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Nonequilibrium Physics: Problem Sheet 1

www.thp.uni-koeln.de/∼as/noneq13.html

1. Diffusion equation

The solution of the diffusion equation as derived in the lecture can be written in the form

$$n(x,t) = \int dx_0 \; W\big((x_0,t_0) \to (x,t)\big) n(x_0,t_0) \tag{1}$$

where we have taken the initial condition at t_0 instead of $t = 0$. In the following we will assume that we do not know the explicit form of $W\big((x_0,t_0) \to (x,t)\big)$.

a) Show that $W\big((x_0,t_0) \to (x,t)\big)$ also satisfies a diffusion equation.

b) Show that $W\big((x_0,t_0) \to (x,t)\big)$ satisfies the definition of a Markov process, i.e.

$$W\big((x_0,t_0) \to (x_2,t_2)\big) = \int dx_1 W\big((x_0,t_0) \to (x_1,t_1)\big) W\big((x_1,t_1) \to (x_2,t_2)\big)\,.$$

2. Langevin equation

The Langevin equation, which describes (for instance) the movement of a Brownian particle, can be written in one dimension as

$$\frac{dv}{dt}(t) = -\gamma v(t) + \xi(t) \tag{2}$$

where γ is a damping constant and $\xi(t)$ is a random term ("white noise") with average $\langle \xi(t) \rangle = 0$ and correlation function $\langle \xi(t)\xi(t') \rangle = c\delta(t-t')$. Noise ξ and velocity v are not correlated with each other.

a) Show that the solution of the stochastic differential equation (2) can be written as

$$v(t) = e^{-\gamma t} \left(v_0 + \int_0^t \xi(\tau) e^{\gamma\tau} \, d\tau \right) .$$

b) Calculate the mean velocity $\langle v(t) \rangle$ and the correlation function $\langle v(t)v(t') \rangle$.
Hint: Be careful in the evaluation of the double integral!

c) Show that in an equilibrium situation the following differential equation holds:

$$\frac{d}{dt}\langle xv \rangle = k_B T - \gamma \langle xv \rangle \,,$$

where $v = \dot{x}$.
Hint: Use the equipartition theorem with $m = 1$.

d) Determine $\langle x(t)v(t) \rangle$ for the initial condition $\langle x(t=0)v(t=0) \rangle = 0$.

e) Determine the mean square displacement $\langle x^2(t) \rangle$ and determine its long and short time limits.