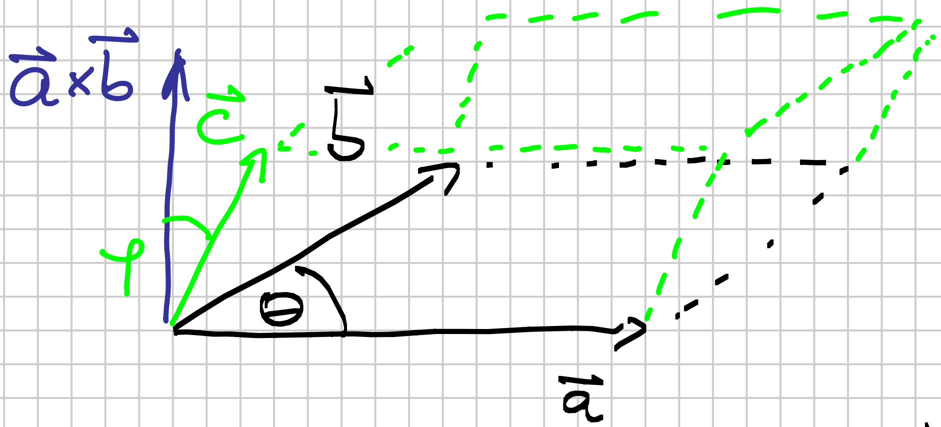
Notiztitel 05.09.2012

7.4. Spatprodukt $(\vec{a} \times \vec{b}) \cdot \vec{c}$



$(\vec{a} \times \vec{b}) \cdot \vec{c} =$

$a\, b \sin\Theta \cdot c \cdot \cos\varphi$

Interpretation:

Volumen = Grundfläche × Höhe

= Länge a · Breite b · sin Θ × Höhe c × cos φ

$$V = \vec{a}\,(\vec{b}\times\vec{c}) = a_i \vec{e}_i \cdot \varepsilon_{jke}\, b_j\, c_k\, \vec{e}_e$$

$$\boxed{\vec{e}_i \cdot \vec{e}_e = \delta_{ie} = \begin{cases} 1 & \text{für } i = e \\ 0 & \text{für } i \neq e \end{cases}}$$

$$= a_i\, b_j\, c_k\, \delta_{ie}\, \varepsilon_{jke} = a_e\, b_j\, c_k\, \varepsilon_{jke}$$

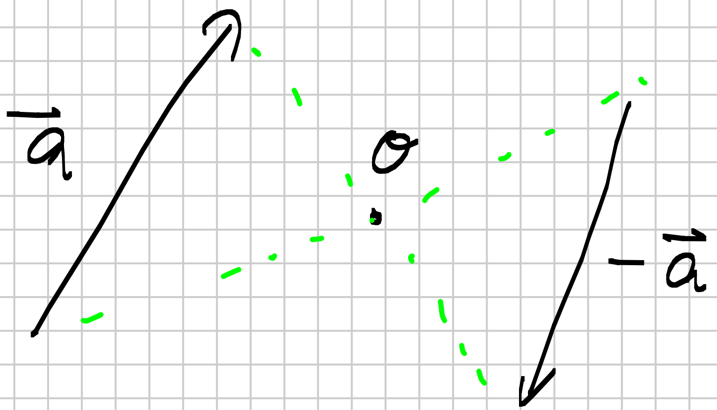
$$= \varepsilon_{jke}\, b_j\, c_k\, a_e = \varepsilon_{kej}\, b_j\, c_k\, a_e = \varepsilon_{ejk}\, b_j\, c_k\, a_e$$

$$= \vec{a}\,(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c} = \vec{b}\,(\vec{c}\times\vec{a})$$

