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Superbosonization of Invariant Random Matrix Ensembles

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Abstract: 'Superbosonization' is a new variant of the method of commuting and anti-commuting variables as used in studying random matrix models of disordered and chaotic quantum systems. We here give a concise mathematical exposition of the key formulas of superbosonization. Conceived by analogy with the bosonization technique for Dirac fermions, the new method differs from the traditional one in that the superbosonization field is dual to the usual Hubbard-Stratonovich field. The present paper addresses invariant random matrix ensembles with symmetry group U_n, O_n, or USp_n, giving precise definitions and conditions of validity in each case. The method is illustrated at the example of Wegner's n-orbital model. Superbosonization promises to become a powerful tool for investigating the universality of spectral correlation functions for a broad class of random matrix ensembles of non-Gaussian and/or non-invariant type.

1. Introduction and Overview

The past 25 years have seen substantial progress in the physical understanding of insulating and metallic behavior in disordered quantum Hamiltonian systems of the random Schrödinger and random band matrix type. A major role in this development, bearing especially on the metallic regime and the metal-insulator transition, has been played by the method of commuting and anti-commuting variables, or supersymmetry method for short. Assuming a Gaussian distribution for the disorder, this method proceeds by making a Hubbard-Stratonovich transformation followed by a saddle-point approximation (or elimination of the massive modes) to arrive at an effective field theory of the non-linear sigma model type. This effective description has yielded many new results including, e.g., the level statistics in small metallic grains, localization in thick disordered wires, and a scaling theory of critical systems in higher dimension [6].

While the method has been widely used and successfully so, there exist some limitations and drawbacks. For one thing, the method works well only for systems with normal-distributed disorder; consequently, addressing the universality question for non-Gaussian distributions (like the invariant ensembles studied via the orthogonal polynomial method) has so far been beyond reach. For another, the symmetries of the effective theory are not easy to keep manifest when using the mathematically well-founded approach of Schäfer and Wegner [17]. A problem of lesser practical relevance is that the covariance matrix of the random variables, which is to be inverted by the Hubbard-Stratonovich transformation, does not always have an inverse [17].

In this paper we introduce a new variant of the supersymmetry approach which is complementary to the traditional one. Inspired by the method of bosonization of Dirac fermions, and following an appellation by Efetov and coworkers, we refer to the new method as 'superbosonization'. As we will see, in order for superbosonization to be useful the distribution of the random Hamiltonians must be invariant under some symmetry group, and this group cannot be 'too small' in a certain sense. We expect the method to be at its best for random matrix ensembles with a local gauge symmetry such as Wegner's n-orbital model [20] with gauge group $K = \mathrm{U}_n$, O_n, or USp_n.

Superbosonization differs in several ways from the traditional method of Schäfer and Wegner [17] based on the Hubbard-Stratonovich (HS) transformation: (i) the superbosonization field has the physical dimension of 1/energy (whereas the HS field has the dimension of energy). (ii) For a fixed symmetry group K, the target space of the superbosonization field is always the *same* product of compact and non-compact symmetric spaces regardless of where the energy parameters are, whereas the HS field of the Schäfer-Wegner method changes as the energy parameters move across the real axis. (iii) The method is not restricted to Gaussian disorder distributions. (iv) The symmetries of the effective theory are manifest at all stages of the calculation.

A brief characterization of what is meant by the physics word of 'superbosonization' is as follows. The object of departure of the supersymmetry method (in its old variant as well as the new one) is the Fourier transform of the probability measure of the given ensemble of disordered Hamiltonians. This Fourier transform is evaluated on a supermatrix built from commuting and anti-commuting variables and thus becomes a superfunction; more precisely a function, say f, which is defined on a complex vector space V_0 and takes values in the exterior algebra $\wedge(V_1^*)$ of another complex vector space V_1. If the probability measure is invariant under a group K, so that the function f is equivariant with respect to K acting on V_0 and V_1, then a standard result from invariant theory tells us that f can be viewed as the pullback of a superfunction F defined on the quotient of $V = V_0 \oplus V_1$ by the group K. The heart of the superbosonization method is a formula which reduces the integral of f to an integral of the lifted function F. Depending on how the dimension of V compares with the rank of K, such a reduction step may or may not be useful for further analysis of the integral. Roughly speaking, superbosonization gets better with increasing value of $\mathrm{rank}(K)$. From a mathematical perspective, superbosonization certainly promises to become a powerful tool for the investigation and proof of the universality of spectral correlations for a whole class of random matrix ensembles that are not amenable to treatment by existing techniques.

Let us now outline the plan of the paper. In Sects. 1.1 and 1.2 we give an informal introduction to our results, which should be accessible to physicists as well as mathematicians. A concise summary of the motivation (driven by random matrix applications) for the mathematical setting of this paper is given in Sect. 2.

In Sect. 3 we present a detailed treatment of the special situation of $V_1 \equiv 0$ (the so-called Boson-Boson sector), where anti-commuting variables are absent. This case was treated in an inspiring paper by Fyodorov [8], and our final formula – the transfer of the integral of f to an integral of F – coincides with his. The details of the derivation, however, are different. While Fyodorov employs something he calls the Ingham-Siegel integral, our approach proceeds directly by push forward to the quotient $V_0//K^{\mathbb{C}}$. Another difference is that our treatment covers each of the three classical symmetry groups $K = \mathrm{U}_n$, O_n, and USp_n, not just the first two.

Section 4 handles the complementary situation $V_0 = 0$ (the so-called Fermion-Fermion sector). In this case the starting point is the Berezin integral of $f \in \wedge(V_1^*)$, i.e., one differentiates once with respect to each of the anti-commuting variables, or projects on the top-degree component of $\wedge(V_1^*)$. From a theoretical physicist's perspective, this case is perhaps the most striking one, as it calls for the mysterious step of transforming the Berezin integral of f to an integral of the lifted function F over a compact symmetric space. The conceptual difficulty here is that many choices of F exist, and any serious theoretical discussion of the matter has to be augmented by a proof that the final answer does not depend on the specific choice which is made.

Finally, Sect. 5 handles the full situation where $V_0 \neq 0$ and $V_1 \neq 0$ (i.e., both types of variable, commuting and anti-commuting, are present). Heuristic ideas as to how one might tackle this situation are originally due to Lehmann, Saher, Sokolov and Sommers [15] and to Hackenbroich and Weidenmüller [10]. These ideas have recently been pursued by Efetov and his group [7] and by Guhr [9], but their papers are short of mathematical detail – in particular, the domain of integration after the superbosonization step is left unspecified – and address only the case of unitary symmetry. In Sect. 5 we supply the details missing from these earlier works and prove the superbosonization formula for the cases of $K = \mathrm{U}_n$, O_n, and USp_n, giving sufficient conditions of validity in each case. While it should certainly be possible to construct a proof based solely on supersymmetry and invariant-theoretic notions including Howe dual pairs, Lie superalgebra symmetries and the existence of an invariant Berezin measure, our approach here is different and more constructive: we use a chain of variable transformations reducing the general case to the cases dealt with in Sects. 3 and 4.

1.1. Basic setting. Motivated by the method of commuting and anti-commuting variables as reviewed in Sect. 2, let there be a set of complex variables Z_c^i with complex conjugates $\tilde{Z}_i^c := \overline{Z_c^i}$, where indices are in the range $i = 1, \ldots, n$ and $c = 1, \ldots, p$. Let there also be two sets of anti-commuting variables ζ_e^i and $\tilde{\zeta}_i^e$ with index range $i = 1, \ldots, n$ and $e = 1, \ldots, q$. (Borrowing the language from the physics context where the method is to be applied, one calls n the number of orbitals and p and q the number of bosonic resp. fermionic replicas.) It is convenient and useful to arrange the variables Z_c^i, ζ_e^i in the form of rectangular matrices Z, ζ with n rows and p resp. q columns. A similar arrangement as rectangular matrices is made for the variables $\tilde{Z}_i^c$, $\tilde{\zeta}_i^e$, but now with p resp. q rows and n columns.

We are going to consider integrals over these variables in the sense of Berezin [1]. Let

$$D_{Z,\tilde{Z};\zeta,\tilde{\zeta}} := 2^{pn} \prod_{c=1}^{p} \prod_{i=1}^{n} |d\mathfrak{Re}(Z_c^i)\, d\mathfrak{Im}(Z_c^i)| \otimes (2\pi)^{-qn} \prod_{e=1}^{q} \prod_{j=1}^{n} \frac{\partial^2}{\partial \zeta_e^j\, \partial \tilde{\zeta}_j^e}, \quad (1.1)$$

where the derivatives are left derivatives, i.e., we use the sign convention $\frac{\partial^2}{\partial\zeta\partial\tilde{\zeta}}\tilde{\zeta}\zeta = 1$, and the product of derivatives projects on the component of maximum degree in the anti-commuting variables. The other factor is Lebesgue measure for the commuting complex variables Z. We here denote such integrals by

$$\int f \equiv \int_{\mathrm{Mat}_{n,p}(\mathbb{C})} D_{Z,\bar{Z};\zeta,\tilde{\zeta}}\, f(Z,\tilde{Z};\zeta,\tilde{\zeta}) \tag{1.2}$$

for short. The domain of integration will be the linear space of all complex rectangular $n \times p$ matrices Z, with $\tilde{Z} = Z^\dagger \in \mathrm{Mat}_{p,n}(\mathbb{C})$ being the Hermitian adjoint of Z. We assume that our integrands f decrease at infinity so fast that the integral $\int f$ exists.

In the present paper we will be discussing such integrals for the particular case where the integrand f has a Lie group symmetry. More precisely, we assume that a Lie group K is acting on $\mathbb{C}^n$ and this group is either the unitary group U_n, or the real orthogonal group O_n, or the unitary symplectic group USp_n. The fundamental K-action on $\mathbb{C}^n$ gives rise to a natural action by multiplication on the left resp. right of our rectangular matrices: $Z \mapsto gZ$, $\zeta \mapsto g\zeta$ and $\tilde{Z} \mapsto \tilde{Z}g^{-1}$, $\tilde{\zeta} \mapsto \tilde{\zeta}g^{-1}$ (where $g \in K$). The functions f to be integrated shall have the property of being K-invariant:

$$f(Z,\tilde{Z};\zeta,\tilde{\zeta}) = f(gZ,\tilde{Z}g^{-1};g\zeta,\tilde{\zeta}g^{-1}) \quad (g \in K). \tag{1.3}$$

We wish to establish a reduction formula for the Berezin integral $\int f$ of such functions. This formula will take a form that varies slightly between the three cases of $K = \mathrm{U}_n$, $K = \mathrm{O}_n$, and $K = \mathrm{USp}_n$.

1.1.1. The case of U_n*-symmetry.* Let then f be an analytic and U_n-invariant function of our basic variables $Z, \tilde{Z}, \zeta, \tilde{\zeta}$ for $\tilde{Z} = Z^\dagger$. We now make the further assumption that f extends to a $\mathrm{GL}_n(\mathbb{C})$-invariant *holomorphic* function when Z and $\tilde{Z}$ are viewed as *independent* complex matrices; which means that the power series for f in terms of Z and $\tilde{Z}$ converge everywhere and that the symmetry relation (1.3) for the extended function f holds for all $g \in \mathrm{GL}_n(\mathbb{C})$, the complexification of U_n. The rationale behind these assumptions about f is that they guarantee the existence of another function F which lies 'over' f in the following sense.

It is a result of classical invariant theory [11] that the algebra of $\mathrm{GL}_n(\mathbb{C})$-invariant polynomial functions in $Z, \tilde{Z}, \zeta, \tilde{\zeta}$ is generated by the quadratic invariants

$$(\tilde{Z}Z)_c^{c'} \equiv \tilde{Z}_i^{c'} Z_c^i, \quad (\tilde{Z}\zeta)_e^{c'} \equiv \tilde{Z}_i^{c'} \zeta_e^i, \quad (\tilde{\zeta}Z)_c^{e'} \equiv \tilde{\zeta}_i^{e'} Z_c^i, \quad (\tilde{\zeta}\zeta)_e^{e'} \equiv \tilde{\zeta}_i^{e'} \zeta_e^i.$$

Here we are introducing the summation convention: an index that appears twice, once as a subscript and once as a superscript, is understood to be summed over.

How does this invariant-theoretic fact bear on our situation? To answer that, let F be a holomorphic function of complex variables $x_c^{c'}$, $y_e^{e'}$ and anti-commuting variables $\sigma_e^{c'}$, $\tau_c^{e'}$ with index range $c, c' = 1, \ldots, p$ and $e, e' = 1, \ldots, q$. Again, let us organize these variables in the form of matrices, $x = (x_c^{c'})$, $y = (y_e^{e'})$, etc., and write F as

$$F(x_c^{c'}, y_e^{e'}; \sigma_e^{c'}, \tau_c^{e'}) \equiv F\begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix}.$$

Then the relevant statement from classical invariant theory in conjunction with [16] is this: given any $\mathrm{GL}_n(\mathbb{C})$-invariant holomorphic function f of the variables $Z, \tilde{Z}, \zeta, \tilde{\zeta}$, it is possible to find a holomorphic function F of the variables x, y, σ, τ so that

$$F\begin{pmatrix} \tilde{Z}Z & \tilde{Z}\zeta \\ \tilde{\zeta}Z & \tilde{\zeta}\zeta \end{pmatrix} = f(Z,\tilde{Z};\zeta,\tilde{\zeta}). \tag{1.4}$$

To be sure, there exists no unique choice of such function F. Indeed, since the top degree of the Grassmann algebra generated by the anti-commuting variables ζ_e^i and $\tilde{\zeta}_i^{e'}$ is $2qn$, any monomial in the matrix variables y of degree higher than qn vanishes identically upon making the substitution $y_e^{e'} = \tilde{\zeta}_i^{e'} \zeta_e^i$.

In the following, we will use the abbreviated notation $F = F(Q)$, where the symbol Q stands for the supermatrix built from the matrices x, σ, τ, y :

$$Q = \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix}. \tag{1.5}$$

1.1.2. Orthogonal and symplectic symmetry. In the case of the symmetry group being $K = \mathrm{O}_n$ the complex vector space $\mathbb{C}^n$ is equipped with a non-degenerate symmetric tensor $\delta_{ij} = \delta_{ji}$ (which you may think of as the Kronecker delta symbol). By definition, the elements k of the orthogonal group O_n satisfy the conditions $k^{-1} = k^\dagger$ and $k^{\mathrm{t}}\delta k = \delta$, where k^{t} means the transpose of the matrix k. Let δ^{ij} denote the components of the inverse tensor, δ^{-1}. In addition to $\tilde{Z}Z$, $\tilde{Z}\zeta$, $\tilde{\zeta}Z$, $\tilde{\zeta}\zeta$ we now have the following independent quadratic K-invariants:

$$\begin{aligned}
(Z^{\mathrm{t}}\delta Z)_{c'c} &= Z^i_{c'}\delta_{ij}Z^j_c, \quad (\tilde{Z}\delta^{-1}\tilde{Z}^{\mathrm{t}})^{c'c} = \tilde{Z}^{c'}_i\delta^{ij}\tilde{Z}^c_j, \\
(Z^{\mathrm{t}}\delta\zeta)_{c'e} &= Z^i_{c'}\delta_{ij}\zeta^j_e, \quad (\tilde{\zeta}\delta^{-1}\tilde{Z}^{\mathrm{t}})^{e'c} = \tilde{\zeta}^{e'}_i\delta^{ij}\tilde{Z}^c_j, \\
(\zeta^{\mathrm{t}}\delta\zeta)_{e'e} &= \zeta^i_{e'}\delta_{ij}\zeta^j_e, \quad (\tilde{\zeta}\delta^{-1}\tilde{\zeta}^{\mathrm{t}})^{e'e} = \tilde{\zeta}^{e'}_i\delta^{ij}\tilde{\zeta}^e_j.
\end{aligned}$$

In the case of symplectic symmetry, the dimension n has to be an even number and $\mathbb{C}^n$ is equipped with a non-degenerate skew-symmetric tensor $\varepsilon_{ij} = -\varepsilon_{ji}$. Elements k of the unitary symplectic group USp_n satisfy the conditions $k^{-1} = k^\dagger$ and $k^{\mathrm{t}}\varepsilon k = \varepsilon$. If $\varepsilon^{ij} = -\varepsilon^{ji}$ are the components of ε^{-1}, the extra quadratic invariants for this case are

$$(Z^{\mathrm{t}}\varepsilon Z)_{c'c} = Z^i_{c'}\varepsilon_{ij}Z^j_c, \quad (\tilde{Z}\varepsilon^{-1}\tilde{Z}^{\mathrm{t}})^{c'c} = \tilde{Z}^{c'}_i\varepsilon^{ij}\tilde{Z}^c_j, \quad \text{etc.}$$

To deal with the two cases of orthogonal and symplectic symmetry in parallel, we introduce the notation $\beta := \delta$ for $K = \mathrm{O}_n$ and $\beta := \varepsilon$ for $K = \mathrm{USp}_n$, and we organize all quadratic invariants as a supermatrix:

$$\begin{pmatrix} \tilde{Z}Z & \tilde{Z}\beta^{-1}\tilde{Z}^{\mathrm{t}} & \tilde{Z}\zeta & \tilde{Z}\beta^{-1}\tilde{\zeta}^{\mathrm{t}} \\ Z^{\mathrm{t}}\beta Z & Z^{\mathrm{t}}\tilde{Z}^{\mathrm{t}} & Z^{\mathrm{t}}\beta\zeta & Z^{\mathrm{t}}\tilde{\zeta}^{\mathrm{t}} \\ \tilde{\zeta}Z & \tilde{\zeta}\beta^{-1}\tilde{Z}^{\mathrm{t}} & \tilde{\zeta}\zeta & \tilde{\zeta}\beta^{-1}\tilde{\zeta}^{\mathrm{t}} \\ -\zeta^{\mathrm{t}}\beta Z & -\zeta^{\mathrm{t}}\tilde{Z}^{\mathrm{t}} & -\zeta^{\mathrm{t}}\beta\zeta & -\zeta^{\mathrm{t}}\tilde{\zeta}^{\mathrm{t}} \end{pmatrix}. \tag{1.6}$$

This particular matrix arrangement is motivated as follows.

Let Q be the supermatrix (1.5) of before, but now double the size of each block; thus x here is a matrix of size $2p \times 2p$, the rectangular matrix σ is of size $2p \times 2q$, and so on. Then impose on Q the symmetry relation $Q = T_\beta Q^{\mathrm{st}}(T_\beta)^{-1}$, where

$$T_\delta = \begin{pmatrix} 0 & 1_p & & \\ 1_p & 0 & & \\ & & 0 & -1_q \\ & & 1_q & 0 \end{pmatrix}, \quad T_\varepsilon = \begin{pmatrix} 0 & -1_p & & \\ 1_p & 0 & & \\ & & 0 & 1_q \\ & & 1_q & 0 \end{pmatrix},$$

and Q^{st} means the supertranspose:

$$Q^{\mathrm{st}} = \begin{pmatrix} x^{\mathrm{t}} & \tau^{\mathrm{t}} \\ -\sigma^{\mathrm{t}} & y^{\mathrm{t}} \end{pmatrix}.$$

It is easy to check that the supermatrix (1.6) obeys precisely this relation $QT_\beta = T_\beta Q^{\mathrm{st}}$.

For the symmetry groups $K = \mathrm{O}_n$ and $K = \mathrm{USp}_n$ – with the complexified groups being $G = \mathrm{O}_n(\mathbb{C})$ and $G = \mathrm{Sp}_n(\mathbb{C})$ – it is still true that the algebra of G-invariant holomorphic functions f of $Z, \tilde{Z}, \zeta, \tilde{\zeta}$ is generated by the invariants that arise at the quadratic level. Thus, if f is any function of such kind, then there exists (though not uniquely so) a holomorphic function $F(Q)$ which pulls back to the given function f :

$$F \begin{pmatrix} \tilde{Z}Z & \tilde{Z}\beta^{-1}\tilde{Z}^{\mathrm{t}} & \tilde{Z}\zeta & \tilde{Z}\beta^{-1}\tilde{\zeta}^{\mathrm{t}} \\ Z^{\mathrm{t}}\beta Z & Z^{\mathrm{t}}\tilde{Z}^{\mathrm{t}} & Z^{\mathrm{t}}\beta\zeta & Z^{\mathrm{t}}\tilde{\zeta}^{\mathrm{t}} \\ \tilde{\zeta}Z & \tilde{\zeta}\beta^{-1}\tilde{Z}^{\mathrm{t}} & \tilde{\zeta}\zeta & \tilde{\zeta}\beta^{-1}\tilde{\zeta}^{\mathrm{t}} \\ -\zeta^{\mathrm{t}}\beta Z & -\zeta^{\mathrm{t}}\tilde{Z}^{\mathrm{t}} & -\zeta^{\mathrm{t}}\beta\zeta & -\zeta^{\mathrm{t}}\tilde{\zeta}^{\mathrm{t}} \end{pmatrix} = f(Z, \tilde{Z}; \zeta, \tilde{\zeta}). \tag{1.7}$$

1.2. Superbosonization formula. A few more definitions are needed to state our main result, which transfers the integral of f to an integral of F.

In (1.2) the definition of the Berezin integral $\int f$ was given. Let us now specify how we integrate the 'lifted' function F, beginning with the case of $K = \mathrm{U}_n$. There, the domain of integration will be $D = D_p^0 \times D_q^1$, where D_p^0 is the symmetric space of positive Hermitian $p \times p$ matrices and D_q^1 is the group of unitary $q \times q$ matrices, $D_q^1 = \mathrm{U}_q$. The Berezin superintegral form to be used for $F(Q)$ is

$$DQ := d\mu_{D_p^0}(x)\, d\mu_{D_q^1}(y)\, (2\pi)^{-pq}\, \Omega_{W_1} \circ \mathrm{Det}^q(x - \sigma y^{-1}\tau)\, \mathrm{Det}^p(y - \tau x^{-1}\sigma), \tag{1.8}$$

where the meaning of the various symbols is as follows. The Berezin form Ω_{W_1} is defined as the product of all derivatives with respect to the anti-commuting variables:

$$\Omega_{W_1} = \prod_{c=1}^{p} \prod_{e=1}^{q} \frac{\partial^2}{\partial\sigma_e^c\, \partial\tau_c^e}. \tag{1.9}$$

The symbol $d\mu_{D_q^1}$ denotes a suitably normalized Haar measure on $D_q^1 = \mathrm{U}_q$ and $d\mu_{D_p^0}$ means a positive measure on D_p^0 which is invariant with respect to the transformation $X \mapsto gXg^\dagger$ for all invertible complex $p \times p$ matrices $g \in \mathrm{GL}_p(\mathbb{C})$. Our precise normalization conventions for these measures are defined by the Gaussian limits

$$\lim_{t \to +\infty} \sqrt{t/\pi}^{\,p^2} \int_{D_p^0} \mathrm{e}^{-t\,\mathrm{Tr}\,(x-\mathrm{Id})^2} d\mu_{D_p^0}(x) = 1 = \lim_{t \to +\infty} \sqrt{t/\pi}^{\,q^2} \int_{D_q^1} \mathrm{e}^{t\,\mathrm{Tr}\,(y-\mathrm{Id})^2} d\mu_{D_q^1}(y).$$

Now assume that $p \leq n$. Then we assert that the *superbosonization formula*

$$\int f = \frac{\mathrm{vol}(\mathrm{U}_n)}{\mathrm{vol}(\mathrm{U}_{n-p+q})} \int_D DQ\ \mathrm{SDet}^n(Q)\, F(Q) \tag{1.10}$$

holds for a large class of analytic functions with suitable falloff behavior at infinity. (In the body of the paper we state and prove this formula for the class of Schwartz functions, i.e., functions that decrease faster than any power. This, however, is not yet the optimal formulation, and we expect the formula (1.10) to hold in greater generality.) Here $\mathrm{vol}(\mathrm{U}_n) := \int d\mu_{D_n^1}(y)$ is the volume of the unitary group, the integrands f and F are assumed to be related by (1.4), and SDet is the superdeterminant function,

$$\mathrm{SDet}\begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = \frac{\mathrm{Det}(x)}{\mathrm{Det}(y - \tau x^{-1}\sigma)}.$$

It should be mentioned at this point that ideas toward the existence of such a formula as (1.10) have been vented in the recent literature [7,9]. These publications, however, do not give an answer to the important question of which integration domain to choose for Q. Noting that the work of Efetov et al. is concerned with the case of $n = 1$ and $p = q \gg 1$, let us emphasize that the inequality $p \leq n$ is in fact necessary in order for our formula (1.10) to be true. (The situation for $p > n$ is explored in a companion paper [3].) Moreover, be advised that analogous formulas for the related cases of $K = \mathrm{O}_n, \mathrm{USp}_n$ have not been discussed at all in the published literature.

Turning to the latter two cases, we introduce two $2r \times 2r$ matrices t_s and t_a :

$$t_s = \begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix}, \quad t_a = \begin{pmatrix} 0 & -1_r \\ 1_r & 0 \end{pmatrix},$$

where $r = p$ or $r = q$ depending on the context. Then let a linear space $\mathrm{Sym}_b(\mathbb{C}^{2r})$ for $b := s$ or $b := a$ be defined by

$$\mathrm{Sym}_b(\mathbb{C}^{2r}) := \left\{ M \in \mathrm{Mat}_{2r,2r}(\mathbb{C}) \mid M = t_b\, M^{\mathrm{t}} (t_b)^{-1} \right\}.$$

Thus the elements of $\mathrm{Sym}_b(\mathbb{C}^{2r})$ are complex $2r \times 2r$ matrices which are symmetric with respect to transposition followed by conjugation with t_b. With this notation, we can rephrase the condition $Q = T_\beta Q^{\mathrm{st}} (T_\beta)^{-1}$ for the blocks x resp. y as saying they are in $\mathrm{Sym}_s(\mathbb{C}^{2p})$ resp. $\mathrm{Sym}_a(\mathbb{C}^{2q})$ for $\beta = \delta$ and in $\mathrm{Sym}_a(\mathbb{C}^{2p})$ resp. $\mathrm{Sym}_s(\mathbb{C}^{2q})$ for $\beta = \varepsilon$. The domain of integration for Q will now be $D_\beta := D^0_{\beta,p} \times D^1_{\beta,q}$, where

$$D^0_{\delta,p} = \mathrm{Herm}^+ \cap \mathrm{Sym}_s(\mathbb{C}^{2p}), \quad D^1_{\delta,q} = \mathrm{U} \cap \mathrm{Sym}_a(\mathbb{C}^{2q}),$$

in the case of $\beta = \delta$ (or $K = \mathrm{O}_n$), and

$$D^0_{\varepsilon,p} = \mathrm{Herm}^+ \cap \mathrm{Sym}_a(\mathbb{C}^{2p}), \quad D^1_{\varepsilon,q} = \mathrm{U} \cap \mathrm{Sym}_s(\mathbb{C}^{2q}),$$

in the case of $\beta = \varepsilon$ (or $K = \mathrm{USp}_n$). Thus in both cases, $\beta = \delta$ and $\beta = \varepsilon$, the integration domains $D^0_{\beta,p}$ and $D^1_{\beta,q}$ are constructed by taking the intersection with the positive Hermitian matrices and the unitary matrices, respectively.

The Berezin superintegral form DQ for the cases $\beta = \delta, \varepsilon$ has the expression

$$DQ := d\mu_{D^0_{\beta,p}}(x)\, d\mu_{D^1_{\beta,q}}(y)\, \Omega_{W_1} \circ \frac{\mathrm{Det}^q(x - \sigma y^{-1}\tau)\,\mathrm{Det}^p(y - \tau x^{-1}\sigma)}{(2\pi)^{2pq}\,\mathrm{Det}^{\frac{1}{2}m_\beta}(1 - x^{-1}\sigma y^{-1}\tau)}, \tag{1.11}$$

where $m_\delta = 1$ and $m_\varepsilon = -1$. The Berezin form Ω_{W_1} now is simply a product of derivatives w.r.t. all of the anti-commuting variables in the matrix σ. (The entries of τ are determined from those of σ by the relation $Q = T_\beta Q^{\mathrm{st}} (T_\beta)^{-1}$.) For $\beta = \delta$ one defines

$$\Omega_{W_1} = \prod_{c=1}^{p} \prod_{e=1}^{q} \frac{\partial^2}{\partial \sigma^c_e\, \partial \sigma^{c+p}_{e+q}} \otimes \prod_{c=1}^{p} \prod_{e=1}^{q} \frac{\partial^2}{\partial \sigma^{c+p}_e\, \partial \sigma^c_{e+q}}, \tag{1.12}$$

while for $\beta = \varepsilon$ the definition is the same except that the ordering of the derivatives $\partial/\partial\sigma_e^{c+p}$ and $\partial/\partial\sigma_{e+q}^c$ in the second product has to be reversed.

