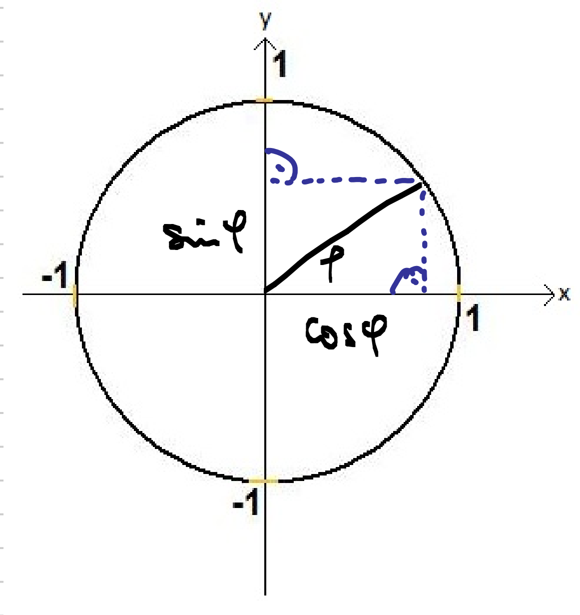
Notiztitel 05.09.2012

4) Trigonometrische Funktionen

Einheitskreis

Umfang $U = 2\pi R = 2\pi$



Für $0 \le \varphi \le 180^\circ$ (π) $\quad 0 \le \sin\varphi \le 1$

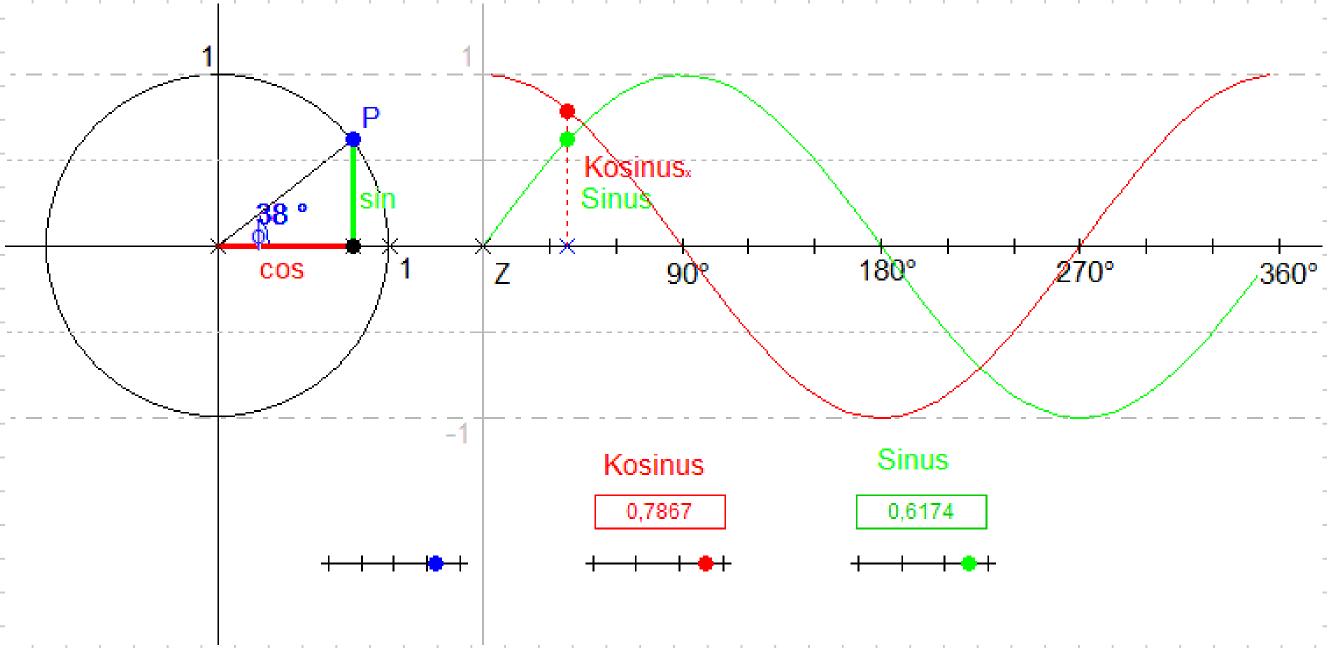
" $180 \le \varphi \le 360^\circ$ $\quad 0 \ge \sin\varphi \ge -1$

$$\left.\begin{array}{l} 0 \le \varphi \le 90^\circ \; (\frac{\pi}{2}) \\ 270^\circ \le \varphi \le 360^\circ \end{array}\right\} \quad 0 \le \cos\varphi \le 1$$