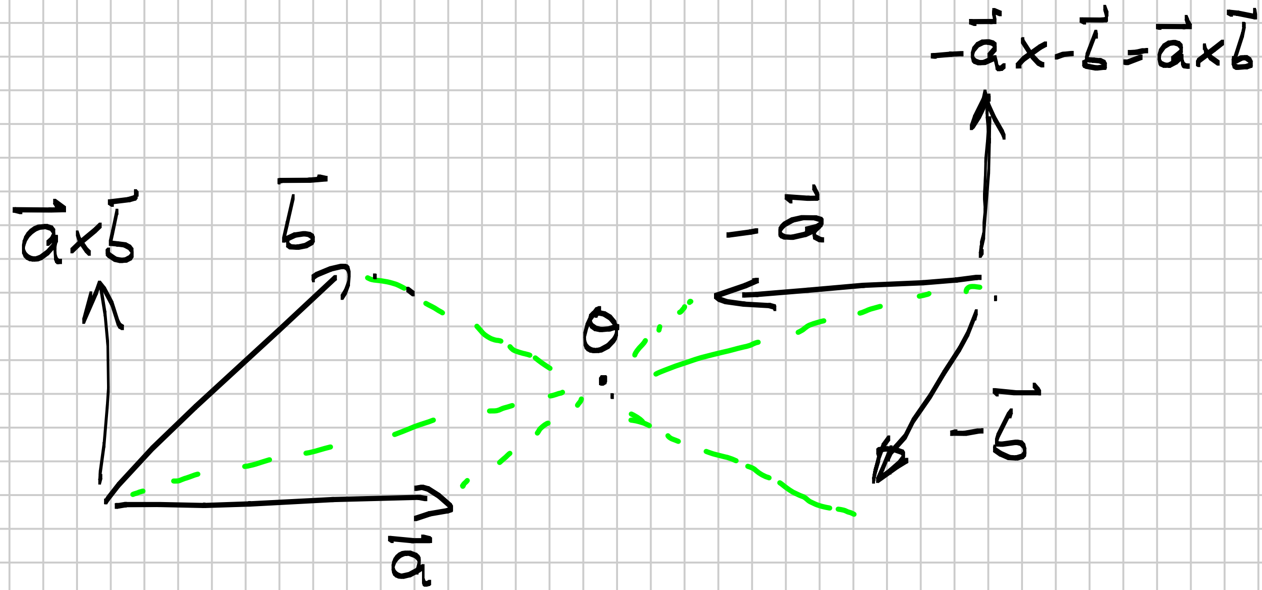
$$= -\varepsilon_{kje}\, b_j\, c_k\, a_e = \ldots$$

$$= -\vec{b}\,(\vec{a}\times\vec{c}) = -\vec{a}\cdot(\vec{c}\times\vec{b}) = -\vec{c}\cdot(\vec{b}\times\vec{a})$$

7.5. Polare und axiale Vektoren



Inversion am Ursprung



Vektorprodukt axialer Vektor

(Pseudovektor, Drehsinn)

8) Drehung von Vektoren (Drehmatrizen)

a) Drehung der Vektoren im festen Koordinatensystem

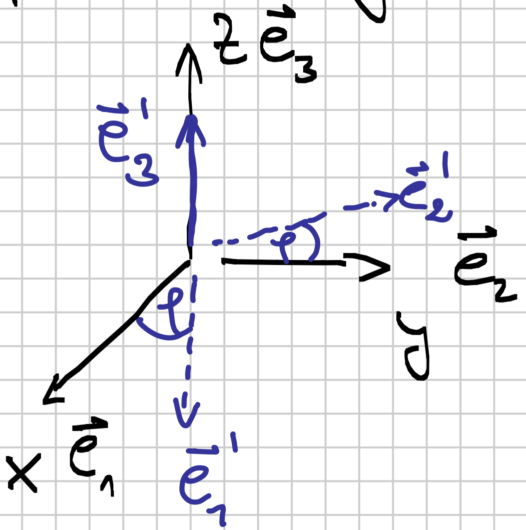
$$\vec{a} = a_i \vec{e}_i \longrightarrow \vec{a}' = a_i' \vec{e}_i$$

b) Drehung des Koordinatensystems

$$\vec{a} = a_i \vec{e}_i \longrightarrow \vec{a}' = a_i' \vec{e}_i' \equiv \vec{a}$$

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ & $\vec{e}_1', \vec{e}_2', \vec{e}_3'$ sind zwei rechtshändige Koord. systeme mit gleichem Ursprung

Bsp: Drehung um alte z-Achse $\vec{e}_3$ um Winkel φ:



$$\vec{a} = a_i \vec{e}_i = a_i' \vec{e}_i{}'$$

$$\vec{e}_i{}' = d_{ij} \vec{e}_j$$ Koordinatentransformation

z.B. $\vec{e}_1{}' = \cos\varphi \, \vec{e}_1 + \sin\varphi \, \vec{e}_2 + 0 \, \vec{e}_3$

$\vec{e}_2{}' = -\sin\varphi \, \vec{e}_1 + \cos\varphi \, \vec{e}_2 + 0 \, \vec{e}_3$