It remains to define the measures $d\mu_{D^0_{\beta,p}}$ and $d\mu_{D^1_{\beta,q}}$. To do so, we first observe that the complex group $\mathrm{GL}_{2p}(\mathbb{C})$ acts on $\mathrm{Sym}_b(\mathbb{C}^{2p})$ by conjugation in a twisted sense:

$$x \mapsto g x \tau_b(g^{-1}), \quad \tau_b(g^{-1}) = t_b\, g^{\mathrm{t}} (t_b)^{-1} \quad (b = s, a).$$

A derived group action on the restriction to the positive Hermitian matrices is then obtained by restricting to the subgroup $G' \subset \mathrm{GL}_{2p}(\mathbb{C})$ defined by the condition

$$\tau_b(g^{-1}) = g^\dagger.$$

This subgroup G' turns out to be $G' \simeq \mathrm{GL}_{2p}(\mathbb{R})$ for $b = s$ and $G' \simeq \mathrm{GL}_p(\mathbb{H})$, the invertible $p \times p$ matrices whose entries are real quaternions, for $b = a$. In the sector of y, the unitary group U_{2q} acts on $D^1_{\beta,q} = \mathrm{U} \cap \mathrm{Sym}_b(\mathbb{C}^{2q})$ by the same twisted conjugation,

$$y \mapsto g y \tau_b(g^{-1}) \quad (b = a, s).$$

Now in all cases, $d\mu_{D^0_{\beta,p}}$ and $d\mu_{D^1_{\beta,q}}$ are measures on $D^0_{\beta,p}$ and $D^1_{\beta,q}$ which are invariant by the pertinent group action. Since the group actions at hand are transitive, all of our invariant measures are unique up to multiplication by a constant. As before, we consider a Gaussian limit in order to fix the normalization constant:

$$\lim_{t\to+\infty} \sqrt{t/\pi}^{\,p(2p+m_\beta)} \int_{D^0_{\beta,p}} \mathrm{e}^{-\frac{t}{2}\,\mathrm{Tr}\,(x-\mathrm{Id})^2} d\mu_{D^0_{\beta,p}}(x) = 1.$$

The normalization of $d\mu_{D^1_{\beta,q}}$ is specified by the corresponding formula where we make the replacements $p \to q$, and $m_\beta \to -m_\beta$, and $-\mathrm{Tr}\,(x - \mathrm{Id})^2 \to +\mathrm{Tr}\,(y - \mathrm{Id})^2$. An explicit expression for each of these invariant measures is given in the Appendix.

We are now ready to state the superbosonization formula for the cases of orthogonal and symplectic symmetry. Let the inequality of dimensions $n \geq 2p$ be satisfied. We then assert that the following is true.

Let the Berezin integral $\int f$ still be defined by (1.2), but now assume the holomorphically extended integrand f to be G-invariant with complexified symmetry group $G = \mathrm{O}_n(\mathbb{C})$ for $\beta = \delta$ and $G = \mathrm{Sp}_n(\mathbb{C})$ for $\beta = \varepsilon$. Let $K_n = \mathrm{O}_n$ in the former case and $K_n = \mathrm{USp}_n$ in the latter case. Then, choosing any holomorphic function $F(Q)$ related to the given function f by (1.7), the integration formula

$$\int f = 2^{(q-p)m_\beta} \frac{\mathrm{vol}(K_n)}{\mathrm{vol}(K_{n-2p+2q})} \int_{D_\beta} DQ \;\mathrm{SDet}^{n/2}(Q)\, F(Q) \tag{1.13}$$

holds true, provided that f falls off sufficiently fast at infinity.

Thus the superbosonization formula takes the same form as in the previous case $K = \mathrm{U}_n$, except that the exponent n now is reduced to $n/2$. The latter goes hand in hand with the size of the supermatrix Q having been expanded by $p \to 2p$ and $q \to 2q$.

Another remark is that the square root of the superdeterminant,

$$\mathrm{SDet}^{n/2}(Q) = \sqrt{\mathrm{Det}^n(x)} \big/ \sqrt{\mathrm{Det}^n(y - \tau x^{-1}\sigma)},$$

Table 1. Isomorphisms between integration domains and symmetric spaces

K	D_p^0	$\tilde{D}_p^0$	D_q^1	$\tilde{D}_q^1$
U_n	$\mathrm{Herm}^+ \cap \mathrm{Mat}_{p,p}(\mathbb{C})$	$\mathrm{GL}_p(\mathbb{C})/\mathrm{U}_p$	$\mathrm{U} \cap \mathrm{Mat}_{q,q}(\mathbb{C})$	U_q
O_n	$\mathrm{Herm}^+ \cap \mathrm{Sym}_s(\mathbb{C}^{2p})$	$\mathrm{GL}_{2p}(\mathbb{R})/\mathrm{O}_{2p}$	$\mathrm{U} \cap \mathrm{Sym}_a(\mathbb{C}^{2q})$	$\mathrm{U}_{2q}/\mathrm{USp}_{2q}$
USp_n	$\mathrm{Herm}^+ \cap \mathrm{Sym}_a(\mathbb{C}^{2p})$	$\mathrm{GL}_p(\mathbb{H})/\mathrm{USp}_{2p}$	$\mathrm{U} \cap \mathrm{Sym}_s(\mathbb{C}^{2q})$	$\mathrm{U}_{2q}/\mathrm{O}_{2q}$

is always analytic in the sector of the matrix y. For the case of orthogonal symmetry this is because $D^1_{\delta,q} = \mathrm{U} \cap \mathrm{Sym}_a(\mathbb{C}^{2q})$ is isomorphic to the unitary skew-symmetric $2q \times 2q$ matrices and for such matrices the determinant has an analytic square root known as the Pfaffian. (In the language of random matrix physics, $D^1_{\delta,q}$ is the domain of definition of the Circular Symplectic Ensemble, which has the feature of Kramers degeneracy.) In the case of symplectic symmetry, where the number n is always even, no square root is being taken in the first place.

As another remark, let us mention that each of our integration domains is isomorphic to a symmetric space of compact or non-compact type. These isomorphisms $D^1_q \simeq \tilde{D}^1_q$ and $D^0_p \simeq \tilde{D}^0_p$ are listed in Table 1. Detailed explanations are given in the main text.

Let us also mention that the expressions (1.8) and (1.11) for the Berezin integration forms DQ can be found from a supersymmetry principle: each DQ is associated with one of three Riemannian symmetric superspaces in the sense of [22] (to be precise, these are the supersymmetric non-linear sigma model spaces associated with the random matrix symmetry classes AIII, BDI, and CII) and is in fact the Berezin integration form which is invariant w.r.t. the action of the appropriate Lie superalgebra $\mathfrak{gl}$ or $\mathfrak{osp}$. We will make no use of this symmetry principle in the present paper. Instead, we will give a direct proof of the superbosonization formulas (1.10) and (1.13), deriving the expressions (1.8) and (1.11) by construction, not from a supersymmetry argument.

Finally, we wish to stress that in random matrix applications, where n typically is a large number, the reduction brought about by the superbosonization formulas (1.10) and (1.13) is a striking advance: by conversion from its original role as the number of integrations to do, the big integer n has been turned into an exponent, whereby asymptotic analysis of the integral by saddle-point methods becomes possible.

1.3. Illustration. To finish this introduction, let us illustrate the new method at the example of Wegner's n-orbital model with n orbitals per site and unitary symmetry.

The Hilbert space V of that model is an orthogonal sum, $V = \oplus_{i \in \Lambda} V_i$, where i labels the sites (or vertices) of a lattice Λ and the $V_i \cong \mathbb{C}^n$ are Hermitian vector spaces of dimension n. The Hamiltonians of Wegner's model are random Hermitian operators $H : V \to V$ distributed according to a Gaussian measure $d\mu(H)$. To specify the latter, let $\Pi_i : V \to V_i$ be the orthogonal projector on V_i. The probability measure of Wegner's model is then given as a Gaussian distribution $d\mu(H)$ with Fourier transform

$$\int \mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}\,(HK)} d\mu(H) = \mathrm{e}^{-\frac{1}{2n}\sum_{ij} C_{ij} \mathrm{Tr}(K\Pi_i K \Pi_j)},$$

where $K \in \mathrm{End}(V)$, and the variances $C_{ij} = C_{ji}$ are non-negative real numbers. We observe that $d\mu(H)$ is invariant under conjugation $H \mapsto gHg^{-1}$ by unitary transformations $g \in \prod_{i \in \Lambda} \mathrm{U}(V_i)$; such an invariance is called a *local gauge symmetry* in physics.

Let us now be interested in, say, the average ratio of characteristic polynomials:

$$R(E_0, E_1) := \int \frac{\mathrm{Det}(E_1 - H)}{\mathrm{Det}(E_0 - H)}\, d\mu(H) \quad (\mathfrak{Im}\, E_0 > 0).$$

To compute $R(E_0, E_1)$ one traditionally uses a supersymmetry method involving the so-called Hubbard-Stratonovich transformation. In order for this approach to work, one needs to assume that the positive quadratic form with matrix coefficients C_{ij} has an inverse. If it does, then the traditional approach leads to the following result [4]:

$$R(E_0, E_1) = \int \mathrm{e}^{-\frac{n}{2}\sum_{ij}(C^{-1})_{ij}\,(x_i x_j - y_i y_j)}\, D_{\mathrm{HS}}(x, y) \prod_{k\in\Lambda} \frac{(y_k - E_1)^{n-1}}{(x_k - E_0)^{n+1}}\, \frac{dy_k\, dx_k}{2\pi/\mathrm{i}},$$

where the integral is over $x_k \in \mathbb{R}$ and $y_k \in \mathrm{i}\mathbb{R}$. The factor $D_{\mathrm{HS}}(x, y)$ is a fermion determinant resulting from integration over the anti-commuting components of the Hubbard-Stratonovich field; it is the determinant of the matrix with elements

$$n \left(\delta_{ij} - (C^{-1})_{ij}\,(x_i - E_0)(y_j - E_1)\right).$$

Notice that the integration variables x_k and y_k carry the physical dimension of energy.

In contrast, using the new approach opened up by the superbosonization formula of the present paper, we obtain

$$R(E_0, E_1) = \int \mathrm{e}^{-\frac{n}{2}\sum_{ij} C_{ij}\,(x_i x_j - y_i y_j)}\, D_{\mathrm{SB}}(x, y) \prod_{k\in\Lambda} \left(\frac{x_k\, \mathrm{e}^{\mathrm{i}E_0 x_k}}{y_k\, \mathrm{e}^{\mathrm{i}E_1 y_k}}\right)^n \frac{dx_k\, dy_k}{2\pi\mathrm{i}\, x_k y_k}.$$

Here the integral is over $x_k \in \mathbb{R}_+$ and $y_k \in \mathrm{U}_1$ (the unit circle in $\mathbb{C}$). These integration variables have the physical dimension of $(\text{energy})^{-1}$. The factor $D_{\mathrm{SB}}(x, y)$ still is a fermion determinant, which now arises from integration over the anti-commuting variables of the superbosonization field; it is the determinant of the matrix with elements

$$n \left(\delta_{ij} + C_{ij}\, x_i\, y_j\right).$$

When both methods (Hubbard-Stratonovich and superbosonization) are applicable, our two formulas for $R(E_0, E_1)$ are exactly equivalent to each other. Please be warned, however, that this equivalence is by no means easy to see directly.

From a practical viewpoint, the main difference between the two formulas is that one of them is expressed by the quadratic form of variances C_{ij} whereas the other one is expressed by the *inverse* of that quadratic form. A rigorous analysis based on the formula from Hubbard-Stratonovich transformation (or, rather, the resulting formula for the density of states) for the case of long-range C_{ij} and $n = 1$, was made in [4]. A similar analysis based on the formula from superbosonization has not yet been done.

2. Motivation: Supersymmetry Method

Imagine some quantum mechanical setting where the Hilbert space is $\mathbb{C}^n$ equipped with its standard Hermitian structure. On that finite-dimensional space, let us consider Hermitian operators H that are drawn at random from a probability distribution or ensemble $d\mu(H)$. We might wish to compute the spectral correlation functions of the ensemble or some other observable quantity of interest in random matrix physics.

One approach to this problem is by the so-called supersymmetry method [6,21]. In that method the observables one wishes to compute are written in terms of Green's functions, i.e., matrix coefficients of the resolvent operator of H, which are then expressed as Gaussian integrals over commuting and anti-commuting variables.

The key object of the method is the characteristic function of $d\mu(H)$,

$$\mathscr{F}(K) = \int \mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}\,(HK)} d\mu(H),$$

where the exact meaning of the Fourier variable K depends on what observable is to be computed. In the general case (with p bosonic and q fermionic 'replicas'), defining the matrix entries of K with respect to the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$ by $K e_j = e_i\, K^i_j$, one lets

$$K^i_j := Z^i_c \tilde{Z}^c_j + \zeta^i_e\, \tilde{\zeta}^e_j,$$

where Z^i_c, $\tilde{Z}^c_j$ and ζ^i_e, $\tilde{\zeta}^e_j$ are the commuting and anti-commuting variables of Sect. 1 and the summation convention is still in force. To compute, say, the spectral correlation functions of $d\mu(H)$, one multiplies $\mathscr{F}(K)$ by the exponential function

$$\exp\left(\mathrm{i} Z^i_c E^c_{c'} \tilde{Z}^{c'}_i + \mathrm{i} \zeta^j_e F^e_{e'} \tilde{\zeta}^{e'}_j\right), \quad E^c_{c'} = E_c \delta^c_{c'}, \quad F^e_{e'} = F_e \delta^e_{e'}, \tag{2.1}$$

where the parameters E_c and F_e have the physical meaning of energies, and one integrates the product against the flat Berezin integration form $D_{Z,\tilde{Z};\zeta,\tilde{\zeta}}$ over the real vector space defined by $\tilde{Z}^c_i = \mathrm{sgn}(\mathfrak{Im}\, E_c)\overline{Z^i_c}$ (for $c = 1, \ldots, p$ and $i = 1, \ldots, n$). The desired correlation functions are then generated by a straightforward process of taking derivatives with respect to the energy parameters at coinciding points. Note that for all $g \in \mathrm{GL}_n(\mathbb{C})$ the exponential (2.1) is invariant under

$$Z^i_c \mapsto g^i_j Z^j_c, \quad \tilde{Z}^c_i \mapsto \tilde{Z}^c_j (g^{-1})^j_i, \quad \zeta^i_e \mapsto g^i_j \zeta^j_e, \quad \tilde{\zeta}^i_e \mapsto \tilde{\zeta}^j_e (g^{-1})^j_i.$$

Let us now pass to a basis-free formulation of this setup. For that we are going to think of the sets of complex variables Z^i_c and $\tilde{Z}^c_i$ as bases of holomorphic linear functions for the complex vector spaces $\mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$ resp. $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$, and we interpret the anti-commuting variables ζ^i_e and $\tilde{\zeta}^e_i$ as generators for the exterior algebras of the vector spaces $\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)^*$ resp. $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q)^*$. Let

$$V_0 := \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p), \quad V_1 := \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n) \oplus \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q).$$

If we now choose some fixed Hermitian operator H drawn from our random matrix ensemble, the exponential $\mathrm{e}^{-\mathrm{i}\,\mathrm{Tr}\,(HK)}$ is seen to be a holomorphic function on V_0 with values in the exterior algebra $\wedge(V_1^*)$. Under mild assumptions on $d\mu(H)$ (e.g., bounded support, or rapid decay at infinity) the holomorphic property carries over to the integral $\int \exp(-\mathrm{i}\mathrm{Tr}\, HK)\, d\mu(H)$. The characteristic function $\mathscr{F}(K)$ in that case is a holomorphic function

$$\mathscr{F}(K) : V_0 \to \wedge(V_1^*),$$

and so is the function resulting from $\mathscr{F}(K)$ by multiplication with the Gaussian factor (2.1). We denote this product of functions by f for short.

Combining V_0 and V_1 to a $\mathbb{Z}_2$-graded vector space $V := V_0 \oplus V_1$, we denote the graded-commutative algebra of holomorphic functions $V_0 \to \wedge(V_1^*)$ by $\mathscr{A}_V$. The main task in the supersymmetry method is to compute the Berezin superintegral of $f \in \mathscr{A}_V$.

This task is rather difficult to carry out for functions f corresponding to a general probability measure $d\mu(H)$. Let us therefore imagine that f has some symmetries. Thus, let a group $G \subset \mathrm{GL}(\mathbb{C}^n)$ be acting on $\mathbb{C}^n$, and let

$$g.(L \oplus \tilde{L}) = L\, g^{-1} \oplus g\tilde{L} \quad (g \in G),$$

for $L \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ and $\tilde{L} \in \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$ be the induced action of G on V_0. We also have the same G-action on V_1, and the latter induces a G-action on $\wedge(V_1^*)$.

Now let the given probability measure $d\mu(H)$ be invariant with respect to conjugation by the elements of (a unitary form of) such a group G. Via the Fourier transform, this symmetry gets transferred to the characteristic function $\mathscr{F}(K)$, and also to the product of $\mathscr{F}(K)$ with the exponential (2.1). Our function f then satisfies the relation $f(v) = g.f(g^{-1}v)$ for all $g \in G$ and $v \in V_0$ and thus is an element of the subalgebra $\mathscr{A}_V^G \subset \mathscr{A}_V$ of G-*equivariant* holomorphic functions.

Following Dyson [5] the complex symmetry groups G of prime interest in random matrix theory are $G = \mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, and $\mathrm{Sp}_n(\mathbb{C})$. These are the complexifications of the compact symmetry groups U_n, O_n, and USp_n, corresponding to ensembles of Hermitian matrices with unitary symmetry, real symmetric matrices with orthogonal symmetry, and quaternion self-dual matrices with symplectic symmetry.

To summarize, in the present paper we will be concerned with the algebra $\mathscr{A}_V^G$ of G-equivariant holomorphic functions

$$f\ :\ V_0 \to \wedge(V_1^*), \quad v \mapsto f(v) = g.f(g^{-1}v) \quad (g \in G), \tag{2.2}$$

for the classical Lie groups $G = \mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$, and $\mathrm{Sp}_n(\mathbb{C})$, and the vector spaces

$$V_0 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n), \tag{2.3}$$
$$V_1 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n). \tag{2.4}$$

Our strategy will be to lift $f \in \mathscr{A}_V^G$ to another algebra $\mathscr{A}_W$ of holomorphic functions $F\ :\ W_0 \to \wedge(W_1^*)$, using a surjective homomorphism $\mathscr{A}_W \to \mathscr{A}_V^G$. The thrust of the paper then is to prove a statement of reduction – the superbosonization formula – transferring the Berezin superintegral of $f \in \mathscr{A}_V^G$ to such an integral of $F \in \mathscr{A}_W$.

The advantage of our treatment (as compared to the orthogonal polynomial method) is that it readily extends to the case of symmetry groups $G \times G \times \cdots \times G$ with direct product structure. This will make it possible in the future to treat such models as Wegner's gauge-invariant model [20] with n orbitals per site and gauge group G.

2.1. Notation. We now fix some notation which will be used throughout the paper.

If A and B are vector spaces and $L\ :\ A \to B$ is a linear mapping, we denote the canonical adjoint transformation between the dual vector spaces B^* and A^* by $L^{\mathrm{t}}\ :\ B^* \to A^*$. We call L^{t} the 'transpose' of L. A Hermitian structure $\langle , \rangle$ on a complex vector space A determines a complex anti-linear isometry $c_A\ :\ A \to A^*$ by $v \mapsto \langle v, \cdot \rangle$. If both A and B carry Hermitian structure, then $L\ :\ A \to B$ has a Hermitian adjoint $L^\dagger\ :\ B \to A$ defined by $L^\dagger = c_A^{-1} \circ L^{\mathrm{t}} \circ c_B$. The operator $(L^\dagger)^{\mathrm{t}}\ :\ A^* \to B^*$ is denoted by $(L^\dagger)^{\mathrm{t}} \equiv \overline{L}$. Note $\overline{L} = c_B \circ L \circ c_A^{-1}$. Finally, if each of A and B is equipped with a non-degenerate pairing $A \times A \to \mathbb{C}$ and $B \times B \to \mathbb{C}$, so that we are given complex linear isomorphisms $\alpha : A \to A^*$ and $\beta : B \to B^*$, then there exists a transpose $L^T : B \to A$ by

$$L^T : B \xrightarrow{\beta} B^* \xrightarrow{L^t} A^* \xrightarrow{\alpha^{-1}} A.$$

To emphasize that this is really a story about linear operators rather than basis-dependent matrices, we use such notations as $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ for the vector space of complex linear transformations from $\mathbb{C}^n$ to $\mathbb{C}^p$. In a small number of situations we will resort to the alternative notation $\mathrm{Mat}_{p,n}(\mathbb{C})$.

From here on in this article, Lie groups by default will be complex Lie groups; thus $\mathrm{GL}_n \equiv \mathrm{GL}(\mathbb{C}^n)$, $\mathrm{O}_n \equiv \mathrm{O}(\mathbb{C}^n)$, and $\mathrm{Sp}_n \equiv \mathrm{Sp}(\mathbb{C}^n)$, with $n \in 2\mathbb{N}$ in the last case.

3. Pushing Forward in the Boson-Boson Sector

In this section, we shall address the special situation of $V_1 = 0$, or fermion replica number $q = 0$. Thus we are now facing the commutative algebra $\mathscr{A}_V^G \equiv \mathscr{O}(V)^G$ of G-invariant holomorphic functions on the complex vector space

$$V \equiv V_0 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n).$$

In order to deal with this function space we will use the fact that $\mathscr{O}(V)$ can be viewed as a completion of the symmetric algebra $\mathrm{S}(V^*)$. Since the G-action on $\mathrm{S}(V^*)$ preserves the $\mathbb{Z}$-grading

$$\mathrm{S}(V^*) = \oplus_{k \geq 0} \mathrm{S}^k(V^*)$$

and is reductive on each symmetric power $\mathrm{S}^k(V^*)$, one has a subalgebra $\mathrm{S}^k(V^*)^G$ of G-fixed elements in $\mathrm{S}^k(V^*)$ for all k.

3.1. G-invariants at the quadratic level. It is a known fact of classical invariant theory (see, e.g., [12]) that for each of the cases $G = \mathrm{GL}_n$, O_n, and Sp_n, all G-invariants in $\mathrm{S}(V^*)$ arise at the quadratic level, i.e., $\mathrm{S}(V^*)^G$ is generated by $\mathrm{S}^2(V^*)^G$. Let us therefore sharpen our understanding of these quadratic invariants $\mathrm{S}^2(V^*)^G$.

3.1.1. The case of $G = \mathrm{GL}_n$. All quadratic invariants are just of a single type here: they arise by composing the elements of $\mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$ with those of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$.

Lemma 3.1. *$\mathrm{S}^2(V^*)^G$ is isomorphic as a complex vector space to $W^* = \mathrm{End}(\mathbb{C}^p)^*$.*

Proof. Using the canonical transpose $\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B^*, A^*)$ we have

$$V \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \oplus \mathrm{Hom}((\mathbb{C}^n)^*, (\mathbb{C}^p)^*).$$

For $G = \mathrm{GL}_n$ there exists no non-zero G-invariant tensor in $\mathbb{C}^n \otimes \mathbb{C}^n$ or $(\mathbb{C}^n)^* \otimes (\mathbb{C}^n)^*$. Therefore $\mathrm{S}^2(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p))^G = 0$ and $\mathrm{S}^2(\mathrm{Hom}((\mathbb{C}^n)^*, (\mathbb{C}^p)^*))^G = 0$, resulting in

$$\mathrm{S}^2(V)^G \simeq \big(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \otimes \mathrm{Hom}((\mathbb{C}^n)^*, (\mathbb{C}^p)^*)\big)^G.$$

The space of G-invariants in $(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ is one-dimensional (with generator $\varphi^i \otimes e_i$ given by the canonical pairing between a vector space and its dual). Since the action of G on $\mathbb{C}^p$ is trivial, it follows that

$$\mathrm{S}^2(V)^G \simeq \left(\mathbb{C}^p \otimes (\mathbb{C}^n)^* \otimes \mathbb{C}^n \otimes (\mathbb{C}^p)^*\right)^G \simeq \mathbb{C}^p \otimes (\mathbb{C}^p)^* \simeq \mathrm{End}(\mathbb{C}^p) \equiv W.$$

The action of $G = \mathrm{GL}_n$ on $\mathrm{S}^2(V)$ and $\mathrm{S}^2(V^*)$ is reductive. Therefore there exists a canonical pairing $\mathrm{S}^2(V)^G \otimes \mathrm{S}^2(V^*)^G \to \mathbb{C}$ and the isomorphism $\mathrm{S}^2(V)^G \to W$ dualizes to an isomorphism $W^* \to (\mathrm{S}^2(V)^G)^* \simeq \mathrm{S}^2(V^*)^G$. $\square$

3.1.2. The cases of $G = \mathrm{O}_n$, Sp_n. Here $\mathbb{C}^n$ is equipped with a G-invariant non-degenerate bilinear form or, equivalently, with a G-equivariant isomorphism

$$\beta : \mathbb{C}^n \to (\mathbb{C}^n)^*,$$

which is symmetric for $G = \mathrm{O}_n$ and alternating for $G = \mathrm{Sp}_n$. To distinguish between these two, we sometimes write $\beta = \delta$ in the former case and $\beta = \varepsilon$ in the latter case.

To describe $\mathrm{S}^2(V^*)^G$ for both cases, we introduce the following notation. On $U := \mathbb{C}^p \oplus (\mathbb{C}^p)^*$ we have two canonical bilinear forms: the symmetric form

$$s(v \oplus \varphi, v' \oplus \varphi') = \varphi'(v) + \varphi(v'),$$

and the alternating form

$$a(v \oplus \varphi, v' \oplus \varphi') = \varphi'(v) - \varphi(v').$$

Definition 3.2. *Let $b = s$ or $b = a$. An endomorphism $L : U \to U$ of the complex vector space $U = \mathbb{C}^p \oplus (\mathbb{C}^p)^*$ is called symmetric with respect to b if $L = L^T$, i.e. if*

$$b(Lx, y) = b(x, Ly)$$

for all $x, y \in U$. We denote the vector space of such endomorphisms by $\mathrm{Sym}_b(U)$.

Lemma 3.3. *If $U = \mathbb{C}^p \oplus (\mathbb{C}^p)^*$, then the space of quadratic invariants $\mathrm{S}^2(V^*)^G$ is isomorphic as a complex vector space to W^*, where $W = \mathrm{Sym}_s(U)$ for $G = \mathrm{O}_n$ and $W = \mathrm{Sym}_a(U)$ for $G = \mathrm{Sp}_n$.*

Proof. We still have $V \simeq \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n) \oplus \mathrm{Hom}((\mathbb{C}^p)^*, (\mathbb{C}^n)^*)$ but now, via the given complex linear isomorphism $\beta : \mathbb{C}^n \to (\mathbb{C}^n)^*$, we even have an identification

$$V \simeq \mathrm{Hom}(U, \mathbb{C}^n) \simeq U^* \otimes \mathbb{C}^n, \quad U = \mathbb{C}^p \oplus (\mathbb{C}^p)^*.$$

Also, letting $\mathrm{Sym}(V, V^*)$ denote the vector space of symmetric linear transformations

$$\sigma : V \to V^*, \quad \sigma(v)(v') = \sigma(v')(v),$$

there is an isomorphism $\mathrm{S}^2(V^*) \to \mathrm{Sym}(V, V^*)$ by

$$\varphi'\varphi + \varphi\varphi' \mapsto \left(v \mapsto \varphi'(v)\varphi + \varphi(v)\varphi'\right).$$

This descends to a vector space isomorphism between $\mathrm{S}^2(V^*)^G$ and $\mathrm{Sym}_G(V, V^*)$, the G-equivariant mappings in $\mathrm{Sym}(V, V^*)$.

Consider now $\mathrm{Hom}_G(V, V^*) \simeq \mathrm{Hom}_G(U^* \otimes \mathbb{C}^n, U \otimes (\mathbb{C}^n)^*)$. As a consequence of the G-action on U and U^* being trivial, one immediately deduces that

$$\mathrm{Hom}_G(U^* \otimes \mathbb{C}^n, U \otimes (\mathbb{C}^n)^*) \simeq \mathrm{Hom}(U^*, U) \otimes \mathrm{Hom}_G(\mathbb{C}^n, (\mathbb{C}^n)^*).$$

The vector space $\mathrm{Hom}_G(\mathbb{C}^n, (\mathbb{C}^n)^*)$ is one-dimensional with generator β. Because β is symmetric for $G = \mathrm{O}_n$ and alternating for $G = \mathrm{Sp}_n$, it follows that

$$\mathrm{S}^2(V^*)^G \simeq \mathrm{Sym}_G(U^* \otimes \mathbb{C}^n, U \otimes (\mathbb{C}^n)^*) \simeq \begin{cases} \mathrm{Sym}(U^*, U), & G = \mathrm{O}_n, \\ \mathrm{Alt}(U^*, U), & G = \mathrm{Sp}_n, \end{cases}$$

where the notation $\mathrm{Alt}(U^*, U)$ means the vector space of alternating homomorphisms $A : U^* \to U$, i.e., $\varphi(A(\varphi')) = -\varphi'(A(\varphi))$. Note that $\mathrm{Sym}(U^*, U) \simeq \mathrm{Sym}(U, U^*)^*$ and $\mathrm{Alt}(U^*, U) \simeq \mathrm{Alt}(U, U^*)^*$ by the trace form $(A, B) \mapsto \mathrm{Tr}\,(AB)$.