$90^\circ \le \varphi \le 270^\circ$ $\quad 0 \ge \cos\varphi \ge -1$

Bogenmaß: $\dfrac{x}{2\pi} = \dfrac{\varphi}{360^\circ}$

$$\cos(-\varphi) = \cos\varphi \qquad \sin(-\varphi) = -\sin\varphi$$



$$\sin(\pi - \varphi) = \sin\varphi$$

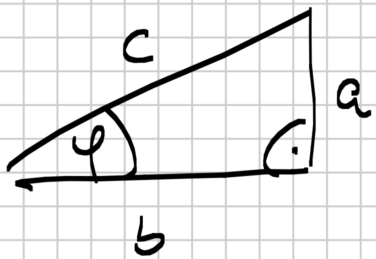
$$\sin(\varphi + 2\pi) = \sin\varphi$$

...

$$\sin\left(\varphi + \frac{\pi}{2}\right) = \cos\varphi$$

$\alpha =$	$\frac{\pi}{2} \pm \varphi$	$\pi \pm \varphi$	$\frac{3\pi}{2} \pm \varphi$	$2\pi \pm \varphi$
$\sin(\alpha)$	$\cos \varphi$	$\mp \sin \varphi$	$-\cos \varphi$	$-\sin \varphi$
$\cos(\alpha)$	$\mp \sin \varphi$	$-\cos \varphi$	$\pm \sin \varphi$	$+\cos \varphi$

Rechtwinkliges Dreieck:



$a = c \sin \varphi$

$b = c \cos \varphi$

$a^2 + b^2 = c^2$ bzw $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$ $\sin^2 \varphi + \cos^2 \varphi = 1$

5) Das Skalarprodukt $\vec{a} \cdot \vec{b}$

„inneres Produkt“

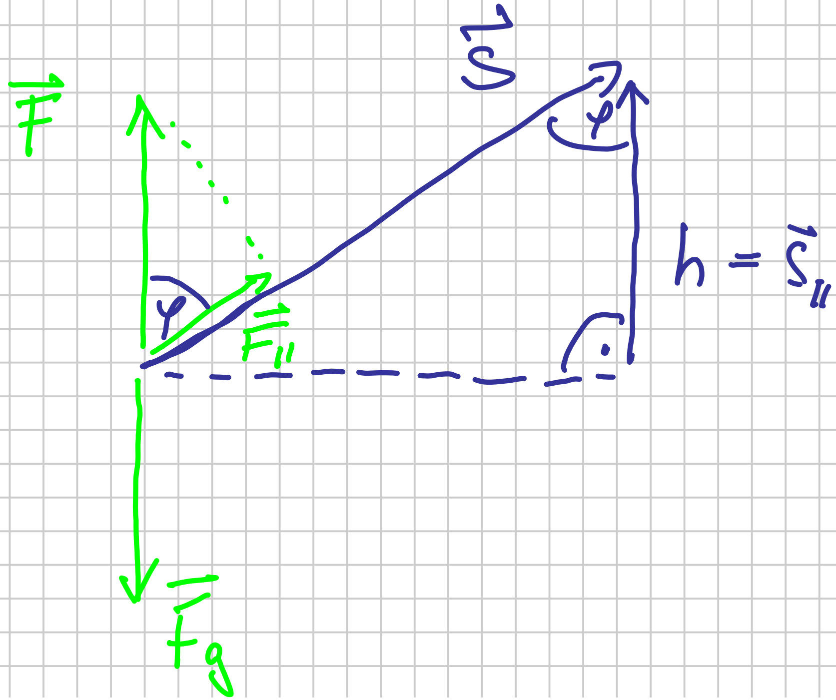
„Arbeit = Kraft mal Weg“

$$W = |\vec{F}| \cdot |\vec{s}_{\parallel}| = +|\vec{F}| \cdot (|\vec{s}| \cdot \cos\varphi)$$

$$= |\vec{F}| \cdot h$$

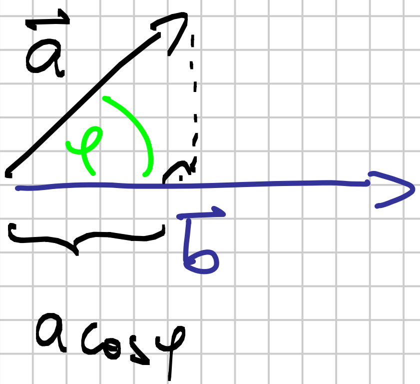
$$= |\vec{F}_{\parallel}| \cdot |\vec{s}| = (|\vec{F}| \cdot \cos\varphi) \cdot |\vec{s}|$$

