (2004).
[11] M. L. Mehta, *Random Matrices* (Elsevier Academic, San Diego, 2004).
[12] A. Borodin, Nucl. Phys. B **536**, 704–732 (1998).
[13] J. D. E. Konhauser, Pac. J. Math. **21**, 303–314 (1967).
[14] K. A. Muttalib, J. Phys. A **28**, L159–L164 (1995).