Now let $L \in \mathrm{Sym}_b(U)$. Since $b = s$ is symmetric and $b = a$ is alternating, the image of $\mathrm{Sym}_b(U)$ in $\mathrm{Hom}(U, U^*)$ under the mapping $L \mapsto \phi_L$ defined by

$$\phi_L(x)(y) := b(Lx, y)$$

is $\mathrm{Sym}(U, U^*)$ for $b = s$ and $\mathrm{Alt}(U, U^*)$ for $b = a$. Moreover, since the bilinear form b is non-degenerate, this mapping is an isomorphism of $\mathbb{C}$-vector spaces. Thus we have

$$\mathrm{S}^2(V^*)^G \simeq \begin{cases} \mathrm{Sym}(U^*, U) \simeq \mathrm{Sym}(U, U^*)^* \simeq \mathrm{Sym}_s(U)^*, & G = \mathrm{O}_n, \\ \mathrm{Alt}(U^*, U) \simeq \mathrm{Alt}(U, U^*)^* \simeq \mathrm{Sym}_a(U)^*, & G = \mathrm{Sp}_n, \end{cases}$$

which is the statement that was to be proved. □

3.2. The quadratic map Q. Summarizing the results of the previous subsection, the vector space W of quadratic G-invariants in $\mathrm{S}(V)$ is

$$W = \mathrm{S}^2(V)^G = \begin{cases} \mathrm{End}(\mathbb{C}^p), & G = \mathrm{GL}_n, \\ \mathrm{Sym}_s(U_p), & G = \mathrm{O}_n, \\ \mathrm{Sym}_a(U_p), & G = \mathrm{Sp}_n, \end{cases}$$

where $U \equiv U_p = \mathbb{C}^p \oplus (\mathbb{C}^p)^*$. For notational convenience, we will sometimes think of $W = \mathrm{End}(\mathbb{C}^p) \hookrightarrow \mathrm{End}(\mathbb{C}^p) \oplus \mathrm{End}((\mathbb{C}^p)^*)$ for the first case ($G = \mathrm{GL}_n$) as the intersection $\mathrm{Sym}_s(U_p) \cap \mathrm{Sym}_a(U_p)$ of the vector spaces W for the last two cases ($G = \mathrm{O}_n, \mathrm{Sp}_n$).

In the following we will repeatedly use the decomposition of elements $w \in W$ as

$$w = \begin{pmatrix} A & B \\ C & A^{\mathrm{t}} \end{pmatrix} \in \begin{pmatrix} \mathrm{End}(\mathbb{C}^p) & \mathrm{Hom}((\mathbb{C}^p)^*, \mathbb{C}^p) \\ \mathrm{Hom}(\mathbb{C}^p, (\mathbb{C}^p)^*) & \mathrm{End}((\mathbb{C}^p)^*) \end{pmatrix}. \tag{3.1}$$

Note that B and C are symmetric for the case of $G = \mathrm{O}_n$ and alternating for $G = \mathrm{Sp}_n$. The case of $G = \mathrm{GL}_n$ is included by setting $B = C = 0$. Note also the dimensions $\dim W = p^2$, $p(2p+1)$, and $p(2p-1)$ for $G = \mathrm{GL}_n$, O_n, and Sp_n, in this order.

Our treatment below is based on the relationship of $\mathscr{O}(V)^G$ with the holomorphic functions $\mathscr{O}(W)$. To make this relation explicit, we now introduce a map

$$Q : V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n) \to \mathrm{End}(U_p)$$

by defining its blocks according to the decomposition (3.1) as

$$Q\,:\,L\oplus\tilde{L}\mapsto\begin{pmatrix} L\,\tilde{L} & L\,\beta^{-1}L^{\mathrm{t}}\\ \tilde{L}^{\mathrm{t}}\beta\tilde{L} & \tilde{L}^{\mathrm{t}}L^{\mathrm{t}}\end{pmatrix}.$$

Recall that the G-equivariant isomorphism $\beta:\,\mathbb{C}^n\to(\mathbb{C}^n)^*$ is symmetric for $G=\mathrm{O}_n$, alternating for $G=\mathrm{Sp}_n$, and non-existent for $G=\mathrm{GL}_n$ in which case the off-diagonal blocks $L\beta^{-1}L^{\mathrm{t}}$ and $\tilde{L}^{\mathrm{t}}\beta\tilde{L}$ are understood to be zero. In all three cases this mapping Q is G-invariant: $Q(L\oplus\tilde{L})=Q(Lg^{-1}\oplus g\tilde{L})$ for all $g\in G$. In the last two cases this is because $g^{\mathrm{t}}\beta g=\beta$ by the very notion of what it means for β to be G-equivariant.

Lemma 3.4. *The G-invariant mapping $Q:\,V\to\mathrm{End}(U)$ is into W.*

Proof. Let G be one of the groups O_n or Sp_n and denote by L and $\tilde{L}$ the elements of $\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^p)$ resp. $\mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^n)$. Introducing two isomorphisms

$$\psi\,:\,V\to\mathrm{Hom}(\mathbb{C}^n,U),\quad L\oplus\tilde{L}\mapsto L\oplus\tilde{L}^{\mathrm{t}}\beta,$$
$$\tilde{\psi}\,:\,V\to\mathrm{Hom}(U,\mathbb{C}^n),\quad L\oplus\tilde{L}\mapsto\tilde{L}\oplus\beta^{-1}L^{\mathrm{t}},$$

we have $Q(v)=\psi(v)\tilde{\psi}(v)$ for $v=L\oplus\tilde{L}$. The two maps ψ and $\tilde{\psi}$ are related by

$$\beta\tilde{\psi}(v)=\psi(v)^{\mathrm{t}}T_b,$$

where $T_b\,:\,U\to U^*$ is the isomorphism given by $x\mapsto b(x,\cdot)$. Note that T_b is symmetric for $b=s$ and alternating for $b=a$. Using the relations above, one computes that

$$Q(v)^{\mathrm{t}}=\tilde{\psi}(v)^{\mathrm{t}}\psi(v)^{\mathrm{t}}=T_b^{\mathrm{t}}\psi(v)(\beta^{\mathrm{t}})^{-1}\beta\tilde{\psi}(v)T_b^{-1}.$$

If parities $\sigma(\beta),\sigma(T_b)\in\{\pm1\}$ are assigned to β and T_b by $\beta^{\mathrm{t}}=\sigma(\beta)\,\beta$ and $T_b^{\mathrm{t}}=\sigma(T_b)\,T_b$, then $\sigma(\beta)=\sigma(T_b)$ by construction, and it follows that $Q(v)^{\mathrm{t}}=T_bQ(v)(T_b)^{-1}$. This is equivalent to saying that $Q(v)=Q(v)^T\in\mathrm{Sym}_b(U)=W$, which proves the statement for the groups $G=\mathrm{O}_n,\mathrm{Sp}_n$. The remaining case of $G=\mathrm{GL}_n$ is included as a subcase by the embedding $\mathrm{End}(\mathbb{C}^p)\hookrightarrow\mathrm{Sym}_s(U_p)\cap\mathrm{Sym}_a(U_p)$. □

While the map $Q\,:\,V\to W$ will not always be surjective, as the rank of $L\in\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^p)$ is at most $\min(n,p)$, there is a pullback of algebras $Q^*:\,\mathscr{O}(W)\to\mathscr{O}(V)^G$ in all cases. Let us now look more closely at Q^* restricted to W^*, the linear functions on W. For this let $\{e_i\}$, $\{f^i\}$, $\{e_c\}$, and $\{f^c\}$ be standard bases of $\mathbb{C}^n$, $(\mathbb{C}^n)^*$, $\mathbb{C}^p$, and $(\mathbb{C}^p)^*$, respectively, and define bases $\{Z_c^i\}$ and $\{\tilde{Z}_i^c\}$ of $\mathrm{Hom}(\mathbb{C}^p,\mathbb{C}^n)^*$ and $\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^p)^*$ by

$$Z_c^i(\tilde{L})=f^i(\tilde{L}\,e_c),\quad\tilde{Z}_i^c(L)=f^c(L\,e_i),$$

where $i=1,\ldots,n$ and $c=1,\ldots,p$. Also, decomposing $w\in W\subset\mathrm{End}(U)$ as in (3.1), define a set of linear functions $x_c^{c'}$, $y^{c'c}$, and $y_{cc'}$ on W by

$$x_c^{c'}(w)=f^{c'}(Ae_c),\quad y^{c'c}(w)=f^{c'}(Bf^c),\quad y_{cc'}(w)=(Ce_{c'})(e_c).$$

Notice that $y^{c'c}=\pm y^{cc'}$ and $y_{cc'}=\pm y_{c'c}$ where the plus sign applies in the case of $G=\mathrm{O}_n$ and the minus sign for $G=\mathrm{Sp}_n$. The set of functions $\{x_c^{c'}\}$ is a basis of $W^*\simeq\mathrm{End}(\mathbb{C}^p)^*$ for the case of $G=\mathrm{GL}_n$. Expanding this set by including the set of functions $\{y^{c'c},y_{cc'}\}_{c\le c'}$ we get a basis of W^* for $G=\mathrm{O}_n$. The same goes for $G=\mathrm{Sp}_n$ if the condition on indices $c\le c'$ is replaced by $c<c'$.

Lemma 3.5. *The pullback of algebras $Q^* : \mathscr{O}(W) \to \mathscr{O}(V)^G$ restricted to the linear functions on W realizes the isomorphism of complex vector spaces $W^* \to \mathrm{S}^2(V^*)^G$.*

Proof. Applying Q^* to the chosen basis of W^* we obtain the expressions

$$Q^* x_c^{c'} = \tilde{Z}_i^{c'} Z_c^i, \quad Q^* y^{c'c} = \tilde{Z}_i^{c'} \beta^{ij} \tilde{Z}_j^c, \quad Q^* y_{cc'} = Z_c^i \beta_{ij} Z_{c'}^j,$$

where $\beta^{ij} = f^i(\beta^{-1} f^j)$ and $\beta_{ij} = (\beta e_j)(e_i)$. Now the p^2 functions $\tilde{Z}_i^{c'} Z_c^i$ are linearly independent and form a basis of $\mathrm{S}^2(V^*)^{\mathrm{GL}_n}$. By including the $p(p \pm 1)$ linearly independent functions $\tilde{Z}_i^{c'} \beta^{ij} \tilde{Z}_j^c$ and $Z_c^i \beta_{ij} Z_{c'}^j$ we get a basis of $\mathrm{S}^2(V^*)^G$ for $G = \mathrm{O}_n$ resp. Sp_n. Thus our linear map $Q^* : W^* \to \mathrm{S}^2(V^*)^G$ is a bijection in all cases. $\square$

Proposition 3.6. *The homomorphism $Q^* : \mathscr{O}(W) \to \mathscr{O}(V)^G$ is surjective.*

Proof. Let $\mathbb{C}[W] = \mathrm{S}(W^*)$ and $\mathbb{C}[V]^G = \mathrm{S}(V^*)^G$ be the rings of polynomial functions on W and G-invariant polynomial functions on V, respectively. Pulling back functions by the G-invariant quadratic map $Q : V \to W$, we have a homomorphism $Q^* : \mathbb{C}[W] \to \mathbb{C}[V]^G$. This map Q^* is surjective because $\mathbb{C}[V]^G = \mathrm{S}(V^*)^G$ is generated by $\mathrm{S}^2(V^*)^G$ and $Q^* : W^* \to \mathrm{S}^2(V^*)^G$ is an isomorphism.

Our holomorphic functions are expressed by power series with infinite radius of convergence. Therefore the surjective property of $Q^* : \mathbb{C}[W] \to \mathbb{C}[V]^G$ carries over to $Q^* : \mathscr{O}(W) \to \mathscr{O}(V)^G$. $\square$

In the sequel, we will establish a finer result, relating the integral of an integrable function $Q^* F \in \mathscr{O}(V)^G$ along a real subspace of V to an integral of $F \in \mathscr{O}(W)$ over a non-compact symmetric space in W. While Prop. 3.6 applies always, this relation between integrals depends on the relative value of dimensions and will here be developed only in the range $n \geq p$ (for $G = \mathrm{GL}_n$) or $n \geq 2p$ (for $G = \mathrm{O}_n, \mathrm{Sp}_n$).

We begin by specifying the integration domain in V. Using the standard Hermitian structures of $\mathbb{C}^n$ and $\mathbb{C}^p$, let a real subspace $V_\mathbb{R} \subset V$ be defined as the graph of

$$\dagger : \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \to \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n).$$

Thus in order for $L \oplus \tilde{L} \in V$ to lie in $V_\mathbb{R}$ the linear transformation $\tilde{L} : \mathbb{C}^p \to \mathbb{C}^n$ has to be the Hermitian adjoint ($\tilde{L} = L^\dagger$) of $L : \mathbb{C}^n \to \mathbb{C}^p$. Note that $V_\mathbb{R} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$.

The real vector space $V_\mathbb{R}$ is endowed with a Euclidean structure by the norm square

$$||L \oplus L^\dagger||^2 := \mathrm{Tr}(L\, L^\dagger).$$

Let then $\mathrm{dvol}_{V_\mathbb{R}}$ denote the canonical volume density of this Euclidean vector space $V_\mathbb{R}$. Our interest will be in the integral over $V_\mathbb{R}$ of $f\, \mathrm{dvol}_{V_\mathbb{R}}$ for $f \in \mathscr{O}(V)^G$. To make sure that the integral exists, we will assume that f is a Schwartz function along $V_\mathbb{R}$.

Note that the anti-linear bijection $c_p : \mathbb{C}^p \to (\mathbb{C}^p)^*, v \mapsto \langle v, \cdot \rangle$, determines a Hermitian structure on $(\mathbb{C}^p)^*$ by $\langle \varphi, \varphi' \rangle := \langle c_p^{-1} \varphi', c_p^{-1} \varphi \rangle$. The canonical Hermitian structure of $U = \mathbb{C}^p \oplus (\mathbb{C}^p)^*$ is then given by the sum $\langle u \oplus \varphi, u' \oplus \varphi' \rangle = \langle u, u' \rangle + \langle \varphi, \varphi' \rangle$.

The following is a first step toward our goal of transferring the integral $\int_{V_\mathbb{R}} f\, \mathrm{dvol}_{V_\mathbb{R}}$ to an integral over a non-compact symmetric space in W.

Lemma 3.7. *The image of $V_{\mathbb{R}}$ under the quadratic map Q lies in the intersection of W and the non-negative Hermitian operators. Thus*

$$Q(V_{\mathbb{R}}) \subset \begin{cases} \mathrm{Herm}^{\geq 0} \cap \mathrm{End}(\mathbb{C}^p), & G = \mathrm{GL}_n, \\ \mathrm{Herm}^{\geq 0} \cap \mathrm{Sym}_s(\mathbb{C}^p \oplus (\mathbb{C}^p)^*), & G = \mathrm{O}_n, \\ \mathrm{Herm}^{\geq 0} \cap \mathrm{Sym}_a(\mathbb{C}^p \oplus (\mathbb{C}^p)^*), & G = \mathrm{Sp}_n. \end{cases}$$

Proof. In the first case this is immediate from $Q(L \oplus L^\dagger) = L\,L^\dagger = (L\,L^\dagger)^\dagger \geq 0$. To deal with the other two cases we recall the expression

$$Q(L \oplus L^\dagger) = \begin{pmatrix} L\,L^\dagger & L\,\beta^{-1} L^{\mathrm{t}} \\ \overline{L}\beta L^\dagger & \overline{L}\,L^{\mathrm{t}} \end{pmatrix}.$$

We already know from Lemma 3.4 that $Q(L \oplus L^\dagger) \in \mathrm{Sym}_b(\mathbb{C}^p \oplus (\mathbb{C}^p)^*)$, where $b = s$ or $b = a$. The operator $Q(L \oplus L^\dagger)$ is self-adjoint because $(L^{\mathrm{t}})^\dagger = \overline{L}$ and $\beta^\dagger = \beta^{-1}$. It is non-negative because $\langle u \oplus \varphi, Q(L \oplus L^\dagger)(u \oplus \varphi)\rangle = |L^\dagger u + \beta^{-1} L^{\mathrm{t}} \varphi|^2 \geq 0$. $\square$

Remark. The condition $n \geq p$ resp. $n \geq 2p$ emerging below, can be anticipated as the condition for the Q-image of a generic element in $V_{\mathbb{R}}$ to have full rank.

3.3. The symmetric space of regular K-orbits in $V_{\mathbb{R}}$. Recall that our groups G act on V by $g.(L \oplus \tilde{L}) = L\,g^{-1} \oplus g\tilde{L}$. By the relation $(L\,g^{-1})^\dagger = gL^\dagger$ for unitary transformations $g \in G$, the G-action on V restricts to an action on $V_{\mathbb{R}}$ by the unitary subgroup $K = \mathrm{U}_n, \mathrm{O}_n(\mathbb{R})$, or USp_n, of $G = \mathrm{GL}_n, \mathrm{O}_n(\mathbb{C})$, resp. Sp_n.

In this subsection we study the regular K-orbit structure of $V_{\mathbb{R}}$. For this purpose we identify $V_{\mathbb{R}} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ by the K-equivariant isomorphism given by $L \oplus L^\dagger \mapsto L$.

3.3.1. $K = \mathrm{U}_n$. Here and elsewhere let $\mathrm{Hom}'(A, B)$ denote the space of homomorphisms of maximal rank between two vector spaces A and B.

Lemma 3.8. *If $n \geq p$ then* $\mathrm{Hom}'(\mathbb{C}^n, \mathbb{C}^p)/\mathrm{U}_n \simeq \mathrm{GL}_p/\mathrm{U}_p$ *(diffeomorphism).*

Proof. Since a regular transformation $L \in \mathrm{Hom}'(\mathbb{C}^n, \mathbb{C}^p)$ is surjective, the space $\mathrm{im}(L^\dagger)$ has dimension p. Thus the decomposition $\mathbb{C}^n = \ker(L) \oplus \mathrm{im}(L^\dagger)$ defines an element of the Grassmannian $(\mathrm{U}_p \times \mathrm{U}_{n-p})\backslash\mathrm{U}_n$ of complex p-planes in $\mathbb{C}^n$. Fixing some unitary basis of $\mathrm{im}(L^\dagger)$, we can identify the restriction $L : \mathrm{im}(L^\dagger) \to \mathbb{C}^p$ with an element of GL_p. In other words,

$$\mathrm{Hom}'(\mathbb{C}^n, \mathbb{C}^p) \simeq \mathrm{GL}_p \times_{\mathrm{U}_p} (\mathrm{U}_{n-p}\backslash\mathrm{U}_n),$$

which gives the desired statement by taking the quotient by the right U_n-action. $\square$

3.3.2. $K = \mathrm{O}_n$. To establish a similar result for the case of orthogonal symmetry, we need the following preparation. (Here and in the remainder of this subsection $\mathrm{O}_n \equiv \mathrm{O}_n(\mathbb{R})$ means the real orthogonal group.) Recalling that we are given a symmetric isomorphism $\delta : \mathbb{C}^n \to (\mathbb{C}^n)^*$, we associate with $L \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ an extended complex linear operator $\psi(L) \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p \oplus (\mathbb{C}^p)^*)$ by

$$\psi(L)v = (L\,v) \oplus \overline{L}\,\delta v.$$

Lemma 3.9. *The mapping* $\psi : L \mapsto L \oplus (\overline{L} \circ \delta)$ *determines an* O_n*-equivariant isomorphism* $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \to \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{2p})$ *of vector spaces with complex structure.*

Proof. Recall that $\overline{L} = c_p \circ L \circ c_n^{-1}$, where $c_p : \mathbb{C}^p \to (\mathbb{C}^p)^*$ and $c_n : \mathbb{C}^n \to (\mathbb{C}^n)^*$ are the canonical anti-linear isomorphisms given by the Hermitian structures of $\mathbb{C}^p$ resp. $\mathbb{C}^n$. Writing $T := c_n^{-1}\delta$ we have

$$\psi(L) = L \oplus (c_p\, L\, T).$$

Because δ is a symmetric isomorphism, the anti-unitary operator $T : \mathbb{C}^n \to \mathbb{C}^n$ squares to $T^2 = 1$. Let $\mathrm{Fix}(T) \subset \mathbb{C}^n$ denote the real subspace of fixed points $v = Tv$, and define on $U = \mathrm{C}^p \oplus (\mathbb{C}^p)^*$ an anti-linear involution C by

$$C(u \oplus \varphi) := (c_p^{-1}\varphi) \oplus c_p\, u.$$

If $\mathrm{Fix}(C) \subset U$ denotes the real subspace of fixed points of C, then from

$$C\psi(L)v = L\,Tv \oplus c_p\, Lv \overset{v=Tv}{=} \psi(L)v$$

we see that the $\mathbb{C}$-linear operator $\psi(L)$ maps $\mathrm{Fix}(T) \simeq \mathbb{R}^n$ into $\mathrm{Fix}(C) \simeq \mathbb{R}^{2p}$. Thus we may identify $\psi(L)$ with an element of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^{2p})$. The correspondence $L \mapsto \psi(L)$ is bijective and transforms multiplication by $\sqrt{-1}$, $L \mapsto \mathrm{i}L$, into $\psi(L) \mapsto J\psi(L)$, where $J : u + c_p\, u \mapsto \mathrm{i}u - \mathrm{i}c_p\, u$ is the complex structure of the real vector space $\mathbb{R}^{2p} \simeq \mathrm{Fix}(C)$.

By definition, the elements of the real orthogonal group O_n commute with T. Thus $\psi(L) \circ k = \psi(L\,k)$ for $k \in \mathrm{O}_n$, which means that ψ is O_n-equivariant. □

As before, let $U = \mathbb{C}^p \oplus (\mathbb{C}^p)^*$ be equipped with the Hermitian structure which is induced from that of $\mathbb{C}^p$ by $\langle u \oplus \varphi, u' \oplus \varphi' \rangle = \langle u, u' \rangle + \langle c_p^{-1}\varphi', c_p^{-1}\varphi \rangle$. Its restriction to $\mathrm{Fix}(C) \simeq \mathbb{R}^{2p}$ is a Euclidean structure defining the real orthogonal group O_{2p}.

Lemma 3.10. *If* $n \geq 2p$ *then* $\mathrm{Hom}'(\mathbb{R}^n, \mathbb{R}^{2p})/\mathrm{O}_n \simeq \mathrm{GL}_{2p}(\mathbb{R})/\mathrm{O}_{2p}$.

Proof. A regular linear operator $L : \mathbb{R}^n \to \mathbb{R}^{2p}$ determines an orthogonal decomposition $\mathbb{R}^n = \ker(L) \oplus \mathrm{im}(L^\dagger)$ into Euclidean subspaces of dimension $n - 2p$, resp. $2p$ and hence a point of the symmetric space $(\mathrm{O}_{2p} \times \mathrm{O}_{n-2p})\backslash\mathrm{O}_n$. Therefore, arguing in the same way as in the proof of Lemma 3.8, we have an identification

$$\mathrm{Hom}'(\mathbb{R}^n, \mathbb{R}^{2p}) \simeq \mathrm{GL}_{2p}(\mathbb{R}) \times_{\mathrm{O}_{2p}} (\mathrm{O}_{n-2p}\backslash\mathrm{O}_n).$$

The desired statement follows by taking the quotient by O_n. □

Remark. Although each of $\mathrm{GL}_{2p}(\mathbb{R})$ and O_{2p} has two connected components, their quotient $\mathrm{GL}_{2p}(\mathbb{R})/\mathrm{O}_{2p} = \mathrm{GL}_{2p}^+(\mathbb{R})/\mathrm{SO}_{2p}$ is connected.

For later purposes note that the anti-unitary map $C : U \to U$ combines with the Hermitian structure of U to give the canonical symmetric bilinear form of U :

$$\langle C(u \oplus \varphi), u' \oplus \varphi' \rangle = \varphi'(u) + \varphi(u') = s(u \oplus \varphi, u' \oplus \varphi').$$

3.3.3. $K = \mathrm{USp}_n$. In the final case to be addressed, we are given an alternating isomorphism $\varepsilon : \mathbb{C}^n \to (\mathbb{C}^n)^*$ and hence an anti-unitary operator $T := c_n^{-1}\varepsilon : \mathbb{C}^n \to \mathbb{C}^n$ which squares to $T^2 = -1$. Note $n \in 2\mathbb{N}$. The Hermitian vector space $\mathbb{C}^n$ now carries the extra structure of a complex symplectic vector space with symplectic form

$$\omega(v, v') := \langle Tv, v'\rangle = \overline{\langle T^2 v, Tv'\rangle} = -\omega(v', v).$$

The symmetry group of the Hermitian symplectic vector space $\mathbb{C}^n$ is $K = \mathrm{USp}_n$.

To do further analysis in this situation, it is convenient to fix some decomposition

$$\mathbb{C}^n = P \oplus T(P)$$

which is orthogonal with respect to the Hermitian structure of $\mathbb{C}^n$ and Lagrangian w.r.t. the symplectic structure. The latter means that P and $T(P)$ are non-degenerately paired by ω, so that we have an isomorphism $T(P) \xrightarrow{\sim} P^*$ by $Tv \mapsto \omega(Tv, \cdot) = -\langle v, \cdot\rangle$.

Writing $U = \mathbb{C}^p \oplus (\mathbb{C}^p)^*$ we still define $\psi : \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \to \mathrm{Hom}(\mathbb{C}^n, U)$ by

$$\psi(L) = L \oplus \overline{L}\,\varepsilon = L \oplus c_p\, L\, T,$$

and invoke the canonical Hermitian structure of U to determine the adjoint $\psi(L)^\dagger$. For future reference we note that the map $L \mapsto \psi(L)$ is USp_n-equivariant.

Lemma 3.11. *The decomposition* $\mathbb{C}^n = \ker \psi(L) \oplus \mathrm{im}\, \psi(L)^\dagger$ *is a decomposition into Hermitian symplectic subspaces.*

Proof. By the definition of the operation of taking the Hermitian adjoint, the space $\mathrm{im}\, \psi(L)^\dagger$ is the orthogonal complement of $\ker \psi(L)$ in the Hermitian vector space $\mathbb{C}^n$. Since $U = \mathbb{C}^p \oplus (\mathbb{C}^p)^*$ is an orthogonal sum and $c_p : \mathbb{C}^p \to (\mathbb{C}^p)^*$ is a bijection, the condition $0 = \psi(L)v = Lv \oplus c_p\, L\, Tv$ implies that if v is in the kernel of $\psi(L)$ then so is Tv. Thus T preserves the subspace $\ker \psi(L)$. Being anti-unitary, the operator T then preserves also the orthogonal complement $\mathrm{im}\, \psi(L)^\dagger$. It therefore follows that ω restricts to a non-degenerate symplectic form on both subspaces. □

Next, let an anti-unitary operator $C : U \to U$ with square $C^2 = -1$ be defined by

$$C(u \oplus \varphi) = (c_p^{-1}\varphi) \oplus (-c_p\, u).$$

The associated symplectic structure of U is given by the canonical alternating form:

$$-\langle C(u \oplus \varphi), u' \oplus \varphi'\rangle = \varphi'(u) - \varphi(u') = a(u \oplus \varphi, u' \oplus \varphi').$$

A short computation shows that the complex linear operator $\psi(L) : \mathbb{C}^n \to U$ satisfies the relation $\psi(L) = C\psi(L)T^{-1}$. Let us therefore decompose $\psi(L)$ according to

$$\psi(L) : \ P \oplus P^* \to \mathbb{C}^p \oplus (\mathbb{C}^p)^*.$$

Recalling $T^2 = -1$ and the fact that the anti-unitary operator T exchanges the subspaces P and P^*, we then see that $\psi(L) = C\psi(L)T^{-1}$ is already determined by its blocks $\alpha_1 := \psi(L)|_{P \to \mathbb{C}^p}$ and $\alpha_2 := \psi(L)|_{P^* \to \mathbb{C}^p}$:

$$\psi(L) = \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\bar{\alpha}_2 & \bar{\alpha}_1 \end{pmatrix}.$$

Table 2. Meaning of the groups K_n, G_p, K_p, $K_{n,p}$ for the three choices of G_n

G_n	K_n	G_p	K_p	$K_{n,p}$
$\mathrm{GL}_n(\mathbb{C})$	U_n	$\mathrm{GL}_p(\mathbb{C})$	U_p	U_{n-p}
$\mathrm{O}_n(\mathbb{C})$	$\mathrm{O}_n(\mathbb{R})$	$\mathrm{GL}_{2p}(\mathbb{R})$	$\mathrm{O}_{2p}(\mathbb{R})$	$\mathrm{O}_{n-2p}(\mathbb{R})$
$\mathrm{Sp}_n(\mathbb{C})$	USp_n	$\mathrm{GL}_p(\mathbb{H})$	USp_{2p}	USp_{n-2p}

This means that the matrix expression of $\psi(L)$ with respect to symplectic bases of $P \oplus P^*$ and $\mathbb{C}^p \oplus (\mathbb{C}^p)^*$ consists of real quaternions $q \in \mathbb{H}$:

$$q = q_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q_1 \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} + q_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + q_3 \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \quad (q_j \in \mathbb{R}).$$

Now assume that $n \geq 2p$ and $\psi(L)$ is regular. Then $\psi(L) : \operatorname{im} \psi(L)^\dagger \to U$ is an isomorphism of Hermitian symplectic vector spaces. On expressing this isomorphism with respect to symplectic bases of $\operatorname{im} \psi(L)^\dagger$ and U, we can identify it with an element of $\mathrm{GL}_p(\mathbb{H})$, the group of invertible $p \times p$ matrices with real quaternions for their entries. Note that another characterization of the elements g of $\mathrm{GL}_p(\mathbb{H})$ as a subgroup of $\mathrm{GL}(U)$ is by the equation $Cg = g\,C$. The subgroup of unitary elements in $\mathrm{GL}_p(\mathbb{H})$ is the unitary symplectic group $\mathrm{USp}(U) \equiv \mathrm{USp}_{2p}$.