$$\boxed{W =: \vec{F} \circ \vec{s}}$$

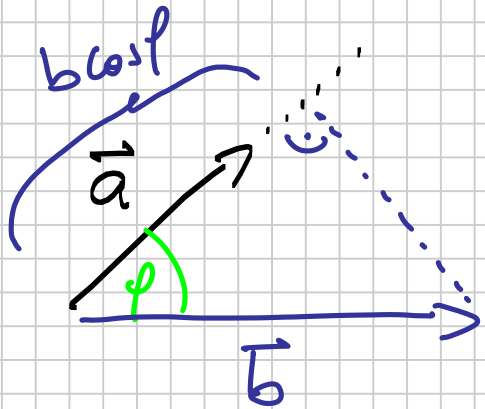


$$(\vec a, \vec b) = \vec a \cdot \vec b := |\vec a| \cdot |\vec b| \cdot \cos\varphi \qquad \varphi = \sphericalangle(\vec a, \vec b)$$

Projektion von $\vec a$ auf $\vec b$



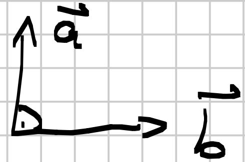
$$\vec a \cdot \vec b = |\vec a| \cos\varphi \, |\vec b| = ab\cos\varphi$$



$$\vec b \cdot \vec a = |\vec b| \cos\varphi \, |\vec a| = ab\cos\varphi$$

Kommutativgesetz: $\vec a \cdot \vec b = \vec b \cdot \vec a$

orthogonale Vektoren $\vec a, \vec b \iff \vec a \cdot \vec b = 0$

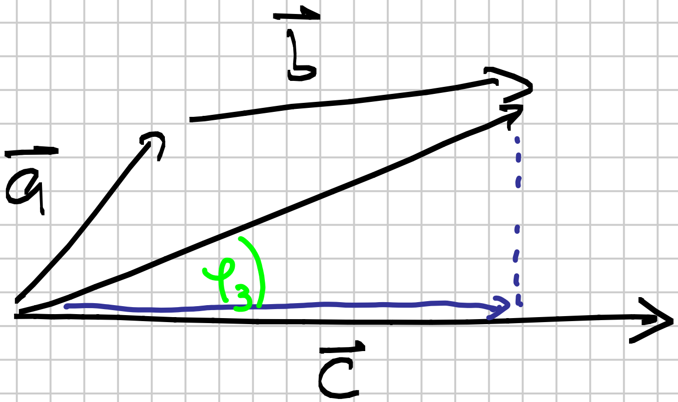


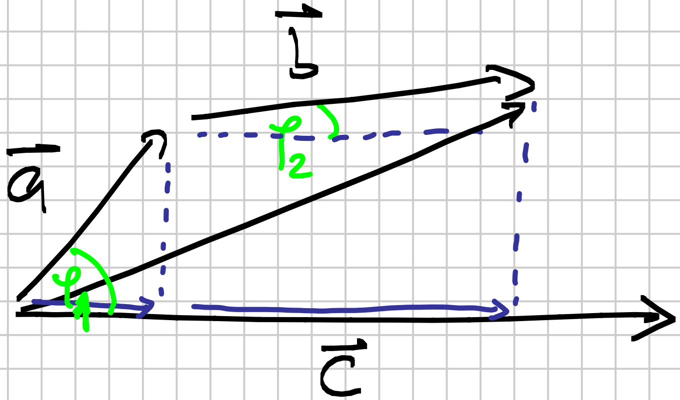
; Projektion $= 0$

Distributivgesetz:

$$(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$$

$$|\vec{a}+\vec{b}|\, c \cos\varphi_3 = a c \cos\varphi_1 + b c \cos\varphi_2$$





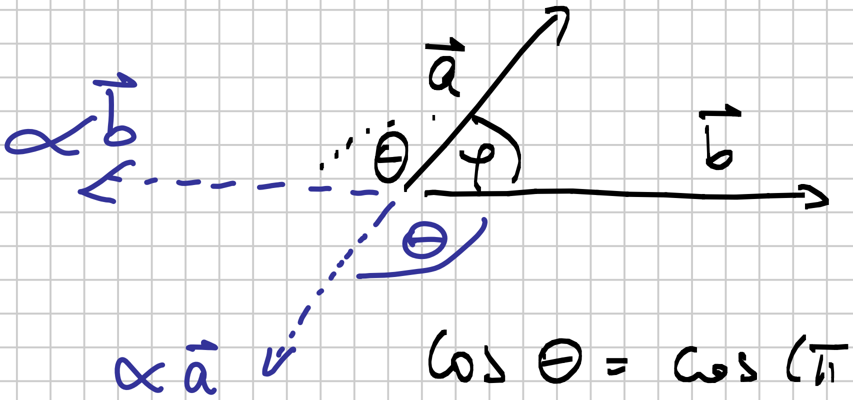
Bilinearität: $\alpha \in \mathbb{R}$ $\quad \vec{a}, \vec{b} \in V$

$$(\alpha\vec{a})\cdot\vec{b} = \vec{a}\,(\alpha\vec{b}) = \alpha\,(\vec{a}\,\vec{b})$$

Beweis: $\alpha > 0$

$$(\alpha a)\cos\varphi\, b = a\cos\varphi(\alpha b)$$

$$= (ab\cos\varphi)\,\alpha$$

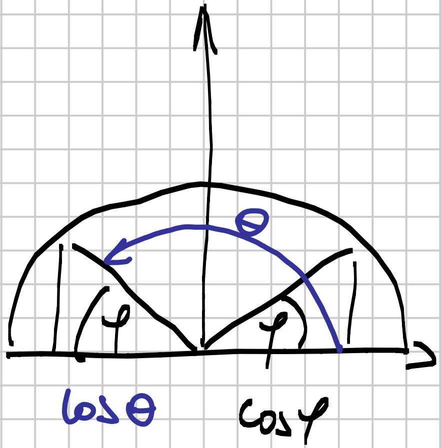


$$\cos\theta = \cos(\pi - \varphi) = -\cos\varphi$$

$\alpha < 0$: $(\alpha\vec a)\vec b = |\alpha|\, ab\cos\theta = -|\alpha|\, ab\cos\varphi$

$$\vec a(\alpha\vec b) = a\cdot|\alpha|\cdot b\cos\theta = -|\alpha|\, ab\cos\varphi$$

$$\alpha(\vec a\vec b) = \alpha\, ab\cos\varphi = -|\alpha|\, ab\cos\varphi$$



Betrag eines Vektors: $a := |\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a^2 \underbrace{\cos 0}_{1}} = a$

Schwarz'sche Ungleichung: $|\vec{a}\cdot\vec{b}| \leq ab$

$|\vec{a}\cdot\vec{b}| = |a \cdot b \cdot \cos\varphi| \leq a \cdot b$ da $|\cos\varphi| \leq 1$

Dreiecksungleichung: $|a-b| \leq |\vec{a}+\vec{b}| \leq a+b$

Beweis: $-ab \leq \vec{a}\cdot\vec{b} \leq ab$ $\quad | \cdot 2 + a^2 + b^2$

$a^2+b^2-2ab \leq a^2+b^2+2\vec{a}\vec{b} \leq a^2+b^2+2ab$

$(a-b)^2 \leq (\vec{a}+\vec{b})^2 \leq (a+b)^2$

$$|a-b| \leq |\vec{a}+\vec{b}| \leq a+b \qquad q.e.d.$$

Rechnen mit Skalarprodukten in kartesischen Koordinaten:

$$\vec{a}\cdot\vec{b} = \left(x_1\vec{e}_x + y_1\vec{e}_y + z_1\vec{e}_z\right)\cdot\left(x_2\vec{e}_x + y_2\vec{e}_y + z_3\vec{e}_z\right)$$

$$= + x_1 x_2 \underbrace{\vec{e}_x^{\,2}}_{1} + x_1 y_2 \underbrace{\vec{e}_x\vec{e}_y}_{=0} + x_1 z_3 \underbrace{\vec{e}_x\vec{e}_z}_{=0}$$

$$\vec{e}_x \perp \vec{e}_y \qquad \vec{e}_x \perp \vec{e}_z$$

$$+ y_1 x_2 \vec{e}_y\vec{e}_x + y_1 y_2 \vec{e}_y^{\,2} + y_1 z_2 \vec{e}_y\vec{e}_z$$

$$+ z_1 x_2 \vec{e}_z\vec{e}_x + z_1 y_2 \vec{e}_z\vec{e}_y + z_1 z_2 \vec{e}_z^{\,2}$$

