Advanced Quantum Mechanics

Prof. Dr. J. Krug, Dr. J. Åberg

Short questions

I. Principles of quantum mechanics

a) Describe the Schrödinger and Heisenberg pictures of time evolution.

b) What are the properties of density operators?

c) What is the difference between a pure and a mixed state?

d) Write down the density operator corresponding to the equal weight *mixture* of the two orthogonal states $|\psi\rangle$ and $|\eta\rangle$.

e) Write down the density operator corresponding to an equal weight *superposition* of the two orthogonal states $|\psi\rangle$ and $|\eta\rangle$.

f) How can mixed states arise from pure states?

g) Explain the Bloch representation of mixed states on a qubit.

II. Second quantization

a) Formulate the *symmetry postulate* for pure quantum states of a system of N identical particles.

b) Given a basis of single particle states, write down the symmetrized and antisymmetrized N-particle states.

c) Suppose that $\psi(x)$ and $\eta(x)$ are orthonormal wave-functions. Consider the following two-particle wave-functions. Which of these could describe two identical bosons, and which could describe two identical (spinless) fermions?

$$\frac{1}{\sqrt{2}}\psi(x_1)\eta(x_2) - \frac{1}{\sqrt{2}}\psi(x_2)\eta(x_1), \tag{1}$$

$$\psi(x_1)\psi(x_2), \tag{2}$$

$$\frac{1}{\sqrt{3}}\psi(x_1)\eta(x_2) + \sqrt{\frac{2}{3}}\psi(x_2)\eta(x_1), \tag{3}$$

$$-\sqrt{\frac{2}{5}}\psi(x_1)\eta(x_2) + \frac{1}{\sqrt{5}}\psi(x_1)\psi(x_2) - \sqrt{\frac{2}{5}}\psi(x_2)\eta(x_1), \tag{4}$$

$$\frac{1}{\sqrt{2}}\psi(x_1)\psi(x_2) - \frac{1}{\sqrt{2}}\eta(x_2)\eta(x_1). \tag{5}$$

d) Suppose that a single particle has the energy spectrum $E_n = \hbar\omega(n + \frac{1}{2})$, for $n = 0, 1, 2, \ldots$. Suppose now that we have 5 identical, non-interacting bosons of this type, what would the ground state energy of these five bosons be?

e) What if we in the above problem let the bosons instead be (spinless) fermions?

f) What if we in the above problem assume that the fermions are spin-1/2 particles (where both spin states have equal energy)?

g) How is the Fock space defined?

h) Let $|2,0,1\rangle$ be a bosonic Fock state with respect to the orthonormal single particle states $\phi_\alpha, \phi_\beta, \phi_\gamma$ (in that order). Write down the corresponding three-particle wave-function $\psi(x_1,x_2,x_3) = \langle x_1,x_2,x_3|2,0,1\rangle$ in terms of the single-particle wave-functions $\phi_\alpha(x)$, $\phi_\beta(x)$, and $\phi_\gamma(x)$.

i) Consider a collection of annihilation operators $a_1,\ldots,a_K$ and creation operators $a_1^\dagger,\ldots,a_K^\dagger$ corresponding to orthonormal single-particle states. Write down the bosonic and the fermionic commutation relations.

j) Write down the Hamiltonian of a system of bosons subject to a single particle potential $V^{(1)}$ and a pair interaction $V^{(2)}$ in second quantization. What would the corresponding expression look like if the particles are fermions?

k) Suppose that $\{a_l\}_l$ is a collection of bosonic or fermionic annihilation operators corresponding to a (complete) orthonormal basis $\{\phi_l(\vec{r})\}_l$ of wave-functions for a single particle (which we imagine to be enclosed in a large box of volume V). In terms of these, write down the field operators $\Psi(\vec{r})$ and $\Psi^\dagger(\vec{r})$.

l) Express the particle-number density operator $n(\vec{r})$ in terms of the field operators.

m) Express the pair correlation function $g^{(2)}$ in terms of the field operators.