The rest of the argument goes the same way as before: a regular transformation $\psi(L)$ is determined by a Hermitian symplectic decomposition $\mathbb{C}^n = \ker \psi(L) \oplus \operatorname{im} \psi(L)^\dagger$ together with a $\mathrm{GL}_p(\mathbb{H})$-transformation from $\operatorname{im} \psi(L)^\dagger$ to U; taking the quotient by the right action of USp_n we directly arrive at the following statement.

Lemma 3.12. *If $n \geq 2p$ then the space of regular USp_n-orbits in the image of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ under ψ is isomorphic to $\mathrm{GL}_p(\mathbb{H})/\mathrm{USp}_{2p}$.*

3.4. Integration formula for K-invariant functions. Let us now summarize the results of the previous section. To do this in a concise way covering all three cases at once, we will employ the notation laid down in Table 2.

Proposition 3.13. *If $\mathrm{rank}(K_p) \leq \mathrm{rank}(K_n)$ so that $K_p \subset K_n$, the space of regular K_n-orbits in $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ is isomorphic to the non-compact symmetric space G_p/K_p.*

Motivated by this result, our next goal is to reduce the integral of a K_n-invariant function on $V_\mathbb{R} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ to an integral over G_p/K_p. To prepare this step we introduce some further notations and definitions as follows.

First of all, let U_p denote the Hermitian vector space

$$U_p = \begin{cases} \mathbb{C}^p & G_n = \mathrm{GL}_n, \\ \mathbb{C}^p \oplus (\mathbb{C}^p)^*;\ s & G_n = \mathrm{O}_n, \\ \mathbb{C}^p \oplus (\mathbb{C}^p)^*;\ a & G_n = \mathrm{Sp}_n. \end{cases}$$

In the second case U_p carries a Euclidean structure (on $\mathbb{R}^{2p} \simeq \mathrm{Fix}(C) \subset U_p$) by the symmetric form s, in the third case it carries a symplectic structure by the alternating form a. Then let us regard U_p (assuming that, depending on the case, the inequality $p \leq n$ or $2p \leq n$ is satisfied) as a subspace of $\mathbb{C}^n$ with orthogonal complement $U_{n,p}$, thereby fixing an orthogonal decomposition $\mathbb{C}^n = U_p \oplus U_{n,p}$. This decomposition is Hermitian, Euclidean, or Hermitian symplectic, respectively.

Let now $X_{p,\,n}$ denote the vector space of structure-preserving linear transformations $U_p \oplus U_{n,p} \to U_p$. Using the language of matrices one would say that

$$X_{p,\,n} \simeq \begin{cases} \mathrm{Mat}_{p,\,n}(\mathbb{C}), & G_n = \mathrm{GL}_n, \\ \mathrm{Mat}_{2p,\,n}(\mathbb{R}), & G_n = \mathrm{O}_n, \\ \mathrm{Mat}_{p,\,n/2}(\mathbb{H}), & G_n = \mathrm{Sp}_n. \end{cases}$$

A special element of $X_{p,\,n}$ is the projector $\Pi : U_p \oplus U_{n,p} \to U_p$ on the first summand. By construction, the symmetry group of the kernel space $\ker(\Pi) = U_{n,p}$ is $K_{n,p}$.

Next, we specify our normalization conventions for invariant measures on the Lie groups and symmetric spaces at hand. For that purpose, let ψ denote the K_n-equivariant isomorphism discussed in the previous subsection:

$$\psi \;:\; \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \to X_{p,\,n}, \quad L \mapsto \begin{cases} L, & G_n = \mathrm{GL}_n, \\ L \oplus \overline{L}\,\delta, & G_n = \mathrm{O}_n, \\ L \oplus \overline{L}\,\varepsilon, & G_n = \mathrm{Sp}_n. \end{cases}$$

To avoid making case distinctions, we introduce an integer m taking the value $m = 0, +1, -1$ for $G = \mathrm{GL}_n, \mathrm{O}_n, \mathrm{Sp}_n$, respectively. Then from $\mathrm{Tr}_{\mathbb{C}^p}\, LL^\dagger = \mathrm{Tr}_{(\mathbb{C}^p)^*}\, \overline{L}L^{\mathrm{t}}$ we have the relation

$$\mathrm{Tr}_{\mathbb{C}^p}\, LL^\dagger = (1+|m|)^{-1}\mathrm{Tr}_{U_p}\, \psi(L)\psi(L)^\dagger,$$

which transfers the Euclidean norm of the vector space $V_{\mathbb{R}} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ to a corresponding norm on $X_{p,\,n}$. In view of the scaling implied by this transfer, we equip the Lie algebra $\mathrm{Lie}(K_p) = T_e K_p$ with the following trace form (or Euclidean structure):

$$\mathrm{Lie}(K_p) \to \mathbb{R}, \quad A \mapsto \parallel A \parallel^2 := -(1+|m|)^{-1}\mathrm{Tr}_{U_p}\, A^2 \geq 0.$$

The compact Lie group K_p is then understood to carry the invariant metric tensor and invariant volume density given by this Euclidean structure on $\mathrm{Lie}(K_p)$. The same convention applies to the compact Lie groups K_n and $K_{n,p}$. Please note that these conventions are standard and natural in that they imply, e.g., $\mathrm{vol}(\mathrm{U}_1) = \mathrm{vol}(\mathrm{SO}_2) = 2\pi$.

By the symbol dg_{K_p} we will denote the G_p-invariant measure on the non-compact symmetric space G_p/K_p. In keeping with the normalization convention we have just defined, the restriction of dg_{K_p} to the tangent space $T_o(G_p/K_p)$ at $o := K_p$ is the Euclidean volume density determined by the trace form $B \mapsto \parallel B \parallel^2 = (1+|m|)^{-1}\mathrm{Tr}_{U_p}\, B^2$, which is positive for Hermitian matrices $B = B^\dagger$.

As a final preparation, we observe that the principal bundle $G_p \to G_p/K_p$ is trivial in all cases. Recall also that the Euclidean vector space $V_{\mathbb{R}} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ comes with a canonical volume form (actually, a density) $\mathrm{dvol}_{V_{\mathbb{R}}}$.

Proposition 3.14. *For $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$ let $f \in \mathcal{O}(V)^{G_n}$ be a holomorphic function on V with the symmetry $f(L \oplus \tilde{L}) = f(L\,h \oplus h^{-1}\tilde{L})$ for all $h \in G_n$. Restrict f to a K_n-invariant function f_r on the real vector subspace $V_{\mathbb{R}} \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ by $f_r(L) := f(L \oplus L^\dagger)$. If f_r is a Schwartz function, then*

$$\int\limits_{\mathrm{Hom}(\mathbb{C}^n,\mathbb{C}^p)} f_r(L)\, \mathrm{dvol}_{V_{\mathbb{R}}}(L) = \frac{\mathrm{vol}(K_n)}{\mathrm{vol}(K_{n,p})} \int\limits_{G_p/K_p} f_r \circ \psi^{-1}(g\Pi)\, J(g)\, dg_{K_p},$$

where the Jacobian function $J : G_p/K_p \to \mathbb{R}$ is given by $J(g) = 2^{p^2 - pn}|\mathrm{Det}(g)|^{2n}$ for $G_n = \mathrm{GL}_n$ and $J(g) = 2^{2p^2 - pn}|\mathrm{Det}(g)|^n$ for $G_n = \mathrm{O}_n, \mathrm{USp}_n$.

Proof. Convergence of the integral on both sides of the equation is guaranteed by the requirement that the integrand f_r be a Schwartz function.

The first step is to transform the integral on the left-hand side to the domain $X_{p,n}$ with integrand $(\psi^{-1})^*(f\,\mathrm{dvol}_{V_\mathbb{R}})$. Of course the space $X'_{p,n}$ of regular elements in $X_{p,n}$ has full measure with respect to $(\psi^{-1})^*(\mathrm{dvol}_{V_\mathbb{R}})$. Now choose a section s of the trivial principal bundle $\pi : G_p \to G_p/K_p$ and parameterize $X'_{p,n}$ by the diffeomorphism

$$\phi \,:\, (G_p/K_p) \times (K_n/K_{n,p}) \to X'_{p,n}, \quad (x, kK_{n,p}) \mapsto s(x)\Pi\, k^{-1}.$$

Using ϕ, transform the integral from $X'_{p,n}$ to $(G_p/K_p) \times (K_n/K_{n,p})$. Right K_n-translations (as well as left K_p-translations) are isometries of $(\psi^{-1})^*(\mathrm{dvol}_{V_\mathbb{R}})$, and varying $\psi(L) = s(x)\Pi\, k^{-1}$ we get $\delta\psi(L) = \delta s(x)\Pi\, k^{-1} - s(x)\Pi\, k^{-1}\delta k\, k^{-1}$. Therefore, the pullback of $\mathrm{dvol}_{V_\mathbb{R}}$ by $\psi^{-1}\circ\phi$ is proportional to the product of invariant measures of G_p/K_p and $K_n/K_{n,p}$ times a Jacobian $j(x)$ which can be computed as the Jacobian of the map

$$\mathscr{L}_{s(x)} : X_{p,n} \to X_{p,n}, \quad L \mapsto s(x)L.$$

In the case of $X_{p,n} = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ this gives $j(x) = |\mathrm{Det}(s(x))|^{2n}$. In the other two cases the dimension of U_p is doubled while the (real) dimension of $X_{p,n}$ stays the same; hence $j(x) = |\mathrm{Det}(s(x))|^n$. In all cases we may replace $|\mathrm{Det}(s(x))|$ by $|\mathrm{Det}(g)|$, where g is any point in the fiber $\pi^{-1}(x)$. Also, by the K_n-invariance of the integrand f one has $f\circ\psi^{-1}(s(x)\Pi\, k^{-1}) = f\circ\psi^{-1}(g\Pi)$ independent of the choice of $g \in \pi^{-1}(x)$. This already proves that the two integrals on the left-hand and right-hand side are proportional to each other, with the constant of proportionality being independent of f.

It remains to ascertain the precise value of this constant. Doing the invariant integral over $K_n/K_{n,p}$ one just picks up the normalization factor of volumes $\mathrm{vol}(K_n)/\mathrm{vol}(K_{n,p})$. The remaining factor 2^{p^2-pn} or 2^{2p^2-pn} in $J(g)$ is determined by the following consideration. Decomposing the elements $\xi \in \mathfrak{k} \equiv \mathrm{Lie}(K_n)$ as

$$\mathfrak{k} \ni \xi = \begin{pmatrix} A & B \\ -B^\dagger & D \end{pmatrix} \in \begin{pmatrix} \mathfrak{k}\cap\mathrm{End}(U_p) & \mathfrak{k}\cap\mathrm{Hom}(U_{n,p}, U_p) \\ \mathfrak{k}\cap\mathrm{Hom}(U_p, U_{n,p}) & \mathfrak{k}\cap\mathrm{End}(U_{n,p}) \end{pmatrix},$$

we have the norm square $\parallel \xi \parallel^2 = (1+|m|)^{-1}(-\mathrm{Tr}\,A^2 + 2\,\mathrm{Tr}\,BB^\dagger - \mathrm{Tr}\,D^2)$. On the other hand, the differential of the mapping ϕ at $(o, eK_{n,p}) \in (G_p/K_p)\times(K_n/K_{n,p})$ is

$$H, \begin{pmatrix} A & B \\ -B^\dagger & 0 \end{pmatrix} \mapsto (H+A)\oplus B \in X_{p,n}\cap\mathrm{End}(U_p) \oplus X_{p,n}\cap\mathrm{Hom}(U_{n,p}, U_p),$$

which gives the norm square $\parallel (H+A)\oplus B \parallel^2 = (1+|m|)^{-1}(\mathrm{Tr}\,H^2 - \mathrm{Tr}\,A^2 + \mathrm{Tr}\,BB^\dagger)$. Thus the term $\mathrm{Tr}\,BB^\dagger$ gets scaled by a factor of two, and by counting the number of independent freedoms in $B \in \mathrm{Hom}(U_{n,p}, U_p)$ we see that the Jacobian $J(g)$ receives an extra factor of $2^{p(n-p)}$ for the case of $G_n = \mathrm{GL}_n$ and $2^{2p(n-2p)/2}$ for $G_n = \mathrm{O}_n, \mathrm{Sp}_n$. □

Remark. For $n = 2p$ and $G_n = \mathrm{O}_n$ the space $K_n/K_{n,p} = \mathrm{O}_n(\mathbb{R})$ consists of two connected components and the volume factor means $\mathrm{vol}(K_n)/\mathrm{vol}(K_{n,p}) = \mathrm{vol}(\mathrm{O}_n(\mathbb{R})) = 2\,\mathrm{vol}(\mathrm{SO}_n(\mathbb{R}))$. On the other hand, for $n > 2p$ and the same case the volume factor is that of the connected space $K_n/K_{n,p} = \mathrm{O}_n(\mathbb{R})/\mathrm{O}_{n-2p}(\mathbb{R}) = \mathrm{SO}_n(\mathbb{R})/\mathrm{SO}_{n-2p}(\mathbb{R})$.

To finish this section, we cast Prop. 3.14 in a form closer in spirit to the rest of paper.

Recall that either we have $G_p = \mathrm{GL}(U_p)$, or else $G_p \subset \mathrm{GL}(U_p)$ is characterized by the commutation rule $g\,C = Cg$. Since $C^\dagger = C^{-1} = \pm C$, all our groups G_p are stabilized by the dagger operation. Thus there exists an involution

$$\theta \,:\, G_p \to G_p, \quad g \mapsto (g^{-1})^\dagger,$$

which is actually a Cartan involution fixing the elements of the maximal compact subgroup K_p. The mapping

$$\gamma \,:\, G_p/K_p \to G_p, \quad g \mapsto g\theta(g^{-1}) = gg^\dagger,$$

embeds the symmetric space into the group. To clarify the connection with the setting of Sect. 3.2, let us understand the image of this embedding as a subspace of W.

Lemma 3.15. *The Cartan embedding $\gamma : G_p/K_p \to G_p \subset \mathrm{End}(U_p)$ projected to the positive Hermitian operators in $W \subset \mathrm{End}(U_p)$ is a bijection.*

Proof. From $\psi(L) = L$ in the first case, and $\psi(L) = L \oplus \overline{L}\,\beta$ in the last two cases, we immediately see that the composition of mappings

$$L \oplus L^\dagger \mapsto \psi(L) \mapsto \psi(L)\psi(L)^\dagger = Q(L \oplus L^\dagger) \in Q(V_\mathbb{R})$$

is the quadratic map $Q : V \to W$ (Sect. 3.2) restricted to $V_\mathbb{R}$, and since $g \in G_p$ arises from decomposing $\psi(L) = g\Pi\,k^{-1}$, the positive Hermitian operator $gg^\dagger = \psi(L)\psi(L)^\dagger$ lies in $Q(V_\mathbb{R}) \subset W$. Thus the embedding $G_p/K_p \to G_p$ is into $\mathrm{Herm}^+ \cap W$.

It remains to be shown that $\gamma : G_p/K_p \to \mathrm{Herm}^+ \cap W$ is one-to-one. In the case of $G_p = \mathrm{GL}(U_p)$, every positive Hermitian operator $h \in \mathrm{Herm}^+ \cap W$ has a unique positive Hermitian square root $\sqrt{h}$, and $h = \sqrt{h}\,\theta(\sqrt{h})^{-1} = \sqrt{h}k\,\theta(\sqrt{h}k)^{-1}$ ($k \in K_p$). Thus there exists a unique inverse $\gamma^{-1}(h) = \sqrt{h}K_p \in G_p/K_p$.

To deal with the other two cases we recall the relation $\langle C_b\,\cdot, \cdot'\rangle = \pm b(\cdot, \cdot')$, i.e., $C \equiv C_b$ combines with the Hermitian structure of U_p to give the bilinear form b, where $b = s$ or $b = a$. This implies that the symmetric transformations $w \in W = \mathrm{Sym}_b(U_p)$ are characterized by the commutation rule $C_b w = w^\dagger C_b$. Indeed,

$$\langle C_b w\,\cdot, \cdot'\rangle = \pm b(w\,\cdot, \cdot') = \pm b(\cdot, w\,\cdot') = \langle C_b\,\cdot, w\,\cdot'\rangle = \langle w^\dagger C_b\,\cdot, \cdot'\rangle,$$

and hence $W \cap \mathrm{Herm}$ are exactly the elements of $\mathrm{End}(U_p)$ that commute with C_b. The desired statement now follows from the definition $G_p = \{g \in \mathrm{GL}(U_p) \mid g\,C_b = C_b\,g\}$ because the squaring map on $\mathrm{Herm}^+ \cap \mathrm{GL}(U_p)$ remains a bijection when restricted to the set of fixed points $\mathrm{Herm}^+ \cap G_p$ of the involution $w \mapsto C_b\,w\,(C_b)^{-1}$. □

In the sequel we will often use the abbreviations

$$n' := (1 + |m|)^{-1} n, \quad \mathrm{Tr}' := (1 + |m|)^{-1} \mathrm{Tr}_{U_p} \quad (m = 0, 1, -1 \text{ for } G = \mathrm{GL}, \mathrm{O}, \mathrm{Sp}).$$

Let now $D_p := \mathrm{Herm}^+ \cap W$ denote the set of positive Hermitian operators in W:

$$D_p = \begin{cases} \mathrm{Herm}^+ \cap \mathrm{End}(\mathbb{C}^p), & G_n = \mathrm{GL}_n, \\ \mathrm{Herm}^+ \cap \mathrm{Sym}_s(\mathbb{C}^p \oplus (\mathbb{C}^p)^*), & G_n = \mathrm{O}_n, \\ \mathrm{Herm}^+ \cap \mathrm{Sym}_a(\mathbb{C}^p \oplus (\mathbb{C}^p)^*), & G_n = \mathrm{Sp}_n. \end{cases}$$

D_p is equipped with a G_p-invariant measure $d\mu_{D_p}$. In keeping with our general conventions, we normalize $d\mu_{D_p}$ so that $(d\mu_{D_p})_o$ agrees with the Euclidean volume density of the Euclidean vector space of Hermitian operators $H \in \mathrm{Lie}(G_p)$ with norm square $\| H \|^2 = (1+|m|)^{-1}\mathrm{Tr}\, H^2 = \mathrm{Tr}' H^2$. The Cartan embedding $g \mapsto g\theta(g^{-1})$ on the Hermitian operators $g = \mathrm{e}^H$ is the squaring map $\mathrm{e}^H \mapsto \mathrm{e}^{2H}$. Thus, pushing the G_p-invariant measure dg_{K_p} forward by the Cartan embedding we get $2^{-\dim(G_p/K_p)} d\mu_{D_p}$. Now since

$$\dim(G_p/K_p) = \begin{cases} p^2 & G_n = \mathrm{GL}_n, \\ p(2p+1) & G_n = \mathrm{O}_n, \\ p(2p-1) & G_n = \mathrm{Sp}_n, \end{cases}$$

the following statement is a straightforward reformulation of Prop. 3.14.

Corollary 3.16. *Given $f \in \mathscr{O}(V)^{G_n}$, and retaining the setup and the conditions of Prop. 3.14, define $F \in \mathscr{O}(W)$ by $Q^* F = f$. Then*

$$\int_{V_{\mathbb{R}}} f \, \mathrm{dvol}_{V_{\mathbb{R}}} = 2^{-p(n+m)} \frac{\mathrm{vol}(K_n)}{\mathrm{vol}(K_{n,p})} \int_{D_p} F(x)\, \mathrm{Det}^{n'}(x)\, d\mu_{D_p}(x).$$

In particular, since the function $x \mapsto F(x) = \mathrm{e}^{-\mathrm{Tr}' x}$ pulls back to $L \mapsto f(L) = \mathrm{e}^{-\mathrm{Tr}\, LL^\dagger}$ and the Gaussian integral $\int \mathrm{e}^{-\mathrm{Tr}\, LL^\dagger} d\mathrm{vol}_{V_{\mathbb{R}}}(L)$ has the value π^{pn}, we infer the formula

$$\int_{D_p} \mathrm{e}^{-\mathrm{Tr}' x}\, \mathrm{Det}^{n'}(x)\, d\mu_{D_p}(x) = (2\pi)^{pn} 2^{pm} \mathrm{vol}(K_{n,p})/\mathrm{vol}(K_n). \tag{3.2}$$

4. Lifting in the Fermion-Fermion Sector

Having settled the case of $V_1 = 0$ (or $q = 0$) we now turn to the complementary case where $V_0 = 0$ (or $p = 0$). Thus, in the present section we consider

$$V \equiv V_1 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n) \simeq \mathbb{C}^{2qn},$$

in which case our basic algebra $\mathscr{A}_V$ becomes an exterior algebra of dimension 2^{2qn}:

$$\mathscr{A}_V = \wedge(V^*) \simeq \wedge(\mathbb{C}^{2N}), \quad N = qn.$$

In the sequel, we will prove an analog of Prop. 3.14 for this situation.

4.1. Quadratic G-invariants. This subsection is closely analogous to Sect. 3.1, the main difference being that the role of the symmetric algebra $\mathrm{S}(V^*)$ is now taken by the exterior algebra $\wedge(V^*)$. It remains true [12] that for each of the classical reductive complex Lie groups $G = \mathrm{GL}_n$, O_n, and Sp_n, a basis of $\wedge^2(V^*)^G$ is a generating system for $\wedge(V^*)^G$. Let us therefore make another study of these quadratic invariants.

Recall that on the direct sum $U_q := \mathbb{C}^q \oplus (\mathbb{C}^q)^*$ we have the canonical symmetric bilinear form s and the canonical alternating bilinear form a.

Lemma 4.1. *Let* $V = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$ *carry the* G*-action induced from the fundamental action on* $\mathbb{C}^n$ *of* $G = \mathrm{GL}_n$, O_n, *or* Sp_n. *If* $U_q = \mathbb{C}^q \oplus (\mathbb{C}^q)^*$, *then the space of quadratic invariants* $\wedge^2(V^*)^G$ *is isomorphic as a complex vector space to* W^*, *where* $W = \mathrm{End}(\mathbb{C}^q)$ *for* $G = \mathrm{GL}_n$, $W = \mathrm{Sym}_a(U_q)$ *for* $G = \mathrm{O}_n$, *and* $W = \mathrm{Sym}_s(U_q)$ *for* $G = \mathrm{Sp}_n$.

Proof. (Sketch). There is no conceptual difference from the proofs of Lemma 3.1 and Lemma 3.3, and we therefore give only a summary of the changes.

In the case of $G = \mathrm{GL}_n$, all quadratic invariants still arise by composing the elements of $\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$ with those of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q)$. Thus $\wedge^2(V)^G \simeq \mathrm{End}(\mathbb{C}^q)$ and, since G acts reductively and $\wedge^2(V)^G$ is paired with $\wedge^2(V^*)^G$, we have $\wedge^2(V^*)^G \simeq \mathrm{End}(\mathbb{C}^q)^*$.

In the other cases we use the isomorphism $\wedge^2(V^*) \to \mathrm{Alt}(V, V^*)$ given by

$$\varphi'\varphi - \varphi\varphi' \mapsto \big(v \mapsto \varphi'(\cdot)\varphi(v) - \varphi(\cdot)\varphi'(v)\big)\,,$$

which descends to an isomorphism $\wedge^2(V^*)^G \to \mathrm{Alt}_G(V, V^*)$. Then, writing $U \equiv U_q$ we make the G-equivariant identification $V \simeq U^* \otimes \mathbb{C}^n$ and have

$$\wedge^2(V^*)^G \simeq \mathrm{Alt}_G(U^* \otimes \mathbb{C}^n, U \otimes (\mathbb{C}^n)^*) \simeq \begin{cases} \mathrm{Alt}(U^*, U), & G = \mathrm{O}_n, \\ \mathrm{Sym}(U^*, U), & G = \mathrm{Sp}_n. \end{cases}$$

This leads to the desired statement by the isomorphisms $\mathrm{Alt}(U^*, U) \simeq \mathrm{Alt}(U, U^*)^* \simeq \mathrm{Sym}_a(U)^*$ and $\mathrm{Sym}(U^*, U) \simeq \mathrm{Sym}(U, U^*)^* \simeq \mathrm{Sym}_s(U)^*$. $\square$

As before, let the elements $w \in W$ be decomposed as

$$w = \begin{pmatrix} A & B \\ C & A^{\mathrm{t}} \end{pmatrix} \in \begin{pmatrix} \mathrm{End}(\mathbb{C}^q) & \mathrm{Hom}((\mathbb{C}^q)^*, \mathbb{C}^q) \\ \mathrm{Hom}(\mathbb{C}^q, (\mathbb{C}^q)^*) & \mathrm{End}((\mathbb{C}^q)^*) \end{pmatrix}.$$

Here $B = -B^{\mathrm{t}}$ and $C = -C^{\mathrm{t}}$ for the case of $G = \mathrm{O}_n$, while $B = +B^{\mathrm{t}}$ and $C = +C^{\mathrm{t}}$ for $G = \mathrm{Sp}_n$, and $B = C = 0$ for $G = \mathrm{GL}_n$. By simple counting, the dimensions of W are $\dim W = q^2$, $q(2q-1)$, and $q(2q+1)$ for $G = \mathrm{GL}_n$, O_n, and Sp_n, respectively.

One can now reconsider the quadratic mapping $Q : V \to W$ defined in Sect. 3.2, with the twist that the elements of V in the present context are to be multiplied with each other in the alternating sense of exterior algebras. However, what matters for our purposes is not the mapping Q but the pullback of algebras $Q^* : \mathscr{O}(W) \to \wedge(V^*)^G$. Let us now specify the latter at the level of the isomorphism $Q^* : W^* \to \wedge^2(V^*)^G$.

For this let $\{e_i\}$, $\{f^i\}$, $\{e_c\}$, and $\{f^c\}$ be standard bases of $\mathbb{C}^n$, $(\mathbb{C}^n)^*$, $\mathbb{C}^q$, and $(\mathbb{C}^q)^*$, respectively, and define bases $\{\zeta_c^i\}$ and $\{\tilde{\zeta}_i^c\}$ of $\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)^*$ and $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q)^*$ by

$$\zeta_c^i(\tilde{L}) = f^i(\tilde{L}\, e_c), \quad \tilde{\zeta}_i^c(L) = f^c(L\, e_i),$$

where $L \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q)$ and $\tilde{L} \in \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$. Of course the index ranges are $i = 1, \ldots, n$ and $c = 1, \ldots, q$. Then, decomposing $w \in W \subset \mathrm{End}(U)$ into blocks A, B, C as above, define a set of linear functions $x_c^{c'}$, $y^{c'c}$, $y_{cc'} : W \to \mathbb{C}$ by

$$x_c^{c'}(w) = f^{c'}(Ae_c), \quad y^{c'c}(w) = f^{c'}(Bf^c), \quad y_{cc'}(w) = (Ce_{c'})(e_c).$$

Notice the symmetry relations $y^{c'c} = \mp y^{cc'}$ and $y_{cc'} = \mp y_{c'c}$ where the upper sign applies in the case of $G = \mathrm{O}_n$ and the lower sign for $G = \mathrm{Sp}_n$. The functions $\{x_c^{c'}\}$ constitute a basis of $W^* \simeq \mathrm{End}(\mathbb{C}^q)^*$ for the case of $G = \mathrm{GL}_n$. Inclusion of the set of functions $\{y^{c'c}, y_{cc'}\}_{c<c'}$ gives a basis of W^* for $G = \mathrm{O}_n$. By including also the functions $y^{c'c}$ and $y_{cc'}$ for $c = c'$ we get a basis of W^* for $G = \mathrm{Sp}_n$.

Recall that we are given a G-equivariant isomorphism $\beta : \mathbb{C}^n \to (\mathbb{C}^n)^*$ which is symmetric ($\beta = \delta$) for $G = \mathrm{O}_n$ and alternating ($\beta = \varepsilon$) for $G = \mathrm{Sp}_n$.

Lemma 4.2. *The isomorphism $W^* \to \wedge^2(V^*)^G$ has a realization by*

$$Q^* x_c^{c'} = \tilde{\zeta}_i^{c'} \zeta_c^i, \quad Q^* y^{c'c} = \tilde{\zeta}_i^{c'} \beta^{ij} \tilde{\zeta}_j^c, \quad Q^* y_{cc'} = \zeta_c^i \beta_{ij} \zeta_{c'}^j,$$

where $\beta_{ij} = (\beta e_j)(e_i)$ are the matrix entries of $\beta : \mathbb{C}^n \to (\mathbb{C}^n)^*$, and $\beta^{ij} = f^i(\beta^{-1} f^j)$.

Proof. All functions $\tilde{\zeta}_i^{c'} \zeta_c^i$, $\tilde{\zeta}_i^{c'} \beta^{ij} \tilde{\zeta}_j^c$ and $\zeta_c^i \beta_{ij} \zeta_{c'}^j$ are G-invariant in the pertinent cases. The q^2 functions $\tilde{\zeta}_i^{c'} \zeta_c^i$ form a basis of $\wedge^2(V^*)^{\mathrm{GL}_n}$. By including the $q(q-1)$ functions $\tilde{\zeta}_i^{c'} \delta^{ij} \tilde{\zeta}_j^c$ and $\zeta_c^i \delta_{ij} \zeta_{c'}^j$ for $c < c'$, we get a basis of $\wedge^2(V^*)^{\mathrm{O}_n}$. Replacing $\beta = \delta$ by $\beta = \varepsilon$ and expanding the index range to $c \leq c'$ we get a basis of $\wedge^2(V^*)^{\mathrm{Sp}_n}$. Thus the linear operator Q^* takes one basis to another one and hence is an isomorphism. □

Next, we review some useful representation-theoretic facts about $\wedge(V^*)^G$.