$\vec{e}_3{}' = \vec{e}_3$

$$\vec{e}_i{}' \cdot \vec{e}_k = d_{ij} \vec{e}_j \vec{e}_k = d_{ij} \delta_{jk} = d_{ik}$$

$d_{ik} = \vec{e}_i{}' \cdot \vec{e}_k = \cos(\vec{e}_i{}', \vec{e}_k)$ Drehmatrix

$$\mathbb{D} = (d_{ik}) = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

zu Bsp:

$$\begin{pmatrix} \cos\varphi & +\sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Betrachte

$$\vec{e}_i{}' \cdot \vec{e}_j{}' = \delta_{ij} = d_{ik}\, \vec{e}_k \cdot d_{jl}\, \vec{e}_l = d_{ik}\, d_{jl}\, \underbrace{\vec{e}_k \vec{e}_l}_{\delta_{kl}}$$

$$= d_{ik}\, d_{jk}$$

$$\Longrightarrow \quad \underline{\sum_{k=1}^{3} d_{ik}\, d_{jk} = \delta_{ij}}$$

$\Longrightarrow$ Zeilenvektoren $\vec{d}_i = \begin{pmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \end{pmatrix}$ bilden Orthonormalsystem wie $\vec{e}_1, \vec{e}_2, \vec{e}_3$

denn $\vec{d}_i \cdot \vec{d}_j = \delta_{ij}$

$$\mathbb{D} = \begin{array}{c} \text{Zeilenvektor} \\ \begin{pmatrix} \boxed{d_{11} \quad d_{12} \quad d_{13}} \\ d_{21} \quad d_{22} \quad d_{23} \\ d_{31} \quad d_{32} \quad d_{33} \end{pmatrix} \end{array} \qquad \mathbb{D} = \begin{array}{c} \text{Spaltenvektor} \\ \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \end{array}$$

$$\begin{pmatrix} \boxed{\cos\varphi \quad +\sin\varphi \quad 0} \\ -\sin\varphi \quad \cos\varphi \quad 0 \\ 0 \quad 0 \quad 1 \end{pmatrix}$$

$$\vec{d}_1 \cdot \vec{d}_2 = (d_{11}, d_{12}, d_{13}) \cdot \begin{pmatrix} d_{21} \\ d_{22} \\ d_{23} \end{pmatrix}$$

$$= \cos\varphi(-\sin\varphi) + \sin\varphi\cos\varphi + 0 = 0$$

$$\vec{d}_1 \cdot \vec{d}_1 = \cos^2\varphi + \sin^2\varphi = 1 \quad \ldots$$

Betrachte die inverse Drehung: $\vec{e}_i = \tilde{d}_{ij}\,\vec{e}_j{}'$

$$D^{-1} = (\tilde{d}_{ij}) = \begin{pmatrix} \tilde{d}_{11} & \tilde{d}_{12} & \tilde{d}_{13} \\ \tilde{d}_{21} & \tilde{d}_{22} & \tilde{d}_{23} \\ \tilde{d}_{31} & \tilde{d}_{32} & \tilde{d}_{33} \end{pmatrix}$$

Einsetzen von $\vec{e}_j' = d_{jk}\,\vec{e}_k$

$$\vec{e}_i = \tilde{d}_{ij}\, d_{jk}\, \vec{e}_k \qquad |\cdot \vec{e}_l$$

$$\vec{e}_i \cdot \vec{e}_l = \delta_{il} = \tilde{d}_{ij}\, d_{jk}\, \vec{e}_k \cdot \vec{e}_l = \tilde{d}_{ij}\, d_{jk}\, \delta_{kl}$$

$$\underline{\delta_{il} = \tilde{d}_{ij}\, d_{jl}}$$

Spaltenvektor der Matrix D

Zeilenvektor der inversen Matrix D^{-1}

bilden Orthonormal-System!