n) Suppose that a bosonic or fermionic quantum field for particles of mass m is described in terms of the following Hamiltonian

$$
\begin{aligned}
H =& H_A + H_B + H_C,\\
H_A =& \int \frac{\hbar^2}{2m} \nabla\Psi^\dagger(\vec{r}) \cdot \nabla\Psi(\vec{r})\, d^3r,\\
H_B =& \int V^{(1)}(\vec{r})\Psi^\dagger(\vec{r})\Psi(\vec{r})\, d^3r,\\
H_C =& \frac{1}{2}\iint V^{(2)}(\vec{r},\vec{r}\,')\Psi^\dagger(\vec{r})\Psi^\dagger(\vec{r}\,')\Psi(\vec{r}\,')\Psi(\vec{r})\, d^3r'\, d^3r,
\end{aligned}
\tag{6}
$$

where $V^{(1)}$ and $V^{(2)}$ are (real-valued) functions. What do the three terms H_A, H_B, and H_C mean physically?

o) For the Hamiltonian in (6), what is the Heisenberg equation of motion for the field operator $\Psi(\vec{r},t)$ in the Heisenberg picture?

III. Bosonic systems

a) What is a Bogoliubov transformation, and what is it used for?

b) Explain the concept of a quasiparticle.

c) What is a dispersion relation? Give three examples.

d) What feature of the dispersion relation of weakly interacting bosons is essential for the phenomenon of superfluidity?

IV. Quantum theory of light

a) Write down the Hamiltonian of the quantized electromagnetic field.

b) Give an example of a physical phenomenon that can be explained in terms of the zero point energy of the electromagnetic field.

c) What can you say about the phase of a photon number state $|n\rangle$ of an electromagnetic field mode?

d) What is the defining relation for a coherent state of a single field mode?

e) What is the probability distribution P_n of the number of photons in a field mode

- in a coherent state $|\alpha\rangle$
- in a chaotic (e.g., thermal) state?

f) Write down the Hamiltonian describing the interaction of an atom with an electromagnetic field.

g) How does the rate at which a photon is absorbed/emitted by an atom depend on the number of photons in the field mode?

h) What does it mean for a field mode to display photon bunching or antibunching?

V. Relativistic quantum mechanics

a) Write down the relation between energy and momentum for a relativistic particle with rest mass m.

b) Write down the Klein-Gordon equation for a free particle, and name two conceptual problems associated with this equation.

c) Why did Paul Dirac insist on his relativistic wave equation to be

- first order in time, and
- linear in the momentum?

d) What are the defining relations that the Dirac matrices α^k and β have to satisfy? Write down a possible realization of these matrices.

e) Give a physical interpretation of the four components of the Dirac spinor. What can you say about the magnitude of these components for a free particle at nonrelativistic velocities?

f) What is the *helicity* of the free Dirac particle, and why is it conserved?

g) Why does Dirac's interpretation of the negative energy states in his equation work only for fermions?

h) Write down the Pauli equation for a charged spin-1/2 particle in an electromagnetic field.

i) Explain the physical meaning of the three symmetry transformations that appear in the CPT-theorem of quantum field theory.

j) Which quantum numbers characterize the eigenstates of the relativistic Coulomb problem? Write down the relativistic expression for the ground state energy of an electron in the field of a nucleus with charge Z.

k) What is the physical mechanism behind the Darwin term?

l) What is the key difference between the (second) quantization of the Klein-Gordon equation and the corresponding procedure applied to the electromagnetic field?

VI. Scattering theory

a) What is the asymptotic form of the wave function describing a stationary scattering process in three dimensions? How is it related to the differential scattering cross section?

b) What is the scattering Green's function?

c) What is the statement of the *optical theorem*, and what physical principle does it rely on?

d) Write down the scattering amplitude for a spherically symmetric potential $V(r)$ in the first Born approximation. Does this approximation satisfy the optical theorem?

e) Explain the concept of the scattering phases δ_l.

f) A particle of momentum $\hbar k$ is incident on a potential with range R. What is the maximal angular momentum quantum number $l_{\max}$ that contributes to the scattering cross section? Give a simple derivation in classical terms.

g) How is the *scattering length* defined?

h) The total scattering cross section of a hard sphere of radius R at high energies is twice the value πR^2 that is expected classically. Explain why.

i) Does the optical theorem hold for inelastic scattering processes?