4.2. G'-irreducibility of $\mathscr{A}_V^G$. Taking $V \oplus V^* \simeq \mathbb{C}^{4N}$ to be equipped with the canonical symmetric form $s(v \oplus \varphi, v' \oplus \varphi') = \varphi'(v) + \varphi(v')$, one defines the Clifford algebra $\mathrm{Cl}(V \oplus V^*)$ to be the associative algebra generated by $V \oplus V^* \oplus \mathbb{C}$ with relations

$$ww' + w'w = s(w, w')1 \qquad (w, w' \in V \oplus V^*).$$

The linear span of the skew-symmetric quadratic elements $(ww' - w'w)$ is closed under the commutator in $\mathrm{Cl}(V \oplus V^*)$ and is canonically isomorphic to the Lie algebra of the orthogonal group of the vector space $V \oplus V^*$ with symmetric bilinear form s. By exponentiating this Lie algebra inside the Clifford algebra, one obtains the spin group, a connected and simply connected Lie group denoted by $\mathrm{Spin}(V \oplus V^*) = \mathrm{Spin}_{4N}$.

Via their actions on V, the complex Lie groups $G = \mathrm{GL}_n$, O_n, and Sp_n, are realized as subgroups of Spin_{4N}. The centralizer of G in Spin_{4N} is another complex Lie group, G', called the Howe dual partner of G [12]. The list of such Howe dual pairs is

$$G \times G' : \quad \mathrm{GL}_n \times \widetilde{\mathrm{GL}}_{2q}, \quad \mathrm{O}_n \times \mathrm{Spin}_{4q}, \quad \mathrm{Sp}_n \times \mathrm{Sp}_{4q}.$$

Note that from $\mathrm{O}_n \subset \mathrm{GL}_n$ and $\mathrm{Sp}_n \subset \mathrm{GL}_n$ one has $\widetilde{\mathrm{GL}}_{2q} \subset \mathrm{Spin}_{4q}$ and $\widetilde{\mathrm{GL}}_{2q} \subset \mathrm{Sp}_{4q}$. In the case of n being odd, $\widetilde{\mathrm{GL}}_{2q}$ is a double covering of GL_{2q} (see below).

A few words of explanation concerning the pairs $G \times G'$ are in order. In the case of the first pair one regards the vector space $V \oplus V^*$ as

$$V \oplus V^* \simeq U_q \otimes \mathbb{C}^n \oplus (U_q)^* \otimes (\mathbb{C}^n)^*, \quad U_q = \mathbb{C}^q \oplus (\mathbb{C}^q)^*,$$

and the centralizer of $G = \mathrm{GL}_n$ in $\mathrm{Spin}(V \oplus V^*)$ is then seen to be $G' = \mathrm{GL}(U_q) \equiv \mathrm{GL}_{2q}$ (or a double cover thereof if n is odd). In the last two cases the G-equivariant isomorphism $\beta : \mathbb{C}^n \to (\mathbb{C}^n)^*$ leads to an identification

$$V \oplus V^* \simeq (U_q \oplus U_q^*) \otimes \mathbb{C}^n.$$

The symmetric bilinear form s on $V \oplus V^*$ in conjunction with β induces a bilinear form on $U_q \oplus (U_q)^*$. For $G = \mathrm{O}_n$ this form is the canonical symmetric bilinear form s and one has the centralizer $G' = \mathrm{Spin}(U_q \oplus U_q^*; s) \equiv \mathrm{Spin}_{4q}$. For $G = \mathrm{Sp}_n$ the induced form is the canonical alternating form a and one has $G' = \mathrm{Sp}(U_q \oplus U_q^*; a) \equiv \mathrm{Sp}_{4q}$.

Now the exterior algebra $\wedge(V^*)$ carries the spinor representation of the Clifford algebra $\mathrm{Cl}(V \oplus V^*)$. This is the representation which is obtained by letting vectors $v \in V$ and linear forms $\varphi \in V^*$ operate by contraction $\iota(v) : \wedge^k(V^*) \to \wedge^{k-1}(V^*)$ and exterior multiplication $\varepsilon(\varphi) : \wedge^k(V^*) \to \wedge^{k+1}(V^*)$.

By the inclusion $G \times G' \subset \mathrm{Spin}_{4N} \subset \mathrm{Cl}(V \oplus V^*)$ the spinor representation of the Clifford algebra gives rise to a representation on $\mathscr{A}_V = \wedge(V^*)$ of each Howe dual pair $G \times G'$. It is known [12] that $\mathscr{A}_V$ decomposes as a direct sum $\oplus_i (U_i \otimes U_i')$ of irreducible $G \times G'$ representations such that $U_i \not\simeq U_j$ and $U_i' \not\simeq U_j'$ for $i \neq j$. In particular, the representation of G' on the algebra of G-invariants $\mathscr{A}_V^G = \wedge(V^*)^G$ is irreducible.

Next we observe that each of our Howe dual groups G' has rank $2q$. Moreover, one can arrange for all of them to share the same maximal torus. This is the Abelian group $T = (\mathbb{C}^\times)^q \times (\mathbb{C}^\times)^q$ acting on $V = V_1 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$ by diagonal transformations

$$(t_1, t_2).(L \oplus \tilde{L}) \mapsto (t_1 L) \oplus (\tilde{L}\, t_2).$$

The induced action of elements $H = (H_1, H_2)$ of the Cartan algebra $\mathfrak{t} = \mathrm{Lie}(T) = \mathbb{C}^q \oplus \mathbb{C}^q$ on the spinor module is by operators

$$\hat{H} = \tfrac{1}{2} \sum\nolimits_c \Big((H_1)_c [\iota(e_i^c), \varepsilon(f_c^i)] + (H_2)_c [\iota(e_c^i), \varepsilon(f_i^c)] \Big) . \tag{4.1}$$

Here $\{e_i^c\}$ means the standard basis of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q)$, and $\{e_c^i\}$ means the standard basis of $\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$, while $\{f_c^i\}$ and $\{f_i^c\}$ are the corresponding dual bases. The factor of $1/2$ in front of the sum reflects the fact that the spinor representation is a "square root" representation.

The zero-degree component $\wedge^0(V^*) = \mathbb{C}$ – the 'vacuum' in physics language – is stabilized by the action of these operators $\hat{H}$. Applying H as $\hat{H}$ to $1 \in \wedge^0(V^*)$ we get

$$H.1 = \lambda(H) 1, \quad \lambda(H) = \frac{n}{2} \sum_{c=1}^{q} \big((H_1)_c + (H_2)_c \big).$$

Note that the weight λ is integral for even n, but half-integral for odd n. (This is why the latter case calls for the group GL_{2q} to be replaced by a double cover $G' = \widetilde{\mathrm{GL}}_{2q}$.) We will denote the integrated weight or character by $\chi := \mathrm{e}^{\lambda \circ \ln}$.

4.3. Berezin integral and lowest weight space. We are now going to think of the irreducible G'-representation space $\mathscr{A}_V^G$ as an irreducible highest-weight module for the Lie algebra of G'. To keep the notation simple we omit the prime and denote this Lie algebra by $\mathfrak{g} := \mathrm{Lie}(G')$. Thus

$$\mathfrak{g} = \begin{cases} \mathfrak{gl}_{2q}, & G = \mathrm{GL}_n, \\ \mathfrak{o}_{4q}, & G = \mathrm{O}_n, \\ \mathfrak{sp}_{4q}, & G = \mathrm{Sp}_n. \end{cases}$$

The vacuum weight λ is a highest weight for the $\mathfrak{g}$-representation $\mathscr{A}_V^G$. We emphasize this fact by making a change of notation $\mathscr{A}_V^G \equiv \mathscr{V}(\lambda)$.

The spinor module comes with a $\mathbb{Z}$-grading by the degree, $\wedge(V^*) = \oplus_{k=0}^{2N} \wedge^k (V^*)$, where $N = \frac{1}{2}\dim V = qn$. Since G is defined on $\mathbb{C}^n$ and acts on V, this grading carries over to the algebra $\mathscr{A}_V^G = \mathscr{V}(\lambda)$:

$$\mathscr{V}(\lambda) = \bigoplus\nolimits_{k\geq 0} \mathscr{V}(\lambda)_k.$$

We denote the highest degree part by $\mathscr{V}(\lambda)_{\text{top}}$. The highest degree part $\wedge^{2N}(V^*)$ of the spinor module is a complex line stable under the symmetry group G. It is easy to check that G in fact acts trivially on $\wedge^{2N}(V^*)$, so $\mathscr{V}(\lambda)_{\text{top}} = \mathscr{V}(\lambda)_{2N} = \wedge^{2N}(V^*)$.

Now there exists a canonical generator $\Omega_V \in \wedge^{2N}(V)$ by the following principle. Since the trace form

$$\text{Hom}(\mathbb{C}^n, \mathbb{C}^q) \otimes \text{Hom}(\mathbb{C}^q, \mathbb{C}^n) \to \mathbb{C}, \quad A \otimes B \mapsto \text{Tr}\, AB,$$

is non-degenerate, the vector spaces $\text{Hom}(\mathbb{C}^n, \mathbb{C}^q)$ and $\text{Hom}(\mathbb{C}^q, \mathbb{C}^n)$ are canonically dual to each other. If $\{e_1, \ldots, e_N\}$ is any basis of $\text{Hom}(\mathbb{C}^n, \mathbb{C}^q)$, let $\{f_1, \ldots, f_N\}$ be the corresponding dual basis of $\text{Hom}(\mathbb{C}^q, \mathbb{C}^n)$. The exterior product

$$\Omega_V = f_N \wedge e_N \wedge \cdots \wedge f_1 \wedge e_1$$

then is independent of the choice of basis and only depends on how we order the two summands in $V = \text{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \text{Hom}(\mathbb{C}^q, \mathbb{C}^n)$. For definiteness, let us say that $\text{Hom}(\mathbb{C}^n, \mathbb{C}^q)$ is the first summand. We then have a canonical element $\Omega_V \in \wedge^{2N}(V)$, and by evaluating the canonical pairing $\wedge^{2N}(V) \otimes \wedge^{2N}(V^*) \to \mathbb{C}$ with fixed argument Ω_V in the first factor, we get an identification $\mathscr{V}(\lambda)_{2N} = \wedge^{2N}(V^*) \simeq \mathbb{C}$.

Definition 4.3. *The projection $\pi : \mathscr{V}(\lambda) \to \mathscr{V}(\lambda)_{2N} \simeq \mathbb{C}$ is called the Berezin integral, and is here denoted by $f \mapsto \Omega_V[f]$.*

Another way to view this projection is as follows. The vacuum $\wedge^0(V^*)$ is the space of highest-weight vectors for $\mathfrak{g}$, whereas the top degree part $\wedge^{2N}(V^*)$ is the space of *lowest-weight vectors*. The latter are the weight vectors of weight $-\lambda$. Indeed, going from zero to top degree amounts to exchanging the operators ε and ι, and since the expression (4.1) is skew-symmetric in these, the weight changes sign.

Now define the subgroup $H \subset G'$ to be the intersection of the stabilizer of $\mathscr{V}(\lambda)_0 = \wedge^0(V^*)$ with the stabilizer of $\mathscr{V}(\lambda)_{2N} = \wedge^{2N}(V^*)$. For $n \in 2\mathbb{N}$ these are the groups

$$H = \begin{cases} \text{GL}_q \times \text{GL}_q \subset G' = \text{GL}_{2q}, & G = \text{GL}_n, \\ \text{GL}_{2q} \subset G' = \text{Spin}_{4q}, & G = \text{O}_n, \\ \text{GL}_{2q} \subset G' = \text{Sp}_{4q}, & G = \text{Sp}_n. \end{cases}$$

If n is odd, we replace H by the double cover forced on us by the square root nature of the spinor representation or the highest weight λ being half-integral.

Let us now specify how the Lie algebra $\text{Lie}(H)$ acts on the spinor module $\wedge(V^*)$. (This will do as a temporary substitute for the more detailed description of the H-action on $W^* = \wedge^2(V^*)^G$ given below.) In the first case, one has $H = \text{GL}(\mathbb{C}^q) \times \text{GL}((\mathbb{C}^q)^*) \equiv \text{GL}_q \times \text{GL}_q$ and $X = (A, D) \in \text{Lie}(H)$ acts on $\wedge(V^*)$ as

$$\hat{X} = \tfrac{1}{2}[\iota(Ae_i^c), \varepsilon(f_c^i)] + \tfrac{1}{2}[\iota(e_c^i), \varepsilon(Df_i^c)],$$

where the notation of (4.1) is being used. Note that $\hat{X}.1 = \frac{n}{2}(\mathrm{Tr}\, A + \mathrm{Tr}\, D)$. In the last two cases, one uses $V \simeq \mathrm{Hom}(\mathbb{C}^n, U_q)$ and fixes a basis $\{e_i^b\}_{i=1,\ldots,n}^{b=1,\ldots,2q}$ of $\mathrm{Hom}(\mathbb{C}^n, U_q)$, with dual basis $\{f_b^i\}$. With these conventions, an element X of the Lie algebra of $H = \mathrm{GL}(U_q) \equiv \mathrm{GL}_{2q}$ acts on $\wedge(V^*)$ as $\hat{X} = \frac{1}{2}[\iota(Xe_i^b), \varepsilon(f_b^i)]$. Note $\hat{X}.1 = \frac{n}{2}\mathrm{Tr}\, X$.

By definition, the roots of H are the roots of $\mathfrak{g}$ which are orthogonal to λ. Since all groups H are connected subgroups of G' of maximal rank, they are in fact characterized by their root systems. Note also that all of our groups H are reductive. Furthermore, the character $\chi : T \to \mathbb{C}^\times$ extends to the character $\chi : H \to \mathbb{C}^\times$, $h \mapsto \mathrm{Det}^{n/2}(h)$.

Being orthogonal to the highest weight λ, the root system of H is orthogonal also to the lowest weight $-\lambda$. It follows that the space of lowest-weight vectors $\mathscr{V}(\lambda)_{2N}$ is stable with respect to H: it is the one-dimensional representation of H corresponding to the reciprocal character $\chi^{-1}(h) = \mathrm{Det}^{-n/2}(h)$. Since H is reductive and the T-weight space $\mathscr{V}(\lambda)_{2N} = \wedge^{\mathrm{top}}(V^*)$ has dimension one, $\mathscr{V}(\lambda)$ decomposes canonically as a H-representation space:

$$\mathscr{V}(\lambda) = \mathscr{V}(\lambda)_{2N} \oplus U,$$

where U is the sum of all other H-subrepresentations in $\mathscr{V}(\lambda)$. From $\dim \mathscr{V}(\lambda)_{2N} = 1$ we then infer that the space of H-equivariant homomorphisms $\mathrm{Hom}_H(\mathscr{V}(\lambda), \mathscr{V}(\lambda)_{2N})$ is one-dimensional. Now the Berezin integral $\pi : \mathscr{V}(\lambda) \to \mathscr{V}(\lambda)_{2N}$ is a non-zero element of that space, and we therefore have the following result.

Lemma 4.4. $\mathrm{Hom}_H(\mathscr{V}(\lambda), \mathscr{V}(\lambda)_{2N}) = \mathbb{C}\pi$.

4.4. Parabolic induction. The $\mathfrak{g}$-representation $\mathscr{V}(\lambda)$ can be constructed in another way, as follows. Decompose $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$, where $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{g}^\pm$ is the direct sum of the root subspaces of $\mathfrak{g}$ corresponding to positive resp. negative roots not orthogonal to λ. Since the highest weight λ is the weight of the vacuum with generator $1 \in \mathscr{V}(\lambda)_0$, this implies that $\mathfrak{g}^+.1 = 0$ and $\mathfrak{g}^-.1 = \mathscr{V}(\lambda)_2$. Or, to put it in yet another way, $\mathfrak{g}^+ \subset \mathfrak{g}$ is the subspace of elements represented on the spinor module by operators of type $\iota\iota$, while $\mathfrak{g}^- \subset \mathfrak{g}$ is the subspace of operators of type $\varepsilon\varepsilon$.

Let $\mathfrak{p} := \mathfrak{h} \oplus \mathfrak{g}^+$. (The notation is to convey that $\mathfrak{p}$ can be viewed as the Lie algebra of a parabolic subgroup of G'.) Since all roots of $\mathfrak{h}$ are orthogonal to λ, the weight $\lambda : \mathfrak{t} \to \mathbb{C}$ extends in the trivial way to a linear function $\lambda : \mathfrak{h} \to \mathbb{C}$; the latter is the function $\lambda(X) = (n/2)\mathrm{Tr}_{\mathbb{C}^{2q}} X$. We further extend λ trivially to all of $\mathfrak{p} = \mathfrak{h} \oplus \mathfrak{g}^+$.

Let $\mathscr{U}(\mathfrak{p})$ be the universal enveloping algebra of $\mathfrak{p}$ and denote by $V_\lambda := \mathscr{V}(\lambda)_0$ the one-dimensional $\mathscr{U}(\mathfrak{p})$-representation defined by $X.v_\lambda = \lambda(X)v_\lambda$ for a generator $v_\lambda \in V_\lambda$ and elements $X \in \mathfrak{p}$. Then by the canonical left action of $\mathfrak{g}$ on $\mathscr{U}(\mathfrak{g})$, the tensor product

$$M(\lambda) := \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} V_\lambda$$

is a $\mathscr{U}(\mathfrak{g})$-representation of highest weight λ and highest-weight vector

$$m_\lambda = 1 \otimes v_\lambda .$$

This representation is called a generalized Verma module or the universal highest-weight $\mathfrak{g}$-representation which is given by parabolic induction from the one-dimensional representation V_λ of $\mathfrak{p}$. The module $M(\lambda)$ has the following universal property.

Lemma 4.5. *Let W be any $\mathfrak{g}$-module with a vector $w \neq 0$ such that i) $X.w = \lambda(X)w$ for all $X \in \mathfrak{p}$, and ii) $\mathscr{U}(\mathfrak{g}).w = W$. Then there exists a surjective $\mathfrak{g}$-equivariant linear map $M(\lambda) \to W$ such that $m_\lambda \mapsto w$.*

In particular, our irreducible $\mathfrak{g}$-representation $\mathscr{V}(\lambda)$ is of this kind. Thus there exists a surjective $\mathfrak{g}$-equivariant map

$$p : M(\lambda) \to \mathscr{V}(\lambda).$$

4.5. H-structure of $M(\lambda)$. The $\mathscr{U}(\mathfrak{g})$-representation $M(\lambda)$ has infinite dimension and cannot be integrated to a representation of G'. As we shall now explain, however, the situation is more benign for the subgroup $H \subset G'$.

From $\mathfrak{g}/\mathfrak{p} \simeq \mathfrak{g}^-$ we have an isomorphism of vector spaces $\mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{p})} V_\lambda \simeq \mathscr{U}(\mathfrak{g}^-)$. By making the identification as

$$\mathscr{U}(\mathfrak{g}^-) \otimes V_\lambda \simeq M(\lambda), \quad n \otimes v_\lambda \mapsto n m_\lambda,$$

we will actually get more, as follows. If α, β are any two roots such that α is orthogonal to λ and β is not, then $\alpha + \beta$ is not orthogonal to λ. From this one directly concludes that if $h \in \mathfrak{h}$ and $n \in \mathfrak{g}^-$, then $[h, n] \in \mathfrak{g}^-$, i.e., $\mathfrak{g}^-$ (or $\mathfrak{g}^+$, for that matter) is normalized by $\mathfrak{h}$. This action of $\mathfrak{h}$ on $\mathfrak{g}^-$ extends to an $\mathfrak{h}$-action on $\mathscr{U}(\mathfrak{g}^-)$: supposing that $n = n_1 \cdots n_r \in \mathscr{U}(\mathfrak{g}^-)$, where $n_i \in \mathfrak{g}^-$ ($i = 1, \ldots, r$), we let

$$\mathrm{ad}(h)n := \sum\nolimits_j n_1 \cdots n_{j-1}[h, n_j] n_{j+1} \cdots n_r,$$

and by this definition we have the following commutation rule of operators in $\mathscr{U}(\mathfrak{g})$:

$$hn = \mathrm{ad}(h)n + nh.$$

If we now let $\mathscr{U}(\mathfrak{h})$ act on $M(\lambda)$ by the canonical left action and on $\mathscr{U}(\mathfrak{g}^-) \otimes V_\lambda$ by

$$h.(n \otimes v_\lambda) := (\mathrm{ad}(h)n) \otimes v_\lambda + \lambda(h) n \otimes v_\lambda,$$

then we see that the identification $\mathscr{U}(\mathfrak{g}^-) \otimes V_\lambda \xrightarrow{\sim} M(\lambda)$ by $n \otimes v_\lambda \mapsto n m_\lambda$ is an isomorphism of $\mathscr{U}(\mathfrak{h})$-representations.

Now every element in $\mathscr{U}(\mathfrak{g}^-)$ lies in an $\mathfrak{h}$-representation of finite dimension. Basic principles therefore entail the following consequence.

Lemma 4.6. *The representation of the Lie algebra $\mathfrak{h}$ on $M(\lambda)$ can be integrated to a representation of the Lie group H.*

In each of the three cases under consideration, $\mathfrak{g}^-$ is commutative and we can identify $\mathscr{U}(\mathfrak{g}^-)$ with the ring of polynomial functions $\mathbb{C}[W]$ on a suitable representation $W \simeq (\mathfrak{g}^-)^*$ of H. From $\mathfrak{g}^- \simeq \mathfrak{g}^-.1 = \mathscr{V}(\lambda)_2 = \wedge^2(V^*)^G$ we have $W = \wedge^2(V)^G$, the subspace of G-fixed vectors in $\wedge^2(V)$. The space W was described in Lemma 4.1 where we saw that $W = \mathrm{End}(\mathbb{C}^q)$, $\mathrm{Sym}_a(U_q)$, and $\mathrm{Sym}_s(U_q)$ for $G = \mathrm{GL}$, O, and Sp, respectively. Note that in all cases W contains the identity element $\mathrm{Id} = \mathrm{Id}_{\mathbb{C}^q}$ or $\mathrm{Id} = \mathrm{Id}_{U_q}$.

We now describe how the group H acts on W. In the last two cases, where $H = \mathrm{GL}(U_q) \equiv \mathrm{GL}_{2q}$ (or a double cover thereof), we associate with each of the two bilinear forms $b = s$ or $b = a$ an involution $\tau_b : H \to H$ by the equation

$$b(\tau_b(g)x, y) = b(x, g^{-1}y) \qquad (x, y \in U_q).$$

The action of H on $W = \mathrm{Sym}_b(U_q)$ is then by twisted conjugation, $g.w = gw\tau_b(g^{-1})$. In the notation of Sect. 1.1.2 we have $\tau_a(g^{-1}) = t_a\, g^{\mathrm{t}}(t_a)^{-1}$ and $\tau_s(g^{-1}) = t_s\, g^{\mathrm{t}}(t_s)^{-1}$. Note that the group of fixed points of τ_b in H is the symplectic group $\mathrm{Sp}(U_q) = \mathrm{Sp}_{2q}$ for $b = a$ and the orthogonal group $\mathrm{O}(U_q) = \mathrm{O}_{2q}$ for $b = s$.

In the first case ($G = \mathrm{GL}_n$) the group H is the subgroup of $\mathrm{GL}(U_q)$ preserving the decomposition $U_q = \mathbb{C}^q \oplus (\mathbb{C}^q)^*$. Here again it will be best to think of the vector space W as the intersection of the vector spaces for the other two cases:

$$W = \mathrm{End}(\mathbb{C}^q) \simeq \mathrm{Sym}_a(U_q) \cap \mathrm{Sym}_s(U_q).$$

This means that we think of $\mathrm{End}(\mathbb{C}^q)$ as being embedded into $\mathrm{End}(U_q)$ as

$$\mathrm{End}(\mathbb{C}^q) \to \begin{pmatrix} \mathrm{End}(\mathbb{C}^q) & \mathrm{Hom}((\mathbb{C}^q)^*, \mathbb{C}^q) \\ \mathrm{Hom}(\mathbb{C}^q, (\mathbb{C}^q)^*) & \mathrm{End}((\mathbb{C}^q)^*) \end{pmatrix}, \quad z \mapsto \begin{pmatrix} z & 0 \\ 0 & z^{\mathrm{t}} \end{pmatrix} =: w.$$

For an element $g = (g_1, g_2) \in H = \mathrm{GL}(\mathbb{C}^q) \times \mathrm{GL}((\mathbb{C}^q)^*)$ one now has $\tau_a(g_1, g_2)^{-1} = \tau_s(g_1, g_2)^{-1} = (g_2^{\mathrm{t}}, g_1^{\mathrm{t}})$, and the action of H on W by twisted conjugation is given by

$$g.w = (g_1, g_2). \begin{pmatrix} z & 0 \\ 0 & z^{\mathrm{t}} \end{pmatrix} = \begin{pmatrix} g_1 z\, g_2^{\mathrm{t}} & 0 \\ 0 & g_2 z^{\mathrm{t}} g_1^{\mathrm{t}} \end{pmatrix}.$$

If we now define an involution τ_0 by $\tau_0(g_1^{-1}, g_2^{-1}) = (g_2^{\mathrm{t}}, g_1^{\mathrm{t}})$, then this action can be written in the short form $g.w = gw\tau_0(g^{-1})$.

To sum up the situation, let $\tau = \tau_0$ for $G = \mathrm{GL}$, $\tau = \tau_a$ for $G = \mathrm{O}$, and $\tau = \tau_s$ for $G = \mathrm{Sp}$. Then the H-action on W always takes the form

$$g.w = gw\tau(g^{-1}).$$

In all three cases it is a well-known fact (see for example [12]) that the ring $\mathbb{C}[W]$ is multiplicity-free as a representation space for H. It then follows that the universal highest-weight representation $M(\lambda) \simeq \mathbb{C}[W] \otimes V_\lambda$ is multiplicity-free.

Lemma 4.7. *Let $V_{-\lambda} = \mathcal{V}(\lambda)_{2N}$ be the one-dimensional H-representation associated to the character $\chi^{-1} = \exp \circ (-\lambda) \circ \ln$. Then the space*

$$\mathrm{Hom}_H(M(\lambda), V_{-\lambda})$$

of H-equivariant homomorphisms from $M(\lambda)$ to $V_{-\lambda}$ has dimension one and is generated by $\pi \circ p$, the composition of the projection $p : M(\lambda) \to \mathcal{V}(\lambda)$ with the Berezin integral $\pi : \mathcal{V}(\lambda) \to V_{-\lambda} = \mathcal{V}(\lambda)_{2N}$.

Proof. Since p is surjective, $\pi \circ p$ is non-trivial and the space $\mathrm{Hom}_H(M(\lambda), V_{-\lambda})$ has at least dimension one. On the other hand, since $M(\lambda)$ is multiplicity-free as an H-representation, the dimension of $\mathrm{Hom}_H(M(\lambda), V')$ cannot be greater than one for any irreducible representation V' of H. □

Corollary 4.8. *Let $P : M(\lambda) = \mathbb{C}[W] \otimes V_\lambda \to V_{-\lambda}$ be any non-trivial H-equivariant linear mapping. Then there exists a non-zero constant c_P such that for every $f \in \mathcal{V}(\lambda)$ and any lift $F \in p^{-1}(f) \subset \mathbb{C}[W] \otimes V_\lambda$ one has*

$$P[F] = c_P\, \Omega[f].$$

We are now going to realize P by integration over a real domain in W.

4.6. Construction of H-equivariant homomorphisms. Let $o \equiv \mathrm{Id}$ denote the identity element of W. Then in all three cases the H-orbit $H.o$ is open and dense in W and can be characterized as the complement of the zero set of a polynomial D :

$$H.o = \{w \in W \mid D(w) \neq 0\},$$

where D will be the Pfaffian function for the case of $G = \mathrm{O}_n$ and will be the determinant function for $G = \mathrm{GL}_n$ and $G = \mathrm{Sp}_n$. Since D does not vanish on $H.o$, the map

$$H.o \hookrightarrow W \oplus \mathbb{C}, \quad w \mapsto (w, D(w)^{-1})$$

defines an inclusion, and one can view $H.o$ as the zero set of a function on $W \oplus \mathbb{C}$:

$$H.o = \{(w,t) \in W \oplus \mathbb{C} \mid D(w)\,t - 1 = 0\}.$$

Hence the ring of algebraic functions on $H.o$, namely $\mathbb{C}[H.o]$, is the same as the ring $\mathbb{C}[W \oplus \mathbb{C}]$ factored by the ideal generated by the function $(w,t) \mapsto D(w)\,t - 1$.

