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Nonequilibrium Physics: Problem Sheet 3

www.thp.uni-koeln.de/∼as/noneq13.html

5. Random Walk and Generating Functions

In the lecture we have derived the master equation for the random walk on a 1d lattice of L sites:

$$\frac{\partial P(n,t)}{\partial t} = pP(n-1,t) + qP(n+1,t) - (p+q)P(n,t)\,, \tag{1}$$

where $P(n,t)$ is the probability to find the particle at position n at time t. p and q are the rates (transition probability per time) for a motion to the right and left neighbour position, respectively.

The standard technique for the solution of iterations is based on generating functions. In our case it is defined by

$$G(z,t) := \frac{1}{L}\sum_{n=1}^{L} P(n,t)z^n\,. \tag{2}$$

In the following we assume periodic boundary conditions, i.e.

$$P(n+L,t) = P(n,t) \tag{3}$$

for $n = 1, 2, \ldots, L$.

a) Convince yourself that $G(z,t)$ is compatible with the periodic boundary condition (3) if z is of the form

$$z_k = e^{2\pi i k/L} \tag{4}$$

with $k = 1, 2, \ldots, L$. Therefore $G(z,t)$ can be considered to be a function of k instead of z, i.e. $G = G(k,t)$, and (2) corresponds to the discrete Fourier transform.

b) Show that the generating function $G(k,t)$ satisfies

$$\frac{\partial G(k,t)}{\partial t} = \lambda_k G(k,t) \qquad \text{with} \qquad \lambda_k = p(z_k - 1) + q\left(\frac{1}{z_k} - 1\right)\,. \tag{5}$$

c) Solve the differential equation (5).

d) Show that $P(n,t)$ is given by

$$P(n,t) = \sum_{k=1}^{L} e^{\lambda_k t} z_k^{-n} G(k,0) \tag{6}$$

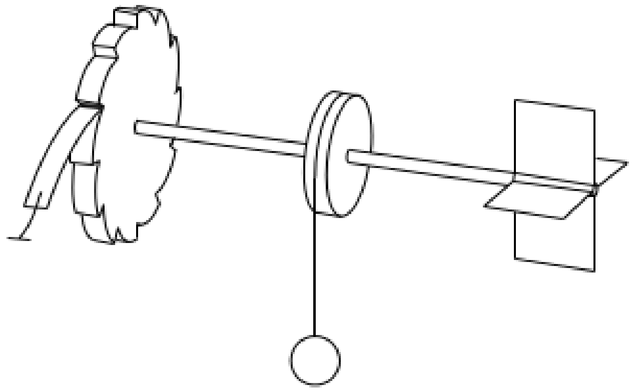
where $G(k,0)$ is related to the initial condition $P(n,t=0)$ by $G(k,0) = \frac{1}{L}\sum_{n=1}^{L} P(n,0)z_k^n$.

e) Show that $\mathrm{Re}\,\lambda_k \le 0$.

f) Determine the stationary distribution $\lim_{t\to\infty} P(n,t)$. How can this result be interpreted?

6. Ratchets

The figure shows Feynman's ratchet and pawl system which is supposed to rectify Brownian motion and thus lifting the weight.



a) Why does Feynman's ratchet and pawl not work?

Hint: Consider the effect of thermal fluctuations on the pawl.

b) Discuss whether or not ratchets violate the second law of thermodynamics, i.e. constitutes a perpetuum mobile of the second kind !

c) In a temperature ratchet the ratchet potential does not change in time. Instead the temperature changes periodically. In which direction does the current flow (for large enough temperature changes)?