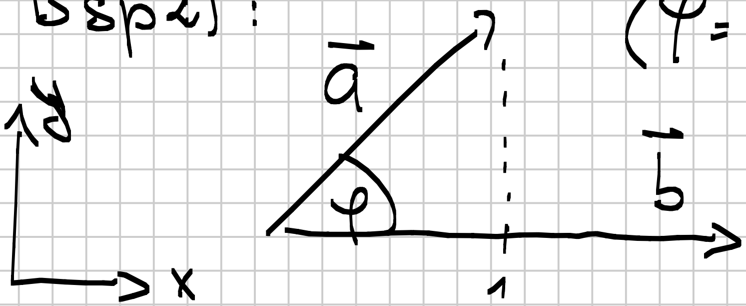
$$= x_1 x_2 + y_1 y_2 + z_1 z_2$$

Bsp 1): $\vec{a} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \vec{a} \cdot \vec{b} = 3 + 4 - 1 = 6$

Längen $\quad a = \sqrt{\vec{a}^2} = \sqrt{9+4+1} = \sqrt{14} \qquad b = \sqrt{\vec{b}^2} = \sqrt{1+4+1} = \sqrt{6}$

$$\cos\varphi = \frac{\vec{a} \cdot \vec{b}}{a \cdot b} = \frac{6}{\sqrt{14}\,\sqrt{6}} = \frac{\sqrt{6}}{\sqrt{14}} \approx 0.65 \qquad \varphi = 49^\circ$$

Bsp 2):



$(\varphi = 45^\circ) \quad \vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \quad \vec{a} \cdot \vec{b} = 2$

$a = \sqrt{2} \qquad b = 2 \qquad \cos\varphi = \frac{\vec{a}\,\vec{b}}{a\,b} = \frac{2}{\sqrt{2}\,2} = \frac{\sqrt{2}}{2}$

Schreibweisen und Abkürzungen:

Einstein'sche Summenkonvention: Summation über mehrfach auftretende Indizes

$$\vec{a} \cdot \vec{b} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} \cdot \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} =: \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \sum_{i=1}^{3} a_i b_i =: a_i b_i$$

Kronecker-Delta:

$$\delta_{ij} = \begin{cases} 0 & \text{für } i \neq j \\ 1 & \text{für } i = j \end{cases}$$

$$a = \sqrt{\vec{a}^2} = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_{k=1}^{3} a_k^2} = \sqrt{a_k a_k}$$

$$\delta_{jj} = \sum_{j=1}^{3} \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

$$\delta_{jl}\,\delta_{lk} = \sum_{l=1}^{3} \delta_{jl}\,\delta_{lk} = \underbrace{\delta_{j1}\,\delta_{1k}}_{\begin{cases} 1 & \text{für } j=k=1 \\ 0 & \text{sonst} \end{cases}} + \underbrace{\delta_{j2}\,\delta_{2k}}_{\begin{cases} 1 & \text{für } j=k=2 \\ 0 & \text{sonst} \end{cases}} + \underbrace{\delta_{j3}\,\delta_{3k}}_{\begin{cases} 1 & \text{für } j=k=3 \\ 0 & \end{cases}}$$

$$= \delta_{jk} = \begin{cases} 1 & \text{für } j=k \\ 0 & \text{sonst} \end{cases}$$

$$a_k \delta_{kl} = \sum_{k=1}^{3} a_k \delta_{kl} = a_1 \delta_{1l} + a_2 \delta_{2l} + a_3 \delta_{3l} = a_l$$

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij} \qquad \left(\vec{e}_i \cdot \vec{e}_i = \delta_{ii} = 3 \right)$$

ohne Summe $\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_x \cdot \vec{e}_y = 1$

$$c_k a_j a_l b_k \delta_{jl} = \sum_{k=1}^{3} \sum_{j=1}^{3} \sum_{l=1}^{3} c_k b_k \underbrace{a_j a_l \delta_{jl}}_{\substack{a_j^2 \text{ für } j=l \\ 0 \text{ sonst}}}$$

$$= \sum_{k=1}^{3} \sum_{j=1}^{3} c_k b_k a_j^2$$

$$= \sum_{k=1}^{3} c_k b_k \sum_{j=1}^{3} a_j^2 = \vec{c} \cdot \vec{b} \cdot \vec{a}^2 = a^2 \vec{b} \vec{c}$$