Let $\mathbb{C}(W)$ be the field of rational functions on W. Thus an element $r \in \mathbb{C}(W)$ can be expressed as a quotient $r = f/g$ of polynomial functions $f, g \in \mathbb{C}[W]$. Denote by $\mathbb{C}[W]_D \subset \mathbb{C}(W)$ the subring of elements which can be written as a quotient $r = f/D^n$,

$$\mathbb{C}[W]_D = \left\{ w \mapsto f(w)/D^n(w) \mid f \in \mathbb{C}[W],\ n \geq 0 \right\}.$$

We now identify $\mathbb{C}[H.o]$ with $\mathbb{C}[W]_D$: an element in $\mathbb{C}[H.o]$ can be represented as a polynomial of the form $(w,t) \mapsto f_0(w) + f_1(w)\,t + \cdots + f_s(w)\,t^s$, where the $f_i \in \mathbb{C}[W]$, and the following map $\mathbb{C}[H.o] \to \mathbb{C}[W]_D$ then defines an isomorphism of rings:

$$\Big((w,t) \mapsto f_0(w) + f_1(w)\,t + \cdots + f_s(w)\,t^s\Big) \mapsto \frac{f_0}{1} + \frac{f_1}{D} + \cdots + \frac{f_s}{D^s}.$$

Our aim here is to construct an H-equivariant homomorphism $M(\lambda) \to V_{-\lambda}$. By the isomorphisms of H-representations: $(V_{-\lambda})^* \simeq V_\lambda$ and $V_\lambda \otimes V_\lambda \simeq V_{2\lambda}$, this is the same as constructing an H-invariant homomorphism

$$M(\lambda) \otimes (V_{-\lambda})^* = \mathbb{C}[W] \otimes V_{2\lambda} \to \mathbb{C}$$

to the trivial representation. For this purpose we identify the representation $\mathbb{C}[W] \otimes V_{2\lambda}$ with the following subspaces of $\mathbb{C}[W]_D$:

$$\mathbb{C}[W] \otimes V_{2\lambda} \simeq \begin{cases} \mathrm{Det}^{-n}\,\mathbb{C}[W] \subset \mathbb{C}[W]_{\mathrm{Det}}, & G = \mathrm{GL}_n, \\ \mathrm{Pfaff}^{\,-n}\,\mathbb{C}[W] \subset \mathbb{C}[W]_{\mathrm{Pfaff}}, & G = \mathrm{O}_n, \\ \mathrm{Det}^{-n/2}\,\mathbb{C}[W] \subset \mathbb{C}[W]_{\mathrm{Det}}, & G = \mathrm{Sp}_n. \end{cases}$$

In the first case, this identification is correct because $g = (g_1, g_2) \in H$ operates on the determinant function $\mathrm{Det}^{-n} : W = \mathrm{End}(\mathbb{C}^q) \to \mathbb{C}$ as

$$(g.\mathrm{Det}^{-n})(z) = \mathrm{Det}^{-n}(g_1^{-1} z\,(g_2^{\mathrm{t}})^{-1}) = \mathrm{Det}^n(g_1)\mathrm{Det}^n(g_2)\mathrm{Det}^{-n}(z),$$

which is the desired behavior since $\mathrm{Det}^n(g_1 g_2^{\mathrm{t}})$ agrees with the character $\chi(g_1,g_2)^2$ associated with the H-representation $V_{2\lambda}$. In the other two cases we have

$$(g.\mathrm{Det}^{-n/2})(w) = \mathrm{Det}^{-n/2}(g^{-1} w\,\tau_b(g)^{-1}) = \mathrm{Det}^n(g)\mathrm{Det}^{-n/2}(w) \quad (b = a, s).$$

Again, this is as it should be since $\mathrm{Det}^n(g) = \chi(g)^2$ is the desired character for $V_{2\lambda}$. Here one should bear in mind that n is always an even number in the third case, and that $\sqrt{\mathrm{Det}} = \mathrm{Pfaff}$ in the second case.

Recall now that $o \equiv \mathrm{Id}$ denotes the identity element in W, let $H_o = \{h \in H \mid h.o = o\}$ be the isotropy group of o, and observe that $H.o$ is isomorphic to H/H_o,

$$H/H_o = \begin{cases} (\mathrm{GL}_q \times \mathrm{GL}_q)/\mathrm{GL}_q, & G = \mathrm{GL}_n, \\ \mathrm{GL}_{2q}/\mathrm{Sp}_{2q}, & G = \mathrm{O}_n, \\ \mathrm{GL}_{2q}/\mathrm{O}_{2q}, & G = \mathrm{Sp}_n. \end{cases}$$

Then fix some maximal compact subgroup $K \subset H$ such that the isotropy group $K_o \subset K$ is a maximal compact subgroup of the stabilizer $H_o \subset H$. Since H_o is a reductive group, we have natural isomorphisms $\mathbb{C}[W]_D = \mathbb{C}[H/H_o] = \mathbb{C}[H]^{H_o}$. The maximal compact subgroup $K \subset H$ is Zariski dense, i.e., a polynomial function that vanishes on K also vanishes on H, so $\mathbb{C}[H] = \mathbb{C}[K]$. For the same reason, given any locally finite-dimensional H-representation, the subspace of H_o-fixed points coincides with the K_o-fixed points. Summarizing, we have

$$\mathbb{C}[W] \otimes V_{2\lambda} \overset{i}{\hookrightarrow} \mathbb{C}[W]_D = \mathbb{C}[H/H_o] = \mathbb{C}[H]^{H_o} = \mathbb{C}[H]^{K_o} = \mathbb{C}[K]^{K_o}.$$

The benefit from this sequence of identifications is that on the space $\mathbb{C}[K]$ there exists a natural and non-trivial K-invariant projection. Indeed, letting dk be a Haar measure on K, we may view $f \in \mathbb{C}[W]_D$ as a function on K and integrate:

$$\mathbb{C}[W]_D \to \mathbb{C}, \quad f \mapsto \int_K f(k.o)\, dk.$$

This projection is K-invariant, and its restriction to our space $\mathbb{C}[W] \otimes V_{2\lambda}$ is still non-trivial. In the first case this is because $\mathrm{Det}^n \in \mathbb{C}[W]$ and hence $\mathbb{C}\,1 \subset \mathrm{Det}^{-n}\,\mathbb{C}[W]$; in the last two cases $\mathrm{Det}^{n/2} \in \mathbb{C}[W]$ and hence $\mathbb{C}\,1 \subset \mathrm{Det}^{-n/2}\,\mathbb{C}[W]$.

We can now reformulate Cor. 4.8 to make the statement more concrete.

Proposition 4.9. *For each case $G = \mathrm{GL}_n$, O_n, or Sp_n, there is a choice of normalized Haar measure dk so that the following holds for all $f \in \wedge(V^*)^G$. If $F \in \mathbb{C}[W]$ is any lift w.r.t. the identification and projection $\mathbb{C}[W] \simeq \mathbb{C}[W] \otimes V_\lambda \to \wedge(V^*)^G$ then*

$$\Omega_V[f] = \int_K F(k.o)\mathrm{Det}^{-n}(k)\, dk.$$

Next, let us give another version of the 'bosonization' formula of Prop. 4.9 in order to get a better match with the supersymmetric formula to be developed below. For that, notice that $K_o = \mathrm{USp}_{2q}$ for $G = \mathrm{O}_n$, $K_o = \mathrm{O}_{2q}(\mathbb{R})$ for $G = \mathrm{Sp}_n$, and $K_o = \mathrm{U}_q$ acting by elements $(k, (k^{-1})^{\mathrm{t}})$ for $G = \mathrm{GL}_n$. From this we see that $\mathrm{Det}^{-n}(k) = 1$ for $k \in K_o$ in all cases. We can therefore push down the integral over K to an integral over the orbit $K.o \simeq K/K_o$. We henceforth denote this orbit by $D_q := K.o$. Writing $y := k\tau(k^{-1})$ for $G = \mathrm{O}, \mathrm{Sp}$ we have $\mathrm{Det}^{-n}(k) = \mathrm{Det}^{-n/2}(k\tau(k)^{-1}) = \mathrm{Det}^{-n/2}(y)$. Similarly, letting $y := k_1(k_2)^{\mathrm{t}}$ for $G = \mathrm{GL}$ we have $\mathrm{Det}^{-n}(k) = \mathrm{Det}^{-n}(k_1)\mathrm{Det}^{-n}(k_2) = \mathrm{Det}^{-n}(y)$. Thus the relation $\mathrm{Det}^{-n}(k) = \mathrm{Det}^{-n'}(y)$ always holds if we set $n' = (1+|m|)^{-1}n$, i.e., $n' = n$ for $G = \mathrm{GL}$ and $n' = n/2$ for $G = \mathrm{O}, \mathrm{Sp}$.

Let now $d\mu_{D_q}$ denote a K-invariant measure on the K-orbit D_q. According to our general conventions, $d\mu_{D_q}$ is normalized in such a way that $(d\mu_{D_q})_o$ coincides with the

Euclidean volume density on $T_o D_q = W \cap \mathrm{Lie}(K)$ which is induced by the quadratic form $A \mapsto -\mathrm{Tr}_{\mathbb{C}^p} A^2$ for $G = \mathrm{GL}$ and $A \mapsto -\frac{1}{2}\mathrm{Tr}_{U_p} A^2$ for $G = \mathrm{O}, \mathrm{Sp}$. We denote this by $A \mapsto -\mathrm{Tr}' A^2$ for short. Pushing the Haar measure dk forward by $K \to K.o = D_q$ we obtain $d\mu_{D_q}$ times a constant. Thus the formula of Prop. 4.9 becomes

$$\Omega_V[f] = \text{const} \times \int_{D_q} F(y)\, \mathrm{Det}^{-n'}(y)\, d\mu_{D_q}(y). \tag{4.2}$$

To determine the unknown constant of proportionality, it suffices to compute both sides of the equation for some special choice of f (and a corresponding function F). If we choose $F(y) = \mathrm{e}^{\mathrm{Tr}' y}$, then f is simply a Gaussian with Berezin integral $\Omega_V[f] = 1$. However, the integral on the right-hand side is not quite so easy to do. We postpone this computation until the end of the paper, where we will carry it out using a supersymmetric reduction technique based on relations developed below. To state the outcome, we recall the definition of the groups $K_{n,p}$ and let the sign of the positive integer p now be reversed; according to Table 2 of Sect. 3.4 this means that $K_{n,-p} = \mathrm{U}_{n+p}$, $\mathrm{O}_{n+2p}(\mathbb{R})$, and USp_{n+2p} for $G = \mathrm{GL}$, O, and Sp.

Lemma 4.10. $\int_{D_q} \mathrm{e}^{\mathrm{Tr}' y}\, \mathrm{Det}^{-n'}(y)\, d\mu_{D_q}(y) = (2\pi)^{-qn} 2^{-qm} \mathrm{vol}(K_{n,-q})/\mathrm{vol}(K_n)$.

Remark. The similarity of this formula with Eq. (3.2) is not an accident; in fact, in Sect. 5.9 we will establish Lemma 4.10 by reduction to the latter result.

Using Lemma 4.10 we can now eliminate the unknown constant of proportionality from (4.2). To state the resulting reformulation of Prop. 4.9, we will use the surjective mapping $Q^* : \mathscr{O}(W) \to \wedge(V^*)^G$ defined in Sect. 4.1.

Theorem 4.11. *For $f \in \wedge(V^*)^{G_n}$, if $F \in (Q^*)^{-1}(f) \in \mathscr{O}(W)$ is any holomorphic function in the inverse image of f, the Berezin integral $f \mapsto \Omega_V[f]$ can be computed as an integral over the compact symmetric space $D_q \simeq K/K_o$:*

$$\Omega_V[f] = (2\pi)^{qn} 2^{qm} \frac{\mathrm{vol}(K_n)}{\mathrm{vol}(K_{n,-q})} \int_{D_q} F(y)\, \mathrm{Det}^{-n'}(y)\, d\mu_{D_q}(y),$$

where $n' = n$ for $G_n = \mathrm{GL}_n$ and $n' = n/2$ for $G_n = \mathrm{O}_n$, Sp_n.

4.7. Shifting by nilpotents. In this last subsection, we derive a result which will be needed in the supersymmetric context of Sects. 5.8 and 5.9. Let $\mathfrak{p} := T_o D_q$ be the tangent space of $D_q = K.o$ at the identity $\mathrm{Id} = o$. Since $H = K^{\mathbb{C}}$, $H_o = K_o^{\mathbb{C}}$, $D_q \simeq K/K_o$, and W is the closure of H/H_o, the tangent space $\mathfrak{p}$ is a real form of W. More precisely,

$$\mathfrak{p} = W \cap \mathrm{Lie}(K).$$

Linearizing $\tau(y) = \tau(k\tau(k)^{-1}) = \tau(k)k^{-1} = y^{-1}$ at $y = o$, we note that $\xi \in \mathfrak{p}$ satisfies $\tau_*(\xi) = -\xi$, where τ_* is the differential $\tau_* := d\tau|_o$.

Let now D_q be equipped with the canonical Riemannian geometry in which K acts by isometries and which is induced by the trace form $-\mathrm{Tr}$ on $\mathfrak{p}$. With each $w \in W$ associate a complex vector field $t_w \in \Gamma(D_q, \mathbb{C} \otimes TD_q)$ by

$$(t_w f)(y) = \frac{d}{dt} f(y + tw)\Big|_{t=0}.$$

Lemma 4.12. *The vector field t_w has the divergence*

$$\mathrm{div}(t_w)(y) = -(q - m/2)\,\mathrm{Tr}\,(y^{-1}w),$$

where $m = 0, +1, -1$ *for* $G = \mathrm{GL}$, O, Sp.

Proof. For $y \in D_q$ choose some fixed element $k \in K$ so that $y = k\tau(k^{-1})$. Fixing an orthonormal basis $\{e_\alpha\}$ of $\mathfrak{p}$, define local coordinate functions x^α in a neighborhood of the point y by the equation

$$y' = k\,\mathrm{e}^{x^\alpha(y')e_\alpha}\tau(k^{-1}).$$

By their construction via the exponential mapping, these are Riemann normal coordinates centered at y. The Riemannian metric expands around y as $\sum_\alpha dx^\alpha \otimes dx^\alpha + \ldots$, with vanishing corrections of linear order in the coordinates x^α.

Let $\partial_\alpha = \partial/\partial x^\alpha$ and express the vector field t_w in this basis as $t_w = t_w^\alpha \partial_\alpha$. Differentiating the equation $k^{-1}(y' + tw)\tau(k) = \mathrm{e}^{x^\alpha(y'+tw)e_\alpha}$ with the help of the relation

$$\frac{d}{dt}x^\alpha(y' + tw)\Big|_{t=0} = (\mathscr{L}_{t_w}x^\alpha)(y') = t_w^\alpha(y'),$$

where $\mathscr{L}_{t_w}$ is the Lie derivative w.r.t. the vector field t_w, and then solving the resulting equation for t_w^α, one obtains the following expansion of t_w^α in powers of the x^β:

$$t_w^\alpha(\cdot) = -\mathrm{Tr}\,(k^{-1}w\tau(k)e_\alpha) + \frac{1}{2}x^\beta(\cdot)\mathrm{Tr}\,\big(k^{-1}w\tau(k)(e_\alpha e_\beta + e_\beta e_\alpha)\big) + \ldots.$$

Since the metric tensor is of the locally Euclidean form $\sum_\alpha dx^\alpha \otimes dx^\alpha + \ldots$, the divergence is now readily computed to be

$$\mathrm{div}(t_w)(k\tau(k)^{-1}) = (\partial_\alpha t_w^\alpha)(k\tau(k)^{-1}) = \sum\nolimits_\alpha \mathrm{Tr}\,(k^{-1}w\tau(k)e_\alpha^2).$$

The sum of squares e_α^2 is independent of the choice of basis e_α. Making any convenient choice, a short computation shows that

$$-\sum\nolimits_\alpha e_\alpha^2 = (q - m/2)\,\mathrm{Id},$$

where $m = 0, +1, -1$ for $G = \mathrm{GL}$, O, Sp. The statement of the lemma now follows by inserting this result in the previous formula and recalling $y = k\tau(k^{-1})$. □

Remark. A check on the formula for the sum of squares $-\sum_\alpha e_\alpha^2$ is afforded by the relations $-\mathrm{Tr}\,e_\alpha^2 = 1$ and $\dim_\mathbb{R}\mathfrak{p} = \dim_\mathbb{C} W = q^2$, $2q(q - 1/2)$, $2q(q + 1/2)$ for $G = \mathrm{GL}$, O, Sp. Note also this: defined by the equation $\mathrm{div}(t_w)d\mu_{D_q} = \mathscr{L}_{t_w}d\mu_{D_q}$, the operation of taking the divergence does not depend on the choice of scale for the metric tensor. Therefore we were free to use a normalization convention for the metric which differs from that used elsewhere in this paper.

Lemma 4.13. *Let $F : D_q \to \mathbb{C}$ be an analytic function, and let $N_0 = \oplus_{k\geq 1} \wedge^{2k}(\mathbb{C}^\bullet)$ be the nilpotent even part of a (parameter) Grassmann algebra $\wedge(\mathbb{C}^\bullet)$. Then for any $w \in N_0 \otimes W$ one has*

$$\int_{D_q} F(y + w)\,d\mu_{D_q}(y) = \int_{D_q} \frac{F(y)\,d\mu_{D_q}(y)}{\mathrm{Det}^{q+m/2}(\mathrm{Id} - y^{-1}w)}.$$

Proof. Let t_w be the vector field generating translations $y \mapsto y + sw$ ($s \in \mathbb{R}$). Since w is nilpotent, the exponential $\exp(s\mathscr{L}_{t_w})$ of the Lie derivative $\mathscr{L}_{t_w}$ is a differential operator of finite order. Applying it to the function F one has $(\mathrm{e}^{s\mathscr{L}_{t_w}} F)(y) = F(y + sw)$.

Now for any density Ω on D_q the integral $\int_{D_q} \mathscr{L}_{t_w} \Omega$ vanishes by Stokes' theorem for the closed manifold D_q. Therefore, partial integration gives

$$\int_{D_q} F(y+sw)\, d\mu_{D_q}(y) = \int_{D_q} F(y)\, \mathrm{e}^{-s\mathscr{L}_{t_w}}\, d\mu_{D_q}(y) = \int_{D_q} F(y)\, J_s(y)\, d\mu_{D_q}(y),$$

where $J_s : D_q \to \mathbb{C} \oplus N_0$ is the function defined by $\mathrm{e}^{-s\mathscr{L}_{t_w}} d\mu_{D_q} = J_s\, d\mu_{D_q}$.

We now set up a differential equation for J_s. For this we consider the derivative

$$\frac{d}{ds}\left(\mathrm{e}^{-s\mathscr{L}_{t_w}} d\mu_{D_q}\right) = \mathrm{e}^{-s\mathscr{L}_{t_w}} \left(-\mathscr{L}_{t_w} d\mu_{D_q}\right).$$

By the relation $\mathscr{L}_{t_w} d\mu_{D_q} = \mathrm{div}(t_w) d\mu_{D_q}$ we then get

$$\frac{d}{ds} J_s\, d\mu_{D_q} = -\mathrm{e}^{-s\mathscr{L}_{t_w}} \left(\mathrm{div}(t_w) d\mu_{D_q}\right) = -\mathrm{e}^{-s\mathscr{L}_{t_w}}\, (\mathrm{div}(t_w))\, J_s\, d\mu_{D_q}.$$

Using the expression for $\mathrm{div}(t_w)$ from Lemma 4.12 we obtain the differential equation

$$\frac{d}{ds} \log J_s(y) = (q - m/2)\mathrm{Tr}\,(w(y - sw)^{-1}) = -(q - m/2)\frac{d}{ds}\mathrm{Tr}\,\log(y - sw).$$

The solution of this differential equation with initial condition $J_{s=0} = 1$ is

$$J_s(y) = \frac{\mathrm{Det}^{q-m/2}(y)}{\mathrm{Det}^{q-m/2}(y - sw)},$$

and setting $s = 1$ yields the statement of the lemma. $\square$

5. Full Supersymmetric Situation

We finally tackle the general situation of $V = V_0 \oplus V_1$, where both V_0 and V_1 are non-trivial. The superbosonization formulas (1.10, 1.13) in this situation will be proved by a chain of variable transformations resulting in reduction to the cases treated in the two preceding sections. This proof has the advantage of being constructive.

5.1. More notation. To continue the discussion in the supersymmetric context we need some more notation. If $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded vector space, one calls $(V_0 \cup V_1)\backslash\{0\}$ the subset of homogeneous elements of V. A vector $v \in V_0\backslash\{0\}$ is called even and $v \in V_1\backslash\{0\}$ is called odd. On the subset of homogeneous elements of V one defines a parity function $|\cdot|$ by $|v| = 0$ for v even and $|v| = 1$ for v odd. Whenever the parity function $v \mapsto |v|$ appears in formulas and expressions, the vector v is understood to be homogeneous even without explicit mention.

There exist two graded-commutative algebras that are canonically associated with $V = V_0 \oplus V_1$. To define them, let $T(V) = \oplus_{k=0}^{\infty} T^k(V)$ be the tensor algebra of V, and let $I_\pm(V) \subset T(V)$ be the two-sided ideal generated by multiplication of $T(V)$ with all combinations $v \otimes v' \pm (-1)^{|v||v'|} v' \otimes v$ for homogeneous vectors $v, v' \in V$. Then the *graded-symmetric* algebra of $V = V_0 \oplus V_1$ is the quotient

$$\mathrm{S}(V) := T(V)/I_-(V) \simeq \mathrm{S}(V_0) \otimes \wedge(V_1),$$

which is isomorphic to the tensor product of the symmetric algebra of V_0 with the exterior algebra of V_1. The *graded-exterior* algebra of V is the quotient

$$\wedge(V) := T(V)/I_+(V) \simeq \wedge(V_0) \otimes \mathrm{S}(V_1).$$

Here we have adopted Kostant's language and notation [14].

Recall that our goal is to prove an integration formula for integrands in $\mathscr{A}_V^G$, the graded-commutative algebra of G-equivariant holomorphic functions $f : V_0 \to \wedge(V_1^*)$ with V_0 and V_1 given in (2.3, 2.4). For that purpose we will view the basic algebra $\mathscr{A}_V$ as a completion of the graded-symmetric algebra

$$\mathrm{S}(V^*) = T(V^*)/I_-(V^*) \simeq \mathrm{S}(V_0^*) \otimes \wedge(V_1^*).$$

The latter algebra is $\mathbb{Z}$-graded by $\mathrm{S}(V^*) = \oplus_{k \geq 0} \mathrm{S}^k(V^*)$, where

$$\mathrm{S}^k(V^*) \simeq \bigoplus\nolimits_{l=0}^{k} \Big(\mathrm{S}^l(V_0^*) \otimes \wedge^{k-l}(V_1^*) \Big).$$

The action of G on V preserves the $\mathbb{Z}_2$-grading $V = V_0 \oplus V_1$. Thus G acts on $T(V^*)$ while leaving the two-sided ideal $I_-(V^*)$ invariant, and it therefore makes sense to speak of the subalgebra $\mathrm{S}(V^*)^G$ of G-fixed elements in $\mathrm{S}(V^*)$.

It is a result of R. Howe – see Theorem 2 of [11] – that for each of the cases $G = \mathrm{GL}_n$, O_n, and Sp_n, the graded-commutative algebra $\mathrm{S}(V^*)^G$ is generated by $\mathrm{S}^2(V^*)^G$. Hence, our attention once again turns to the subspace $\mathrm{S}^2(V^*)^G$ of quadratic invariants.

5.2. Quadratic invariants. It follows from the definition of the graded-symmetric algebra $\mathrm{S}(V^*)$ that the subspace of quadratic elements decomposes as $\mathrm{S}^2(V_0^* \oplus V_1^*) = \mathrm{S}^2(V_0^*) \oplus \wedge^2(V_1^*) \oplus (V_0^* \otimes V_1^*)$. So, since G acts on V_0^* and V_1^* we have a decomposition

$$\mathrm{S}^2(V_0^* \oplus V_1^*)^G = \mathrm{S}^2(V_0^*)^G \oplus \wedge^2(V_1^*)^G \oplus (V_0^* \otimes V_1^*)^G.$$

To describe the components let us recall the notation $U_r = \mathbb{C}^r \oplus (\mathbb{C}^r)^*$ for $r = p, q$.

Lemma 5.1. $\mathrm{S}^2(V^*)^G$ *is isomorphic as a* $\mathbb{Z}_2$*-graded complex vector space to* W^**, where the even and odd components of* $W = W_0 \oplus W_1$ *are*

$$W_0 = W_{00} \times W_{11} = \begin{cases} \mathrm{End}(\mathbb{C}^p) \times \mathrm{End}(\mathbb{C}^q), & G = \mathrm{GL}_n, \\ \mathrm{Sym}_s(U_p) \times \mathrm{Sym}_a(U_q), & G = \mathrm{O}_n, \\ \mathrm{Sym}_a(U_p) \times \mathrm{Sym}_s(U_q), & G = \mathrm{Sp}_n, \end{cases}$$

$$W_1 = \begin{cases} \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q), & G = \mathrm{GL}_n, \\ \mathrm{Hom}(U_q, U_p), & G = \mathrm{O}_n, \mathrm{Sp}_n. \end{cases}$$

Proof. Writing $S^2(V_0^*)^G \simeq W_{00}$ and $\wedge^2(V_1^*)^G \simeq W_{11}$, the statement concerning the even part $W_0 \subset W$ is just a summary of Lemmas 3.1, 3.3, and 4.1. Thus what remains to be done is to prove the isomorphism $(V_0^* \otimes V_1^*)^G \simeq W_1^*$ between odd components. Let us prove the equivalent statement $(V_0 \otimes V_1)^G \simeq W_1$.

In the case of $G = \mathrm{GL}_n$ there are two types of invariant: we can compose an element $\tilde{L} \in \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$ with an element $K \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q)$ to form $K\tilde{L} \in \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$, or else compose $\tilde{K} \in \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$ with $L \in \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$ to form $L\tilde{K} \in \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p)$. This already gives the desired statement $(V_0 \otimes V_1)^{\mathrm{GL}_n} \simeq \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$.

In the cases of $G = \mathrm{O}_n, \mathrm{Sp}_n$ we use $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^r) \simeq \mathrm{Hom}((\mathbb{C}^r)^*, (\mathbb{C}^n)^*)$ and the G-equivariant isomorphism $\beta : \mathbb{C}^n \to (\mathbb{C}^n)^*$ to make the identifications

$$V_0 \simeq \mathrm{Hom}(\mathbb{C}^n, U_p), \quad V_1 \simeq \mathrm{Hom}(U_q, \mathbb{C}^n).$$

After this, the G-invariants in $V_0 \otimes V_1$ are seen to be in one-to-one correspondence with composites $L\tilde{K} \in \mathrm{Hom}(U_q, U_p)$, where $\tilde{K} \in \mathrm{Hom}(U_q, \mathbb{C}^n)$ and $L \in \mathrm{Hom}(\mathbb{C}^n, U_p)$. □

Remark. Defining $\mathbb{Z}_2$-graded vector spaces $\mathbb{C}^{p|q} := \mathbb{C}^p \oplus \mathbb{C}^q$ and $U_{p|q} := U_p \oplus U_q$ we could say that $S^2(V)^{\mathrm{GL}_n} \simeq \mathrm{End}(\mathbb{C}^{p|q})$, while $S^2(V)^{\mathrm{O}_n} \simeq S^2(U_{p|q})$ and $S^2(V)^{\mathrm{Sp}_n} \simeq \wedge^2(U_{p|q})$. We will not use these identifications here.

5.3. Pullback from $\mathscr{A}_W$ to $\mathscr{A}_V^G$. With $W = W_0 \oplus W_1$ as specified in Lemma 5.1, consider now the algebra $\mathscr{A}_W$ of holomorphic functions

$$F : W_0 \to \wedge(W_1^*).$$

At the linear level we have the isomorphism of Lemma 5.1, which we here denote by

$$Q_2^* : W^* \to S^2(V^*)^G.$$

This extends in the natural way to an isomorphism of tensor algebras

$$Q_T^* : T(W^*) \to T\big(S^2(V^*)^G\big).$$

Since Q_2^* is an isomorphism of $\mathbb{Z}_2$-graded vector spaces, Q_T^* sends the ideal $I_-(W^*) \subset T(W^*)$ generated by the graded-skew elements $w \otimes w' - (-1)^{|w||w'|} w' \otimes w$ into the ideal $I_-(V^*) \subset T(V^*)$ generated by the same type of element $v \otimes v' - (-1)^{|v||v'|} v' \otimes v$.

Now, taking the quotient of $T(V^*)$ by $I_-(V^*)$ is compatible with the reductive action of G, and it therefore follows that Q_T^* descends to a mapping

$$Q^* : S(W^*) \to S(V^*)^G.$$

Because $Q^*(W^*) = Q_2^*(W^*) = S^2(V^*)^G$ and $S(V^*)^G$ is generated by $S^2(V^*)^G$, the map $Q^* : S(W^*) \to S(V^*)^G$ is surjective.

