Nonequilibrium Physics: Problem Sheet 8

www.thp.uni-koeln.de/∼as/noneq13.html

17. TASEP and Burgers equation

In the following we consider the TASEP in continuous time coupled to reservoirs with entrance rate α and exit rate β.

a) Justify the (discrete) equation of continuity for bulk sites ($\ell = 2, 3, \ldots, L-1$):

$$\frac{\partial \rho_\ell}{\partial t} = \langle J_{\ell-1,\ell} - J_{\ell,\ell+1} \rangle \tag{1}$$

where $\rho_j = \langle n_j \rangle$ and $J_{\ell,\ell+1} = p n_\ell (1 - n_{\ell+1})$ is the current from site ℓ to $\ell + 1$, and for the boundary sites

$$\frac{\partial \rho_1}{\partial t} = \alpha(1-\rho_1) - \langle J_{1,2} \rangle \ , \qquad \frac{\partial \rho_L}{\partial t} = \langle J_{L-1,L} \rangle - \beta \rho_L \, .$$

b) Replace the average currents $\langle J_{\ell-1,\ell} \rangle$ and $\langle J_{\ell,\ell+1} \rangle$ in the bulk equation (1) by their mean-field approximations.

c) Perform the continuum limit $a \to 0$ of the result in b) where a is the lattice constant, i.e. the position of site ℓ is $x_\ell = \ell a$. Then the density becomes a function $\rho_\ell \to \rho(x)$. Derive the *Burgers equation*

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left(\nu \frac{\partial \rho}{\partial x} - \lambda \rho (1-\rho) \right)$$

which is well-known as a simple equation for the dynamics of fluids in one dimension. Determine the constants ν and λ.
Hint: $\rho_{\ell \pm 1} = \rho(x \pm a) \approx \ldots$.

d) Compare the Burgers equation with usual equation of continuity to determine how the current $J(x) = J(\rho(x)$ depends on the density profile $\rho(x)$.

e) Show that the Burgers equation is related to the linear diffusion equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

by using the *Cole-Hopf transformation*

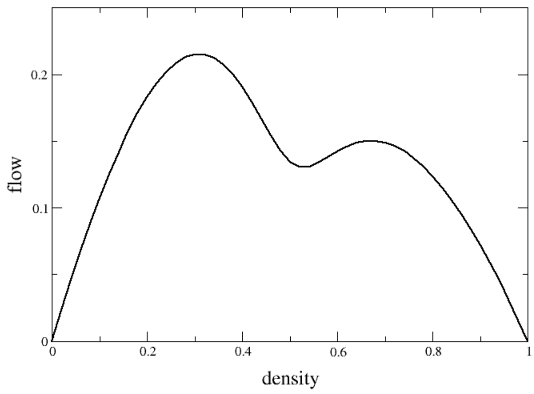
$$\rho(x,t) = \frac{1}{2} + \frac{\nu}{\lambda} \frac{1}{u} \frac{\partial u}{\partial x} \, .$$

f) Show that the diffusion equation has solutions of the form

$$u(x,t) = \sum_{m=1}^{M+1} A_m e^{a_m x + a_m^2 \nu t} \, .$$

18. Extremal principle

Assume that the fundamental diagram of a generalized TASEP is given by the following figure:



How many different phases can one observe in the open system?
Hint: Use the extremal principle.

19. Calibration of Nagel-Schreckenberg model

There are different methods to calibrate the NaSch model, i.e. to determine its timescale in terms of real time. In the following we consider the NaSch model with $v_{\max} = 5$ and $p = 0.5$ Determine the timescale from the following considerations:

a) Empirically a jam moves with 15 km/h. Determine the jam velocity in the NaSch model and identify it with the empirical value.
Hint: What is the probability that the first car in a jam moves? Use a car length of 7.5 m.

b) Identify the empirical value of 2000 vehicles/hour with the maximal flow (0.32 vehicles/timestep).

20. Deterministic NaSch model

Determine the structure of the free flow states in the deterministic limit $p = 0$ of the Nagel-Schreckenberg model. Which configurations lead to an unhindered motion of *all* cars? Up to which density do such states exist? What is the flow in these states?

Hint: Here a formal calculation is not necessary (or helpful!).