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Information Theory: From Statistical Physics to Quantitative Biology

7. exercise class – 7. January 2009

1. The law of large numbers I

The following steps lead to a proof of the (weak) law of large numbers.

a) For any non-negative random variable X and any positive δ, show that

$$Pr(X \geq \delta) \leq \langle X \rangle / \delta, \tag{1}$$

where $\langle X \rangle$ denotes the expectation value of X.

Hint: Consider the indicator function $I(X \geq \delta)$ which equals one if $X \geq \delta$ and zero otherwise. Argue that $\delta I(X \geq \delta) \leq X$.

b) Consider a random variable Y with mean μ and variance σ^2. Use the result from a) with $X = (Y-\mu)^2$ to show

$$Pr(|Y - \mu| > \epsilon) \leq \sigma^2/\epsilon^2 \ . \tag{2}$$

c) Consider a sequence of i.i.d. random variables $Y_1, Y_2, \dots, Y_n$ with mean μ and variance σ^2. Let $S = \frac{1}{n}\sum_{i=1}^n Y_n$ and show that

$$Pr(|S - \mu| > \epsilon) \leq \frac{\sigma^2}{n\epsilon^2} \ . \tag{3}$$

(60 points)

2. The law of large numbers II

For a distribution $P(x)$ with finite mean and variance, and $x_1, x_2, \dots x_N$ drawn independently from this probability distribution, the law of large numbers states that $X = \frac{1}{\sqrt{N}}\sum_{i=1}^N x_i$ is a random number whose distribution approaches a Gaussian for large N.

Investigate this statement numerically for different values of N if

a) $P(x)$ is a Gaussian distribution of mean zero and variance one.

b) $P(x)$ is a uniform distribution of mean zero and variance one.

c) $P(x) = \frac{1}{\pi(1+x^2)}$, called the Cauchy-Lorentz distribution.

Hint: The Matlab-commands `rand` and `randn` produce random numbers with Gaussian and uniform distribution respectively. (Vectors are produced by `rand(1,n)`.)Arbitrary distributions $p(x)$ can be generated by drawing y uniformly on the interval $[0,1]$ and choosing x such that $\int_{-\infty}^{x} dx' p(x') = y$ and solving the integral.
(100 points)