The same holds true [18] at the level of our holomorphic functions $\mathscr{A}_W$ and $\mathscr{A}_V^G$:

Proposition 5.2. *The homomorphism of algebras $Q^* : \mathscr{A}_W \to \mathscr{A}_V^G$ is surjective.*

5.4. Berezin superintegral form. For a $\mathbb{Z}_2$-graded complex vector space such as our space $V = V_0 \oplus V_1$ with dimensions $\dim V_0 = 2pn$ and $\dim V_1 = 2qn$, we denote by $\mathrm{Ber}(V)$ the complex one-dimensional space

$$\mathrm{Ber}(V) = \wedge^{2pn}(V_0^*) \otimes \wedge^{2qn}(V_1).$$

Let now each of the Hermitian vector spaces V_0 and V_1 be endowed with an orientation. Then there is a canonical top-form $\tilde{\Omega}_{V_0} \in \wedge^{2pn}(V_0^*)$ and a canonical generator $\Omega_{V_1} \in \wedge^{2qn}(V_1)$. Their tensor product $\Omega_V := \tilde{\Omega}_{V_0} \otimes \Omega_{V_1} \in \mathrm{Ber}(V)$ is called the (flat) Berezin superintegral form of V. Such a form Ω_V determines a linear mapping

$$\Omega_V : \mathscr{A}_V \to \Gamma(V_0, \wedge^{2pn}(V_0^*)), \quad f \mapsto \Omega_V[f],$$

from the algebra of holomorphic functions $f : V_0 \to \wedge(V_1^*)$ to the space of top-degree holomorphic differential forms on V_0. Indeed, if v is any element of V_0, then by pairing $f(v) \in \wedge(V_1^*)$ with the second factor Ω_{V_1} of Ω_V we get a complex number, and subsequent multiplication by the first factor $\tilde{\Omega}_{V_0}$ results in an element of $\wedge^{2pn}(V_0^*)$.

In keeping with the approach taken in Sect. 3, we want to integrate over the real vector space $V_{0,\mathbb{R}}$ defined as the graph of $\dagger : \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \to \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$. For this, let $\mathrm{dvol}_{V_{0,\mathbb{R}}}$ denote the positive density $\mathrm{dvol}_{V_{0,\mathbb{R}}} := |\tilde{\Omega}_{V_0}|$ restricted to $V_{0,\mathbb{R}}$. (This change from top-degree forms to densities is made in anticipation of the fact that we will transfer the integral to a symmetric space which in certain cases is non-orientable; see the Appendix for more discussion of this issue.)

The Berezin superintegral of $f \in \mathscr{A}_V$ over the integration domain $V_{0,\mathbb{R}}$ is now defined as the two-step process of first converting the integrand $f \in \mathscr{A}_V$ into a holomorphic function $\Omega_{V_1}[f] : V_0 \to \mathbb{C}$ and then integrating this function against $\mathrm{dvol}_{V_{0,\mathbb{R}}}$ over the real subspace $V_{0,\mathbb{R}}$:

$$f \mapsto \int_{V_{0,\mathbb{R}}} \Omega_{V_1}[f] \, \mathrm{dvol}_{V_{0,\mathbb{R}}}.$$

Our interest in the following will be in this kind of integral for the particular case of G-equivariant holomorphic functions $f : V_0 \to \wedge(V_1^*)$ (i.e., for $f \in \mathscr{A}_V^G$).

5.5. Exploiting equivariance. Recall from Sect. 3 the definition of the groups K_n, K_p, $K_{n,p}$, and G_p. Recall also that $X_{p,n} = \psi(\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p))$ denotes the vector space of structure-preserving linear transformations $\mathbb{C}^n \to U_p$. To simplify the notation, let the isomorphism ψ now be understood, i.e., write $\psi(L) \equiv L$.

The subset of regular elements in $X_{p,n}$ is denoted by $X'_{p,n}$. Taking $\Pi \in X'_{p,n}$ to be the orthogonal projector $\mathbb{C}^n = U_p \oplus U_{n,p} \to U_p$ we have the isomorphism

$$G_p \times_{(K_p \times K_{n,p})} K_n \xrightarrow{\sim} X'_{p,n}, \quad (g, k) \mapsto g \Pi k.$$

Note that $X'_{p,n}$ is a left G_p-space and a right K_n-space. Note also the relations $\Pi k = k \Pi$ for $k \in K_p$ and $\Pi k = 0$ for $k \in K_{n,p}$.

Since the compact subgroup $K_n \subset G$ acts on $V_{0,\mathbb{R}}$, the given integrand $f \in \mathscr{A}_V^G$ restricts to a function $f : X'_{p,n} \to \wedge(V_1^*)$ which has the property of being K_n-equivariant:

$$f(L) = f(g \Pi k) = k^{-1}.f(g \Pi) \quad (k \in K_n).$$

Now notice that since the action of G on $\wedge^{2qn}(V_1)$ is trivial, the Berezin form Ω_{V_1} is invariant under G and hence invariant under the subgroup K_n :

$$k(\Omega_{V_1}) = \Omega_{V_1} \circ k^* = \Omega_{V_1} \quad (k \in K_n).$$

Consequently, applying Ω_{V_1} to the K_n-equivariant function f we obtain

$$\Omega_{V_1}[f(g\Pi\, k)] = \Omega_{V_1}[k^{-1}.f(g\Pi)] = \Omega_{V_1}[f(g\Pi)],$$

and this gives the following formula for the integral of f,

$$\int \Omega_{V_1}[f(L)]\,\mathrm{dvol}_{V_{0,\mathbb{R}}}(L) = \frac{\mathrm{vol}(K_n)}{\mathrm{vol}(K_{n,p})} \int\limits_{G_p/K_p} \Omega_{V_1}[f(g\Pi)]\, J(g) dg_{K_p}, \tag{5.1}$$

as an immediate consequence of Prop. 3.14.

Based on this formula, our next step is to process the integrand $\Omega_{V_1}[f(g\Pi)]$.

5.6. Transforming the Berezin integral. It will now be convenient to regard the odd vector space $V_1 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$ for the case of $G = \mathrm{GL}_n$ as

$$V_1 \simeq \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n) \oplus \mathrm{Hom}((\mathbb{C}^q)^*, (\mathbb{C}^n)^*) \qquad (G = \mathrm{GL}).$$

In the other cases, using the isomorphism $\beta : \mathbb{C}^n \to (\mathbb{C}^n)^*$ we make the identification

$$V_1 \simeq \mathrm{Hom}(\mathbb{C}^q \oplus (\mathbb{C}^q)^*, \mathbb{C}^n) = \mathrm{Hom}(U_q, \mathbb{C}^n), \qquad (G = \mathrm{O}, \mathrm{Sp}).$$

Following Sect. 3.4 we fix an orthogonal decomposition $\mathbb{C}^n = U_p \oplus U_{n,p}$ for $G = \mathrm{O}_n$, Sp_n and $\mathbb{C}^n = \mathbb{C}^p \oplus \mathbb{C}^{n-p}$ for $G = \mathrm{GL}_n$, which is Euclidean, Hermitian symplectic, and Hermitian, respectively, and let this induce a vector space decomposition $V_1 = V_\parallel \oplus V_\perp$ in the natural way. For $G = \mathrm{O}_n, \mathrm{Sp}_n$ the summands are

$$V_\parallel = \mathrm{Hom}(U_q, U_p), \quad V_\perp = \mathrm{Hom}(U_q, U_{n,p}) \qquad (G = \mathrm{O}, \mathrm{Sp}),$$

and in the case of $G = \mathrm{GL}_n$ we have

$$\begin{aligned} V_\parallel &= \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}((\mathbb{C}^q)^*, (\mathbb{C}^p)^*), \\ V_\perp &= \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^{n-p}) \oplus \mathrm{Hom}((\mathbb{C}^q)^*, (\mathbb{C}^{n-p})^*) \qquad (G = \mathrm{GL}). \end{aligned}$$

From the statement of Lemma 5.1 we see that $V_\parallel$ is isomorphic to W_1 as a complex vector space (though not as a K_p-space) in all three cases.

The decomposition $V_1 = V_\parallel \oplus V_\perp$ induces a factorization

$$\wedge(V_1^*) \simeq \wedge(V_\parallel^*) \otimes \wedge(V_\perp^*) \tag{5.2}$$

of the exterior algebra of V_1^*. In all three cases (GL, O, Sp) the decomposition of V_1 and that of $\wedge(V_1^*)$ is stabilized by the group $K_p \times K_{n,p}$. We further note that $K_p \hookrightarrow K_n$ acts trivially on $\wedge(V_\perp^*)$ while $K_{n,p} \hookrightarrow K_n$ acts trivially on $\wedge(V_\parallel^*)$.

To compute $\Omega_{V_1}[f(g\Pi)]$, we are going to disect the Berezin form Ω_{V_1} according to the decomposition (5.2). For this, recall that if $V = A \oplus A^*$ is the direct sum of an N-dimensional vector space A and its dual A^*, then there exists a canonical generator $\Omega_V = \Omega_{A\oplus A^*} \in \wedge^{2N}(A \oplus A^*)$ which is given by $\Omega_{A\oplus A^*} = f_N \wedge e_N \wedge \ldots \wedge f_1 \wedge e_1$ for any basis $\{e_j\}$ of A with dual basis $\{f_j\}$ of A^*. Note that $\Omega_{A\oplus A^*} = (-1)^N \Omega_{A^*\oplus A}$.

The following statement is an immediate consequence of the properties of $\wedge$.

Lemma 5.3. *If A, B, C are vector spaces and $A = B \oplus C$ then*

$$\Omega_{A\oplus A^*} = \Omega_{B\oplus B^*} \wedge \Omega_{C\oplus C^*}.$$

Now V_1 and all of our spaces $V_\perp$ and $V_\parallel$ are the direct sum of a vector space and its dual. Recall from Sect. 4.3 that $\Omega_{V_1} = \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,(\mathbb{C}^n)^*)\oplus\mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^n)}$, and let

$$\Omega_{V_\parallel} = \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,(\mathbb{C}^p)^*)\oplus\mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p)}, \quad \Omega_{V_\perp} = \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,(\mathbb{C}^{n-p})^*)\oplus\mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^{n-p})}$$

for $G = \mathrm{GL}$, while in the case of $G = \mathrm{O}, \mathrm{Sp}$ the corresponding definitions are

$$\Omega_{V_\parallel} = \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,\,U_p)\oplus\mathrm{Hom}(\mathbb{C}^q,\,U_p)}, \quad \Omega_{V_\perp} = \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,\,U_{n,p})\oplus\mathrm{Hom}(\mathbb{C}^q,\,U_{n,p})}.$$

Here the vector spaces $\mathrm{Hom}((\mathbb{C}^q)^*,\ U_p)$ and $\mathrm{Hom}(\mathbb{C}^q,\ U_p)$ are regarded as dual to each other by the symmetric bilinear form $s:\ U_p \times U_p \to \mathbb{C}$ for $G = \mathrm{O}$ and the alternating bilinear form $a:\ U_p \times U_p \to \mathbb{C}$ for $G = \mathrm{Sp}$. This means that for $G = \mathrm{O}$ we have

$$\Omega_{V_\parallel} = \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,(\mathbb{C}^p)^*)\oplus\mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p)} \wedge \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,\mathbb{C}^p)\oplus\mathrm{Hom}(\mathbb{C}^q,(\mathbb{C}^p)^*)},$$

while the same Berezin form $\Omega_{V_\parallel}$ for $G = \mathrm{Sp}$ has an extra sign factor $(-1)^{pq}$ due to the alternating property of a (cf. the sentence after the definition of Ω_{W_1} in Eq. (1.12)). The same conventions hold good in the case of $\Omega_{V_\perp}$. For this we need only observe that the given symmetric or alternating bilinear form on $U_{n,p}$ induces such a form on $V_\perp$.

Now, applying Lemma 5.3 to the present situation we always have

$$\Omega_{V_1} = \Omega_{V_\parallel} \wedge \Omega_{V_\perp}.$$

5.6.1. Transformation of $\Omega_{V_\parallel}$. Recall the isomorphism of vector spaces $V_\parallel \simeq W_1$, which we now realize as follows. Using the identifications $V_\parallel \simeq \mathrm{Hom}(U_q, U_p)$ for the case of $G = \mathrm{O}, \mathrm{Sp}$ and $V_\parallel \simeq \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}((\mathbb{C}^q)^*, (\mathbb{C}^p)^*)$ for $G = \mathrm{GL}$, we apply $g \in G_p$ to $v \in V_\parallel$ to form gv, where gv for $G = \mathrm{GL}$ means $gv = g.(\tilde{L} \oplus L^{\mathrm{t}}) = (g\tilde{L}) \oplus (\overline{g}L^{\mathrm{t}})$. Note that the mapping $(g, v) \mapsto gv$ has the property of being K_p-invariant.

Given this isomorphism $g:\ V_\parallel \to W_1$, let $(g^{-1})^*:\ \wedge(V_\parallel^*) \to \wedge(W_1^*)$, $f \mapsto g.f$, be the induced isomorphism preserving the pairing between vectors and forms. Then

$$\Omega_{V_1}[f(g\Pi)] = \big(g(\Omega_{V_\parallel}) \wedge \Omega_{V_\perp}\big)\,[g.f(g\Pi)],$$

so our next step is to compute $g(\Omega_{V_\parallel})$. Here it should be stressed that we define the Berezin form Ω_{W_1} by the same ordering conventions we used to define $\Omega_{V_\parallel}$ above.

Lemma 5.4. *Under the isomorphism $V_\parallel \to W_1$ by $v \mapsto gv$ the Berezin forms Ω_{W_1} and $\Omega_{V_\parallel}$ are related by $g(\Omega_{V_\parallel}) = \mathrm{Det}^q(gg^\dagger)\Omega_{W_1}$.*

Proof. Consider first the case of $G = \mathrm{O}, \mathrm{Sp}$, where $V_\parallel = \mathrm{Hom}(U_q, U_p)$ and the same choice of polarization $\mathrm{Hom}(U_q, U_p) = \mathrm{Hom}((\mathbb{C}^q)^*, U_p) \oplus \mathrm{Hom}(\mathbb{C}^q, U_p)$ determines both $\Omega_{V_\parallel}$ and Ω_{W_1}. Applying $g \in G_p$ to $\Omega_{V_\parallel} \in \wedge^{4pq}(V_\parallel)$ we obtain

$$g(\Omega_{V_\parallel}) = \mathrm{Det}^{2q}(g)\Omega_{W_1}.$$

The groups at hand are $G_p = \mathrm{GL}_{2p}(\mathbb{R}), \mathrm{GL}_p(\mathbb{H})$, and $\mathrm{Det}(g)$ is real for these. Hence

$$\mathrm{Det}^{2q}(g) = \big(\mathrm{Det}(g)\overline{\mathrm{Det}(g)}\,\big)^q = \mathrm{Det}^q(gg^\dagger).$$

In the case of $G = \mathrm{GL}$ we have $\Omega_{V_\parallel} = \Omega_{\mathrm{Hom}((\mathbb{C}^q)^*,(\mathbb{C}^p)^*)\oplus\mathrm{Hom}(\mathbb{C}^q,\mathbb{C}^p)}$. Transforming the second summand by $\tilde{L} \mapsto g\tilde{L}$ for $g \in \mathrm{GL}_p(\mathbb{C})$ we get the Jacobian $\mathrm{Det}^q(g)$, transforming the first summand by $L^{\mathrm{t}} \mapsto \overline{g}L^{\mathrm{t}}$ we get $\mathrm{Det}^q(\overline{g})$. Thus, altogether we obtain again $g(\Omega_{V_\parallel}) = \mathrm{Det}^q(g)\mathrm{Det}^q(\overline{g})\Omega_{W_1} = \mathrm{Det}^q(gg^\dagger)\Omega_{W_1}$. $\square$

5.6.2. Bosonization of $\Omega_{V_\perp}$. We turn to the Berezin form $\Omega_{V_\perp}$ for the factor $\wedge(V_\perp^*)$ of the decomposition (5.2). Recall that the elements of this exterior algebra $\wedge(V_\perp^*)$ are fixed under the action of K_p. Since $\Pi\, k = 0$ for $k \in K_{n,p}$, the K_n-equivariance of $f \in \mathscr{A}_V^G$ implies that $g.f(g\Pi)$ (for any fixed $g \in G_p$) lies in $\wedge(W_1^*) \otimes \wedge(V_\perp^*)^{K_{n,p}}$.

For future reference, we are now going to record a (bosonization) formula for the Berezin integral $\Omega_{V_\perp} : \wedge(V_\perp^*)^{K_{n,p}} \to \mathbb{C}$. For this notice that, since the action of $K_{n,p}$ is complex linear we have $\wedge(V_\perp^*)^{K_{n,p}} = \wedge(V_\perp^*)^{G_{n,p}}$, where $G_{n,p}$ is the complexification of $K_{n,p}$. From Table 2 of Sect. 3.4 we read off that $G_{n,p} = \mathrm{GL}_{n-p}(\mathbb{C})$, $\mathrm{O}_{n-2p}(\mathbb{C})$, and $\mathrm{Sp}_{n-2p}(\mathbb{C})$ for our three cases of GL, O, and Sp, respectively.

The subalgebra $\wedge(V_\perp^*)^{G_{n,p}}$ is generated, once again, by $\wedge^2(V_\perp^*)^{G_{n,p}}$, the quadratic invariants. Applying Lemma 4.1 with $V^* \equiv V_\perp^*$, $G \equiv G_{n,p}$, and $W \equiv W_{11}$ we get

$$\wedge^2(V_\perp^*)^{K_{n,p}} = \wedge^2(V_\perp^*)^{G_{n,p}} \simeq W_{11}^*.$$

Now by the principles expounded in Sect. 4 we lift a given element $f_\perp \in \wedge(V_\perp^*)^{G_{n,p}}$ to a holomorphic function $F : W_{11} \to \mathbb{C}$. To formalize this step, let

$$P_\perp^* \; : \; \mathscr{O}(W_{11}) \to \wedge(V_\perp^*)^{K_{n,p}}$$

be the surjective mapping which was introduced in Sect. 4.1 and denoted by the generic symbol Q^* there. For $F \in \mathscr{O}(W_{11})$ we then have with $n' = n/(1+|m|)$ the result

$$\Omega_{V_\perp}[P_\perp^* F] = (2\pi)^{qn(1-p/n')} 2^{qm} \frac{\mathrm{vol}(K_{n,p})}{\mathrm{vol}(K_{n,p-q})} \int_{D_q^1} \frac{F(y)\, d\mu_{D_q^1}(y)}{\mathrm{Det}^{n'-p}(y)}, \tag{5.3}$$

as an immediate consequence of the formula of Thm. 4.11. Here we refined our notation by writing D_q^1 for the compact symmetric spaces D_q of Sect. 4.6. The non-compact symmetric spaces D_p introduced in Sect. 3.4 will henceforth be denoted by D_p^0.

5.7. Decomposition of pullback. Recall from Sect. 5.3 that we have a pullback of graded-commutative algebras $Q^* : \mathscr{A}_W \to \mathscr{A}_V^G$. To go further, we should decompose Q^* according to the manipulations carried out in the previous two subsections. This, however, will only be possible in a restricted sense, as some of our transformations require that the even part of $w \in W$ be invertible.

We start with a summary of the sequence of operations we have carried out so far. Recall that the elements of $\mathscr{A}_W$ are holomorphic functions $F : W_0 \to \wedge(W_1^*)$, where W_1 and $W_0 = W_{00} \times W_{11}$ were described in Lemma 5.1. Since our domain of integration will be $D_p \equiv D_p^0 \simeq G_p/K_p$, given $F \in \mathscr{A}_W$ let F_1 denote F restricted to $D_p^0 \subset W_{00}$:

$$F_1 : D_p^0 \times W_{11} \to \wedge(W_1^*). \tag{5.4}$$

Now we use the Cartan embedding $G_p/K_p \to D_p^0 \subset G_p$ by $g \mapsto g\theta(g^{-1}) = gg^\dagger$ to pull back F_1 in its first argument from D_p^0 to G_p/K_p. Applying also the mapping $P_\perp^* : \mathscr{O}(W_{11}) \to \wedge(V_\perp^*)^{K_{n,p}}$ we go to the second function

$$F_2 : G_p/K_p \to \wedge(W_1^*) \otimes \wedge(V_\perp^*)^{K_{n,p}}. \tag{5.5}$$

In the next step, employing the isomorphism $V_\| \to W_1$, $v \mapsto gv$ (pointwise for each coset $gK_p \in G_p/K_p$) we pull back F_2 to a K_p-equivariant function

$$F_3 \, : \, G_p \to \wedge(V_\|^*) \otimes \wedge(V_\perp^*)^{K_{n,p}}. \tag{5.6}$$

Be advised that we are now at the level of the integrand $F_3(g) = f(g\Pi)$ of (5.1). In the final step, we pass to the unique extension of F_3 to a K_n-equivariant function

$$F_4 \, : \, X'_{p,n} \stackrel{K_n-\text{eqvt}}{\longrightarrow} \wedge(V_1^*) \tag{5.7}$$

by $F_4(L) = F_4(g\Pi k) := F_3(g)$. Let us give a name to this sequence of steps.

Definition 5.5. *We denote by P^* the homomorphism of graded-commutative algebras taking $F_1 \, : \, D_p^0 \times W_{11} \to \wedge(W_1^*)$ to the K_n-equivariant function $F_4 \, : \, X'_{p,n} \to \wedge(V_1^*)$.*

The main point of this subsection will be to show that Q^* (restricted to $D_p^0 \times W_{11}$) is the composition of P^* with another homomorphism, S^*, which we describe next.

Consider first the case of $G = \mathrm{GL}$, where $W = W_0 \oplus W_1$ and

$$\begin{aligned} W_0 &= W_{00} \oplus W_{11} = \mathrm{End}(\mathbb{C}^p) \oplus \mathrm{End}(\mathbb{C}^q), \\ W_1 &= W_{01} \oplus W_{10} = \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q), \end{aligned}$$

and let W'_{00} denote the subset of regular elements in W_{00}. On $W' := (W'_{00} \times W_{11}) \times (W_{01} \oplus W_{10})$ define a non-linear mapping $S \, : \, W' \to W'$ by

$$S(x, y\,; \sigma, \tau) = (x, y + \tau x^{-1} \sigma; \sigma, \tau).$$

This mapping is compatible with the structure of the graded-commutative algebra $\mathscr{A}_W$ which is induced from the $\mathbb{Z}_2$-grading $W = W_0 \oplus W_1$. Therefore, viewing the entries of σ and τ as anti-commuting generators, S determines an automorphism $S^* \, : \, \mathscr{A}'_W \to \mathscr{A}'_W$ of the superalgebra $\mathscr{A}'_W$ of holomorphic functions from $W'_{00} \times W_{11}$ to $\wedge(W_1^*)$. Adopting the supermatrix notation commonly used in physics one would write

$$(S^* F) \begin{pmatrix} x & \sigma \\ \tau & y \end{pmatrix} = F \begin{pmatrix} x & \sigma \\ \tau & y + \tau x^{-1} \sigma \end{pmatrix}.$$

Next consider the case of $G = \mathrm{O}_n$, where $W_{00} = \mathrm{Sym}_s(U_p)$, $W_{11} = \mathrm{Sym}_a(U_q)$, and $W_1 = \mathrm{Hom}(U_q, U_p)$. Here we have the bilinear forms s on U_p and a on U_q, and these determine an isomorphism $\mathrm{Hom}(U_q, U_p) \to \mathrm{Hom}(U_p, U_q)$, $\sigma \mapsto \sigma^T$ by

$$s(\sigma v, u) = a(v, \sigma^T u) \quad (u \in U_p, \ v \in U_q).$$

However, since σ and v in this definition are to be considered as odd and σ moves past v, the good isomorphism to use (the 'supertranspose') has an extra minus sign:

$$\sigma \mapsto \sigma^{\mathrm{s}T} := -\sigma^T.$$

Restricting again to the regular elements W'_{00} of W_{00}, define a mapping $S \, : \, W' \to W'$ on $W' = (W'_{00} \times W_{11}) \times W_1$ by

$$S(x, y\,; \sigma) = (x, y + \sigma^{\mathrm{s}T} x^{-1} \sigma; \sigma).$$

From $x^{-1} \in \mathrm{Sym}_s(U_p)$ and the definition of the transposition operation $\sigma \mapsto \sigma^{sT}$ via the bilinear forms s and a, it is immediate that $\sigma^{sT} x^{-1} \sigma \in W_{11}$. Now for the same reasons as before, S determines an automorphism $S^* : \mathscr{A}'_W \to \mathscr{A}'_W$.

The definitions for the last case $G = \mathrm{Sp}_n$ are the same as for $G = \mathrm{O}_n$ but for the fact that the two bilinear forms s and a exchange roles.

From here on we consider S^* to be restricted to the functions with domain $D_p^0 \times W_{11}$.

Lemma 5.6. *The homomorphism of superalgebras $Q^* : \mathscr{A}_W \to \mathscr{A}_V^G$, when restricted to a homomorphism Q^* taking functions $D_p^0 \times W_{11} \to \wedge(W_1^*)$ to K_n-equivariant functions $X'_{p,n} \to \wedge(V_1^*)$, decomposes as*

$$Q^* = P^* S^*.$$

Proof. Since the isomorphism $Q^* : W^* \xrightarrow{\sim} \mathrm{S}^2(V^*)^G$ determines $Q^* : \mathscr{A}_W \to \mathscr{A}_V^G$, it suffices to check $Q^* = P^* S^*$ at the level of the quadratic map $Q : V \to W$.

Let us write out the proof for the case of $G = \mathrm{GL}$ (the other cases are no different). Recall that the quadratic map $Q : V \to W$ in this case is given by

$$Q : (L \oplus \tilde{L}) \oplus (K \oplus \tilde{K}) \mapsto \begin{pmatrix} L\tilde{L} & L\tilde{K} \\ K\tilde{L} & K\tilde{K} \end{pmatrix}.$$

Now, fixing a regular element $(L, L^\dagger) \in X'_{p,n}$, we have an orthogonal decomposition

$$\mathbb{C}^n = \ker(L) \oplus \mathrm{im}(L^\dagger),$$

where $\mathrm{im}(L^\dagger) \simeq \mathbb{C}^p$ and $\ker(L) \simeq \mathbb{C}^{n-p}$. Let $\Pi_L := L^\dagger (L L^\dagger)^{-1} L$ denote the orthogonal projection $\Pi_L : \mathbb{C}^n \to \mathrm{im}(L^\dagger)$. If we decompose $K, \tilde{K}$ as

$$K = K_\parallel(L) + K_\perp(L), \quad K_\parallel(L) = K\,\Pi_L, \quad \tilde{K} = \tilde{K}_\parallel(L) + \tilde{K}_\perp(L), \quad \tilde{K}_\parallel(L) = \Pi_L\,\tilde{K},$$

then our homomorphism P^* is the pullback of algebras determined by the map

$$P : X'_{p,n} \times V_1 \to W',$$
$$((L, L^\dagger), (K \oplus \tilde{K})) \mapsto \begin{pmatrix} L L^\dagger & L\,\tilde{K}_\parallel(L) \\ K_\parallel(L) L^\dagger & K_\perp(L)\tilde{K}_\perp(L) \end{pmatrix} = \begin{pmatrix} L L^\dagger & L\,\tilde{K} \\ K L^\dagger & K(\mathrm{Id} - \Pi_L)\tilde{K} \end{pmatrix}.$$

When the second map $S : W' \to W'$ is applied to this result, all blocks remain the same but for the W_{11}-block, which transforms as

$$K(\mathrm{Id} - \Pi_L)\tilde{K} \mapsto K(\mathrm{Id} - \Pi_L)\tilde{K} + (K L^\dagger)(L L^\dagger)^{-1}(L\,\tilde{K}) = K\tilde{K}.$$

Thus $S \circ P$ agrees with Q on $X'_{p,n} \times V_1$, which implies the desired result $Q^* = P^* S^*$. □

We now state an intermediate result en route to the proof of the superbosonization formula. Let $f \in \mathscr{A}_V^G$ and $F \in \mathscr{A}_W$ be related by $f = Q^* F$. We then do the following steps: (i) start from formula (5.1) for $\int \Omega_{V_1}[P^* S^* F]\,\mathrm{dvol}_{V_{0,\mathbb{R}}}$; (ii) transform the Berezin integral $\Omega_{V_1}[(P^* S^* F)(g\Pi)]$ by Lemma 5.4 for the part $\Omega_{V_\parallel}$ and Eq. (5.3) for

$\Omega_{V_\perp}$; (iii) use Cor. 3.16 to push the integral over G_p/K_p forward to D_p^0 by the Cartan embedding; (iv) use $\mathrm{Det}^q(gg^\dagger) = \mathrm{Det}^q(x)$. The outcome of these steps is the formula

$$\int \Omega_{V_1}[f]\,\mathrm{dvol}_{V_{0,\mathbb{R}}} = 2^{(q-p)(n+m)}\pi^{qn}(2\pi)^{-pq(1+|m|)}\frac{\mathrm{vol}(K_n)}{\mathrm{vol}(K_{n,p-q})} \times \int_{D_p^0}\left(\int_{D_q^1}\Omega_{W_1}[S^*F(x,y)]\,\mathrm{Det}^{p-n'}(y)\,d\mu_{D_q^1}(y)\right)\mathrm{Det}^{q+n'}(x)\,d\mu_{D_p^0}(x). \tag{5.8}$$

Let us recall once more that $n' = n$ for $G = \mathrm{GL}$ and $n' = n/2$ for $G = \mathrm{O}, \mathrm{Sp}$.