Es gilt: „Neu" durch „Alt":

1) $\vec{e}_i{}' = d_{ij}\,\vec{e}_j \;\rightsquigarrow\; \vec{e}_i{}'\,\vec{e}_k = d_{ij}\,\vec{e}_j\,\vec{e}_k = d_{ij}\,\delta_{jk} = d_{ik}$

$$\boxed{\vec{e}_i{}'\,\vec{e}_k = d_{ik}}$$

„Alt" durch „Neu":

2) $\vec{e}_i = \tilde{d}_{ij}\,\vec{e}_j{}' \;\rightsquigarrow\; \vec{e}_i \cdot \vec{e}_k{}' = \tilde{d}_{ij}\,\vec{e}_j{}'\,\vec{e}_k{}' = \tilde{d}_{ij}\,\delta_{jk}$

$$\boxed{\vec{e}_i\,\vec{e}_k{}' = \tilde{d}_{ik}}$$ oder $$\boxed{\vec{e}_k\,\vec{e}_i{}' = \tilde{d}_{ki}}$$

$$= \vec{e}_i{}' \cdot \vec{e}_k = d_{ik}$$

$$\Rightarrow \boxed{\tilde{d}_{ki} = d_{ik}}$$

$$\Rightarrow D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \qquad D^{-1} = \begin{pmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{pmatrix} = D^T$$

transponierte Matrix D^T

$$D^{-1} \cdot D = \begin{pmatrix} \boxed{d_{11} \quad d_{21} \quad d_{31}} \\ d_{12} \quad d_{22} \quad d_{32} \\ d_{13} \quad d_{23} \quad d_{33} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} d_{11} \\ d_{21} \\ d_{31} \end{matrix}} & \begin{matrix} d_{12} \\ d_{22} \\ d_{32} \end{matrix} & \begin{matrix} d_{13} \\ d_{23} \\ d_{33} \end{matrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zeilen- und Spaltenvektoren von D^{-1} und D bilden Orthonormalsystem

Bsp: Drehung um φ um z-Achse:

$$D_z(\varphi) = \begin{pmatrix} \cos\varphi & +\sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D_z^{-1}(\varphi) = D_z(-\varphi) = \begin{pmatrix} \cos(-\varphi) & \sin(-\varphi) & 0 \\ -\sin(-\varphi) & \cos(-\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_z^{-1}(\varphi) = D_z^{T}(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ +\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ +\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) Drehung des Koordinatensystems

$$\vec{a} = a_i \vec{e}_i \longrightarrow \vec{a}\,' = a_i' \vec{e}_i\,' \equiv \vec{a}$$

$$\vec{e}_i\,' = d_{ij} \vec{e}_j = \sum_{j=1}^{3} d_{ij} \vec{e}_j$$ Koordinatentransformation

Bestimmung der Komponente a_j des Vektors $\vec{a}$ aus den Komponenten a_i' im gedrehten KOS:

$$\vec{a}\,' = \sum_{i=1}^{3} a_i' \vec{e}_i\,' = \vec{a} = \sum_{j=1}^{3} a_j e_j \qquad | \cdot \vec{e}_j$$

$$= \sum_{i=1}^{3} a_i' \underbrace{\vec{e}_i{}' \cdot \vec{e}_j} = a_j$$

$$d_{ij} = \vec{e}_i{}' \cdot \vec{e}_j = \cos(\vec{e}_i{}', \vec{e}_j)$$

$$\boxed{a_j = \sum_{i=1}^{3} a_i' \, d_{ij}}$$

Bestimmung der Komponente a_i' des Vektors $\vec{a}'$ aus den Koordinaten a_j im ungedrehten KOS:

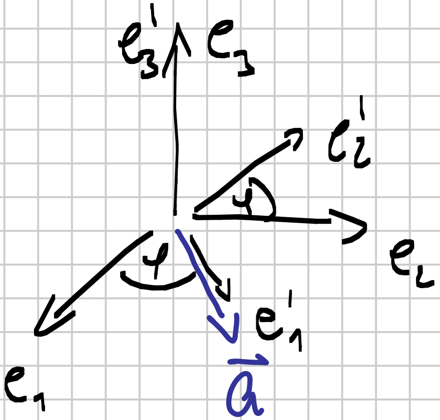
$$\vec{a}' = \sum_{i=1}^{3} a_i' \vec{e}_i' = \vec{a} = \sum_{j=1}^{3} a_j e_j \qquad | \cdot \vec{e}_i'$$

$$= a_i' = \sum_{j=1}^{3} a_j \underbrace{\vec{e}_j \cdot \vec{e}_i'}_{\sim}$$

$$\vec{e}_j \cdot \vec{e}_i' = \tilde{d}_{ji} = d_{ij}$$

$$a_i' = \sum_{j=1}^{3} a_j \tilde{d}_{ji} = \boxed{\sum_{j=1}^{3} a_j d_{ij} = a_i'}$$

Bsp: $\vec{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad D = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad \varphi = 45°$



$$D = \begin{pmatrix} \sqrt{2}/2 & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{a}\,' = \left(d_{ij} \right) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$