5.8. Superbosonization formula. We are now in a position to reap the fruits of all our labors. Introducing the notation $(S_x^*\,\mathrm{Det})(y) = \mathrm{Det}(y + \tau x^{-1}\sigma)$ for $G = \mathrm{GL}$ and $(S_x^*\,\mathrm{Det})(y) = \mathrm{Det}(y + \sigma^{sT}x^{-1}\sigma)$ for $G = \mathrm{O}$ and $G = \mathrm{Sp}$, we note that the superdeterminant function $\mathrm{SDet} : D_p^0 \times D_q^1 \to \wedge(W_1^*)$ is given by

$$\mathrm{SDet}(x,y) = \frac{\mathrm{Det}(x)}{((S_x^*)^{-1}\mathrm{Det})(y)}.$$

We define a related function $J : D_p^0 \times D_q^1 \to \wedge(W_1^*)$ by

$$J(x,y) = \frac{\mathrm{Det}^q(x)\,\mathrm{Det}^{q-m/2}(y)}{((S_x^*)^{-1}\mathrm{Det}^{q-m/2-p})(y)}.$$

Theorem 5.7. *Let $f : V_0 \to \wedge(V_1^*)^G$ be a G-equivariant holomorphic function which restricts to a Schwartz function along the real subspace $V_{0,\mathbb{R}}$. If $F : W_0 \to \wedge(W_1^*)$ is any holomorphic function that pulls back to $Q^*F = f$, then*

$$\int \widetilde{\Omega}_{V_1}[f]\,\mathrm{dvol}_{V_{0,\mathbb{R}}} = 2^{(q-p)m}\frac{\mathrm{vol}(K_n)}{\mathrm{vol}(K_{n,p-q})} \times \int_{D_p^0}\int_{D_q^1}\widetilde{\Omega}_{W_1}[(J\cdot\mathrm{SDet}^{n'}\cdot F)(x,y)]\,d\mu_{D_q^1}(y)\,d\mu_{D_p^0}(x),$$

where $n' = n/(1+|m|) \geq p$, and $\widetilde{\Omega}_{V_1} := 2^{pn}(2\pi)^{-qn}\Omega_{V_1}$ and $\widetilde{\Omega}_{W_1} := (2\pi)^{-pq(1+|m|)}$ Ω_{W_1} are Berezin integral forms with adjusted normalization.

Proof. We first observe that in the present context the formula of Lemma 4.13 can be written as

$$\int_{D_q^1}(S^*F)(x,y)\,d\mu_{D_q^1}(y) = \int_{D_q^1}\frac{F(x,y)\,\mathrm{Det}^{q-m/2}(y)}{((S_x^*)^{-1}\mathrm{Det}^{q-m/2})(y)}\,d\mu_{D_q^1}(y).$$

Our starting point now is Eq. (5.8). We interchange the linear operations of doing the ordinary integral $\int_{D_q}(\ldots)\,d\mu_{D_q}$ and the Berezin integral $\Omega_{W_1}[\ldots]$. The inner integral over y is then transformed as

$$\int_{D_q^1}(S^*F)(x,y)\,\mathrm{Det}^{p-n'}(y)\,d\mu_{D_q^1}(y) = \int_{D_q^1}\frac{F(x,y)\,\mathrm{Det}^{q-m/2}(y)\,d\mu_{D_q^1}(y)}{((S_x^*)^{-1}\mathrm{Det}^{n'-p+q-m/2})(y)}.$$

The factor $((S_x^*)^{-1}\mathrm{Det}^{-n'})(y)$ combines with the factor $\mathrm{Det}^{n'}(x)$ of the outer integral over x to give the power of a superdeterminant:

$$((S_x^*)^{-1}\mathrm{Det}^{-n'})(y)\,\mathrm{Det}^{n'}(x) = \mathrm{SDet}^{n'}(x, y).$$

Then, restoring the integrations to their original order (i.e., Berezin integral first, integral over y second) we immediately arrive at the formula of the theorem. □

Remark. The function $J(x, y)$ is just the factor that appears in the definition of the Berezin measure DQ in Sect. 1.2. Using supermatrix notation, this is seen from the following computation:

$$\begin{aligned} J(x, y) &= \frac{\mathrm{Det}^q(x)\,\mathrm{Det}^{q-m/2}(y)}{\mathrm{Det}^{q-m/2-p}(y-\tau\,x^{-1}\sigma)} = \frac{\mathrm{Det}^q(x)\,\mathrm{Det}^p(y-\tau\,x^{-1}\sigma)}{\mathrm{Det}^{q-m/2}(1-y^{-1}\tau\,x^{-1}\sigma)} \\ &= \frac{\mathrm{Det}^q(x)\,\mathrm{Det}^p(y-\tau\,x^{-1}\sigma)}{\mathrm{Det}^{-q+m/2}(1-x^{-1}\sigma\,y^{-1}\tau)} = \frac{\mathrm{Det}^q(x-\sigma\,y^{-1}\tau)\,\mathrm{Det}^p(y-\tau\,x^{-1}\sigma)}{\mathrm{Det}^{m/2}(1-x^{-1}\sigma\,y^{-1}\tau)}. \end{aligned}$$

We also have adjusted the normalization constants, so that $\mathrm{dvol}_{V_{0,\mathbb{R}}} \otimes \widetilde{\Omega}_{V_1}$ agrees with the Berezin superintegral form $D_{Z,\tilde{Z};\zeta,\tilde{\zeta}}$ of Eq. (1.1), and $d\mu_{D_p^0}\, d\mu_{D_q^1} \otimes \widetilde{\Omega}_{W_1} \circ J$ agrees with DQ as defined in Eqs. (1.8, 1.11). Thus, assuming the validity of Thm. 4.11 we have now completed the proof of our main formulas (1.10) and (1.13). To complete the proof of Thm. 4.11 we have to establish the normalization given by Lemma 4.10.

5.9. Proof of Lemma 4.10. Lemma 4.10 states the value of the integral

$$\int_{D_q^1} \mathrm{e}^{\mathrm{Tr}' y}\, \mathrm{Det}^{-n'}(y)\, d\mu_{D_q^1}(y)$$

over the compact symmetric space D_q^1. To verify that statement, we are now going to compute this integral by supersymmetric reduction to a related integral,

$$\int_{D_q^0} \mathrm{e}^{-\mathrm{Tr}' x}\, \mathrm{Det}^{n'+q}(x)\, d\mu_{D_q^0}(x),$$

over the corresponding non-compact symmetric space D_q^0. For that purpose, consider

$$C_{n,q} := \int_{D_q^0}\int_{D_q^1} \Omega_{W_1}[(J\cdot \mathrm{SDet}^{n'+q})(x, y)]\, \mathrm{e}^{\mathrm{Tr}' y - \mathrm{Tr}' x} d\mu_{D_q^1}(y)\, d\mu_{D_q^0}(x), \quad (5.9)$$

(for each of the three cases $G = \mathrm{GL}, \mathrm{O}, \mathrm{Sp}$) and first process the inner integral:

$$\begin{aligned} &\int_{D_q^1} \Omega_{W_1}[(J\cdot \mathrm{SDet}^{n'+q})(x, y)]\, \mathrm{e}^{\mathrm{Tr}' y}\, d\mu_{D_q^1}(y) \\ &= \int_{D_q^1} \Omega_{W_1}\left[\frac{\mathrm{Det}^{n'+2q}(x)\,\mathrm{Det}^{q-m/2}(y)}{((S_x^*)^{-1}\mathrm{Det}^{n'+q-m/2})(y)}\right] \mathrm{e}^{\mathrm{Tr}' y}\, d\mu_{D_q^1}(y) \\ &= \mathrm{Det}^{n'+2q}(x) \int_{D_q^1} \Omega_{W_1}[S_x^*(\exp\circ \mathrm{Tr}')(y)]\, \mathrm{Det}^{-n'}(y)\, d\mu_{D_q^1}(y). \end{aligned}$$

Here, after inserting the definitions of $\mathrm{SDet}^{n'+q}$ and J for $p = q$, we again made use of the formula of Lemma 4.13, reading it backwards this time.

The next step is to calculate the Berezin integral Ω_{W_1} of $S_x^*(\exp\circ\mathrm{Tr}')(y)$. By the definition of the shift operation S_x^* this is a Gaussian integral. Its value is

$$\Omega_{W_1}[S^*(\exp\circ\mathrm{Tr}')(y)] = \mathrm{e}^{\mathrm{Tr}'y}\,\mathrm{Det}^q(x^{-1})$$

in all three cases. Inserting this result into the above expression for $C_{n,q}$ we get the following product of two ordinary integrals:

$$C_{n,q} = \int_{D_q^0} \mathrm{e}^{-\mathrm{Tr}'x}\,\mathrm{Det}^{n'+q}(x)\,d\mu_{D_q^0}(x) \times \int_{D_q^1} \mathrm{e}^{\mathrm{Tr}'y}\,\mathrm{Det}^{-n'}(y)\,d\mu_{D_q^1}(y).$$

The first one is known to us from Eq. (3.2), while the second one is the integral that we actually want. The formula claimed for this integral in Lemma 4.10 is readily seen to be equivalent to the statement that $C_{n,q} = (2\pi)^{(1+|m|)q^2}$. Thus our final task now is to show that $C_{n,q} = (2\pi)^{(1+|m|)q^2}$. This is straightforward to do by the localization technique for supersymmetric integrals [19], as follows.

To get a clear view of the supersymmetries of our problem, let us go back to our starting point: the algebra $\mathscr{A}_V^G$ of G-equivariant holomorphic functions $V_0 \to \wedge(V_1^*)$ of the $\mathbb{Z}_2$-graded vector space $V = V_0 \oplus V_1$ for $V_0 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p) \oplus \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$ and $V_1 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q) \oplus \mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$. There exists a canonical action of the Lie superalgebra $\mathfrak{gl}_{p|q}$ on $\mathbb{C}^{p|q}$, hence on $V \simeq \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{p|q}) \oplus \mathrm{Hom}(\mathbb{C}^{p|q}, \mathbb{C}^n)$, and hence on the algebra $\mathscr{A}_V^G$. To describe this $\mathfrak{gl}_{p|q}$-action on $\mathscr{A}_V^G$, let $\{E_i^a\}$, $\{\tilde{E}_a^i\}$, $\{e_i^b\}$, and $\{\tilde{e}_b^i\}$ with index range $i = 1, \ldots, n$ and $a = 1, \ldots, p$ and $b = 1, \ldots, q$ be bases of $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^p)$, $\mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^n)$, $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^q)$, and $\mathrm{Hom}(\mathbb{C}^q, \mathbb{C}^n)$, in this order. If $\{F_a^i\}$, $\{\tilde{F}_i^a\}$, etc., denote the corresponding dual bases, then the odd generators of $\mathfrak{gl}_{p|q}$ (the even ones will not be needed here) are represented on $\mathscr{A}_V^G$ by odd derivations

$$d_b^a = \varepsilon(f_b^i)\delta(E_i^a) + \mu(\tilde{F}_i^a)\iota(\tilde{e}_b^i), \quad \tilde{d}_a^b = \varepsilon(\tilde{f}_i^b)\delta(\tilde{E}_a^i) - \mu(F_a^i)\iota(e_i^b),$$

where the operators $\varepsilon(f)$, $\delta(v)$, $\mu(f)$, and $\iota(v)$ mean exterior multiplication by the anti-commuting generator f, the directional derivative w.r.t. the vector v, (symmetric) multiplication by the function f, and alternating contraction with the odd vector v. Clearly, all of these derivations are G-invariant (for $G = \mathrm{GL}_n, \mathrm{O}_n, \mathrm{Sp}_n$) and have vanishing squares $(d_b^a)^2 = (\tilde{d}_a^b)^2 = 0$. Using the coordinate language introduced in Sect. 1.1 one could also write

$$d_b^a = \zeta_b^i \frac{\partial}{\partial Z_a^i} + \tilde{Z}_i^a \frac{\partial}{\partial \tilde{\zeta}_i^b}, \quad \tilde{d}_a^b = \tilde{\zeta}_i^b \frac{\partial}{\partial \tilde{Z}_i^a} - Z_a^i \frac{\partial}{\partial \zeta_b^i}.$$

It will be of importance below that the flat Berezin superintegral form $\mathrm{dvol}_{V_{0,\mathbb{R}}} \otimes \Omega_{V_1}$ is $\mathfrak{gl}_{p|q}$-invariant, which means in particular that

$$\int \Omega_{V_1}[d_b^a f]\,\mathrm{dvol}_{V_{0,\mathbb{R}}} = \int \Omega_{V_1}[\tilde{d}_a^b f]\,\mathrm{dvol}_{V_{0,\mathbb{R}}} = 0$$

for any $f \in \mathscr{A}_V$ with rapid decay when going toward infinity along $V_{0,\mathbb{R}}$.

Superbosonization involves the step of lifting $f \in \mathscr{A}_V^G$ to $F \in \mathscr{A}_W$ by the surjective mapping $Q^* : \mathscr{A}_W \to \mathscr{A}_V^G$. Now, since $W = \mathrm{S}^2(V)^G = \big(T^2(V)/I_-(V)\big)^G$ and the Lie superalgebra $\mathfrak{gl}_{p|q}$ acts on $T^2(V)$ by G-invariant derivations stabilizing $I_-(V)$, we also have a $\mathfrak{gl}_{p|q}$-action by linear transformations $W \to W$. Realizing this action by derivations of $\mathscr{A}_W$ we obtain a $\mathfrak{gl}_{p|q}$-action on $\mathscr{A}_W$. In particular, there exist such derivations $\mathscr{D}^a_b$ and $\tilde{\mathscr{D}}^b_a$ that for every $F \in \mathscr{A}_W$ we have

$$Q^* \mathscr{D}^a_b F = d^a_b Q^* F, \quad Q^* \tilde{\mathscr{D}}^b_a F = \tilde{d}^b_a Q^* F.$$

(In other words, our homomorphism of algebras $Q^* : \mathscr{A}_W \to \mathscr{A}_V^G$ is $\mathfrak{gl}_{p|q}$-equivariant.)

For any positive integers p, q, n with $n' \geq p$ consider now the Berezin superintegral

$$I^n_{p,q}[F] := \int_{D^0_p} \int_{D^1_q} \Omega_{W_1}[J \cdot \mathrm{SDet}^{n'} \cdot F]\, d\mu_{D^1_q}\, d\mu_{D^0_q},$$

which includes our integral $C_{n,q}$ of interest as a special case by letting $p = q$ and $F(x, y) = \mathrm{SDet}^q(x, y)\, \mathrm{e}^{\mathrm{Tr}' y - \mathrm{Tr}' x}$.

Lemma 5.8. *The odd derivations $\mathscr{D}^a_b$ and $\tilde{\mathscr{D}}^b_a$ are symmetries of $F \mapsto I^n_{p,q}[F]$; i.e., the integrals of $\mathscr{D}^a_b F$ and $\tilde{\mathscr{D}}^b_a F$ vanish,*

$$I^n_{p,q}[\mathscr{D}^a_b F] = I^n_{p,q}[\tilde{\mathscr{D}}^b_a F] = 0 \quad (a = 1, \ldots, p;\ b = 1, \ldots, q),$$

for any integrand $F \in \mathscr{A}_W$ such that $Q^* F|_{V_{0,\mathbb{R}}}$ is a Schwartz function.

Proof. While some further labor would certainly lead to a direct proof of this statement, we will prove it here using the superbosonization formula of Thm. 5.7 in reverse. (Of course, to avoid making a circular argument, we must pretend to be ignorant of the constant of proportionality between the two integrals, which will remain an unknown until the proof of Lemma 4.10 has been completed. Such ignorance does not cause a problem here, as we only need to establish a null result.) Thus, applying the formula of Thm. 5.7 in the backward direction with an unknown constant, we have

$$I^n_{p,q}[\mathscr{D}^a_b F] = \text{const} \times \int \Omega_{V_1}[Q^* \mathscr{D}^a_b F] \mathrm{dvol}_{V_{0,\mathbb{R}}}.$$

We now use the intertwining relation $Q^* \mathscr{D}^a_b = d^a_b Q^*$ of $\mathfrak{gl}_{p|q}$-representations. The integral on the right-hand side is then seen to vanish because the integral form $\mathrm{dvol}_{V_{0,\mathbb{R}}} \otimes \Omega_{V_1}$ is $\mathfrak{gl}_{p|q}$-invariant. Thus $I^n_{p,q}[\mathscr{D}^a_b F] = 0$. By same argument also $I^n_{p,q}[\tilde{\mathscr{D}}^b_a F] = 0$. □

Thus we have $2pq$ odd $\mathscr{A}_W$-derivations (or vector fields) $\mathscr{D}^a_b$ and $\tilde{\mathscr{D}}^b_a$ which are symmetries of $I^n_{p,q}$. We mention in passing that for the cases of $G = \mathrm{O}_n$ and $G = \mathrm{Sp}_n$ there exist further symmetries which promote the full symmetry algebra from $\mathfrak{gl}_{p|q}$ to $\mathfrak{osp}_{2p|2q}$. This fact will not concern us here.

Let now $p = q$. Then there exists a distinguished symmetry

$$\mathscr{D} := \mathscr{D}^a_a = \mathscr{D}^1_1 + \mathscr{D}^2_2 + \ldots + \mathscr{D}^q_q,$$

which still satisfies $\mathscr{D}^2 = 0$.

Lemma 5.9. *Viewed as a vector field on the supermanifold of functions $D^0_q \times D^1_q \to \wedge(W_1^*)$, the numerical part of $\mathscr{D}$ vanishes at a single point $o \equiv (\mathrm{Id}, \mathrm{Id}) \in D^0_q \times D^1_q$.*

Proof. We sketch the idea of the proof for $G = \mathrm{GL}_n$. In that case one verifies that $\mathscr{D}$ has the coordinate expression

$$\mathscr{D} = \sigma_b^a \left(\frac{\partial}{\partial x_b^a} + \frac{\partial}{\partial y_b^a} \right) + \left(x_b^a - y_b^a \right) \frac{\partial}{\partial \tau_b^a}.$$

The second summand, the numerical part of $\mathscr{D}$, is zero only when the coordinate functions x_b^a and y_b^a are equal to each other for all $a, b = 1, \ldots, q$. Since $D_q^1 = \mathrm{U}_q$ and D_q^0 is the set of positive Hermitian $q \times q$ matrices, this happens only for $x = \mathrm{Id} \in D_q^0$ and $y = \mathrm{Id} \in D_q^1$. The same strategy of proof works for the cases of $G = \mathrm{O}_n, \mathrm{Sp}_n$. $\square$

We are now in a position to apply the localization principle for supersymmetric integrals [19]. Let $F \in \mathscr{A}_W$ be a $\mathscr{D}$-invariant function which is a Schwartz function on D_q^0. Choose a $\mathscr{D}$-invariant function $g_{\mathrm{loc}} : D_q^0 \times D_q^1 \to \wedge(W_1^*)$ with the property that $g_{\mathrm{loc}} = 1$ on some neighborhood $U(o) \subset V(o)$ of o and $g_{\mathrm{loc}} = 0$ outside of $V(o)$. (Such "localizing" functions do exist.) Then according to Theorem 1 of [19] we have

$$I_{q,q}^n[F] = I_{q,q}^n[g_{\mathrm{loc}} F],$$

since $I_{q,q}^n$ is $\mathscr{D}$-invariant. (Although that theorem is stated and proved for compact supermanifolds, the statement still holds for our non-compact situation subject to the condition that integrands be Schwartz functions.)

Taking $V(o)$ to be arbitrarily small we conclude that $F \to I_{q,q}^n[F]$ depends only on the numerical part of the value of F at o :

$$I_{q,q}^n[F] = \mathrm{const} \times \mathrm{num}(F(o)).$$

To determine the value of the constant for $G = \mathrm{GL}_n$ we consider the special function

$$F = \mathrm{e}^{-\frac{t}{2}(x_a^b x_b^a - y_a^b y_b^a + 2\sigma_a^b \tau_b^a)}.$$

An easy calculation in the limit $t \to +\infty$ then gives $I_{q,q}^n[F] = (2\pi)^{q^2} \mathrm{num}(F(o))$ due to our choice of normalization for $d\mu_{D_q^0}$ and $d\mu_{D_q^1}$. The same calculation for the cases of $G = \mathrm{O}_n, \mathrm{Sp}_n$ gives $I_{q,q}^n[F] = (2\pi)^{2q^2} \mathrm{num}(F(o))$.

These considerations apply to the integrand in Eq. (5.9) with $\mathrm{num}(F(o)) = 1$. Thus we do indeed get $C_{n,q} = (2\pi)^{(1+|m|)q^2}$, and the proof of Lemma 4.10 is now finished.

6. Appendix: Invariant Measures

In the body of this paper we never gave any explicit expressions for the invariant measures $d\mu_{D_p^0}$ and $d\mu_{D_q^1}$. There was no need for that, as these measures are in fact determined (up to multiplication by constants) by invariance with respect to a transitive group action, and this invariance really was the only property that was required.

Nevertheless, we now provide assistance to the practical user by writing down explicit formulas for our measures (or positive densities) $d\mu_{D_p^0}$ and $d\mu_{D_q^1}$. For that purpose, we will use the correspondence between densities and differential forms of top degree. (Recall what the difference is: densities transform by the absolute value of the Jacobian, whereas top-degree differential forms transform by the Jacobian including sign.) Thus we shall give formulas for the differential forms corresponding to $d\mu_{D_p^0}$ and $d\mu_{D_q^1}$. This is a convenient mode of presentation, as it allows us to utilize complex coordinates for the complex ambient spaces as follows.

Consider first the case of $G = \mathrm{GL}_n(\mathbb{C})$, where $D_p^0 = \mathrm{Herm}^+ \cap \mathrm{End}(\mathbb{C}^p)$ and $D_q^1 = \mathrm{U} \cap \mathrm{End}(\mathbb{C}^q)$. Then for $r = p$ or $r = q$ consider $\mathrm{End}(\mathbb{C}^r)$ and let $z_{cc'} : \mathrm{End}(\mathbb{C}^r) \to \mathbb{C}$ (with $c, c' = 1, \ldots, r$) be the canonical complex coordinates of $\mathrm{End}(\mathbb{C}^r)$, i.e., the set of matrix elements with respect to the canonical basis of $\mathbb{C}^r$. On the set of regular points of $\mathrm{End}(\mathbb{C}^r)$ define a holomorphic differential form $\omega^{(r)}$ by

$$\omega^{(r)} = \mathrm{Det}^{-r}(z) \bigwedge_{c,\,c'=1}^{r} dz_{cc'},$$

where $z = (z_{cc'})$ is the matrix of coordinate functions. By the multiplicativity of the determinant and the alternating property of the wedge product, $\omega^{(r)}$ is invariant under transformations $z \mapsto g_1 z\, g_2^{-1}$ for $g_1, g_2 \in \mathrm{GL}_r(\mathbb{C})$. The desired invariant measures (up to multiplication by an arbitrary normalization constant) are

$$d\mu_{D_p^0} \propto \omega^{(p)}\big|_{\mathrm{Herm}^+ \cap \mathrm{End}(\mathbb{C}^p)}, \quad d\mu_{D_q^1} \propto \omega^{(q)}\big|_{\mathrm{U} \cap \mathrm{End}(\mathbb{C}^q)}, \tag{6.10}$$

where we restrict $\omega^{(r)}$ as indicated and reinterpret $d\mu_{D_r^\bullet}$ as a positive density on the orientable manifold $D_r^\bullet$ $(r = p, q)$. For example, for $r = 1$ we have $\omega^{(1)} = z^{-1} dz$. In this case we get an invariant positive density $|dx|$ on the positive real numbers $\mathrm{Herm}^+ \cap \mathbb{C} = \mathbb{R}_+$ by setting $z = \mathrm{e}^x$ with $x \in \mathbb{R}$, and a Haar measure $|dy|$ on the unit circle $\mathrm{U} \cap \mathbb{C} = \mathrm{U}_1 = \mathrm{S}^1$ by setting $z = \mathrm{e}^{\mathrm{i}y}$ with $0 \le y \le 2\pi$. Our normalization conventions for the invariant measures $d\mu_{D_r^\bullet}$ are those described in Sect. 1.2.

We turn to the cases of $G = \mathrm{O}_n(\mathbb{C})$ and $G = \mathrm{Sp}_n(\mathbb{C})$ and recall that the condition on elements M of the complex linear space $\mathrm{Sym}_b(\mathbb{C}^{2r})$ is $M = t_b\, M^{\mathrm{t}} (t_b)^{-1}$. On making the substitution $M = L\, t_b$ this condition turns into

$$\begin{aligned} L &= +L^{\mathrm{t}} \quad \text{for } \ b = s, \\ L &= -L^{\mathrm{t}} \quad \text{for } \ b = a, \end{aligned}$$

while the $\mathrm{GL}_{2r}(\mathbb{C})$-action on $\mathrm{Sym}_b(\mathbb{C}^{2r})$ by twisted conjugation becomes $g.L = gL\, g^{\mathrm{t}}$ in both cases. Define the coordinate function $z_{cc'} : \mathrm{Sym}_b(\mathbb{C}^{2r}) \to \mathbb{C}$ to be the function that assigns to M the matrix element of $M(t_b)^{-1} = L$ in row c and column c'. We then have $z_{cc'} = z_{c'c}$ for $b = s$ and $z_{cc'} = -z_{c'c}$ for $b = a$. As before let $z = (z_{cc'})$ be the matrix made from these coordinate functions (where the transpose $z^{\mathrm{t}} = z$ for $b = s$ and $z^{\mathrm{t}} = -z$ for $b = a$). Then let top-degree differential forms $\omega^{(r;\, b)}$ be defined *locally* on the regular points of $\mathrm{Sym}_b(\mathbb{C}^{2r})$ by

$$\begin{aligned} \omega^{(r;\, s)} &= \mathrm{Det}^{-r-1/2}(z) \bigwedge_{1 \le c \le c' \le r} dz_{cc'}, \\ \omega^{(r;\, a)} &= \mathrm{Det}^{-r+1/2}(z) \bigwedge_{1 \le c < c' \le r} dz_{cc'}. \end{aligned}$$

These are invariant under pullback by $L \mapsto gL\, g^{\mathrm{t}}$, as the transformation behavior of $\mathrm{Det}^{-r \pm 1/2}$ is contragredient to that of the wedge product of differentials in both cases. We emphasize that this really is just a local definition so far, as the presence of the square root factors may be an obstruction to the global existence of such a form.

Now focus on the case of $G = \mathrm{O}_n(\mathbb{C})$. There, restriction to the domains $D^0_{\delta,p}$ and $D^1_{\delta,q}$ gives the differential forms $\omega^{(p;\,s)}\big|_{\mathrm{Herm}^+\cap\mathrm{Sym}_s(\mathbb{C}^{2p})}$ and $\omega^{(q;\,a)}\big|_{\mathrm{U}\cap\mathrm{Sym}_a(\mathbb{C}^{2q})}$. Both of these are globally defined. Indeed, we can take the factor $L \mapsto \mathrm{Det}^{-1/2}(L)$ in the first differential form to be the reciprocal of the positive square root of the positive Hermitian matrix $M = L\,t_s$, and the square root $L \mapsto \mathrm{Det}^{1/2}(L)$ appearing in the second form makes global sense as the Pfaffian of the unitary skew-symmetric matrix $L^{\mathrm{t}} = -L$. Reinterpreting these differential forms as densities we arrive at a $\mathrm{GL}_{2p}(\mathbb{R})$-invariant measure on $D^0_{\delta,p}$ and a U_{2q}-invariant measure on $D^1_{\delta,q}$:

$$d\mu_{D^0_{\delta,p}} \propto \omega^{(p;\,s)}\big|_{\mathrm{Herm}^+\cap\mathrm{Sym}_s(\mathbb{C}^{2p})}, \quad d\mu_{D^1_{\delta,q}} \propto \omega^{(q;\,a)}\big|_{\mathrm{U}\cap\mathrm{Sym}_a(\mathbb{C}^{2q})}. \tag{6.11}$$

Again, our normalization conventions for $d\mu_{D^\bullet_{\delta,r}}$ are those of Sect. 1.2.

In the final case of $G = \mathrm{Sp}_n(\mathbb{C})$ the roles of $\omega^{(\bullet;\,s)}$ and $\omega^{(\bullet;\,a)}$ are reversed. This immediately leads to a good definition of $d\mu_{D^0_{\varepsilon,p}}$ for the non-compact symmetric space $D^0_{\varepsilon,p}$. However, the remaining case of $D^1_{\varepsilon,q} = \mathrm{U}\cap\mathrm{Sym}_s(\mathbb{C}^{2q})$ is problematic because there exists no global definition of $\mathrm{Det}^{1/2}$ on the unitary symmetric matrices. Thus the locally defined differential form $\omega^{(q;\,s)}$ does *not* extend to a globally defined form on $D^1_{\varepsilon,q}$. (Please be advised that this is inevitable, as the compact symmetric space $D^1_{\varepsilon,q} \simeq \mathrm{U}_{2q}/\mathrm{O}_{2q}$ lacks the property of orientability and on a non-orientable manifold any globally defined top-degree differential form must have at least one zero and therefore cannot be both non-zero and invariant in the required sense.)

Of course $d\mu_{D^1_{\varepsilon,q}}$ still exists as a density on the non-orientable manifold $D^1_{\varepsilon,q}$. The discussion above is just saying that there exists no globally defined differential form corresponding to $d\mu_{D^1_{\varepsilon,q}}$. Locally, we have $d\mu_{D^1_{\varepsilon,q}} \propto \omega^{(q;\,s)}\big|_{\mathrm{U}\cap\mathrm{Sym}_s(\mathbb{C}^{2q})}$.

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Note added in proof. After submission of this paper, it was brought to our attention that a bosonization formula for the fermion-fermion sector with U_n symmetry had been developed by Kawamoto and Smit [13]. A supersymmetric generalization was suggested in Ref. [2].

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