Advanced Quantum Mechanics

Prof. Dr. J. Krug, Dr. J. Åberg

Exercise sheet 10 (Due: Monday January, 9$^{\text{th}}$.)

10.1 A reminder about special relativity

The time and space coordinates for one and the same event do generally depend on the relative speed of the observers, and the relation is determined by the Lorentz transformation. Suppose that observer S assigns coordinates (ct, x, y, z) to a particular event. If observer S' moves with speed v in the x-direction relative to observer S, then S' assigns the coordinates (ct', x', y', z') given by

$$t' = \left(t - \frac{vx}{c^2}\right)\gamma(v), \quad x' = (x - vt)\gamma(v), \quad y' = y, \quad z' = z, \quad \gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

(a) Students A and B sleep happily in their beds on Tuesday morning. We assume that their beds are not moving relative to each other (and are not subject to any significant accelerations). They live 5 kilometers apart, and in their frame of reference their alarm clocks go off with a time-difference of 5μs. Student C is very eager to reach a lecture on advanced quantum mechanics, and is already traveling at speed v relative to A and B. Moreover, it so happens that C travels parallel to the line joining A and B. From the perspective of C, the two alarm-clocks go off at the same time. What is the speed v? What is the distance between A and B according to C?

Hint: A and B share the same (inertial) frame of reference, and from their point of view the two events have the space-time coordinates (ct_1, x_1, y_1, z_1) and (ct_2, x_2, y_2, z_2), where you can choose these such that $y_1 = y_2$ and $z_1 = z_2$. Hence, their beds lie along the x-axis. C moves in the x-direction, and assigns the coordinates $(ct_1', x_1', y_1', z_1')$ and $(ct_2', x_2', y_2', z_2')$ to these two events. Although C indeed is eager, we still assume $v < c$. Nevertheless, the speed is not particularly realistic for the public transport system of Cologne. **(4 points)**

(b) The energy momentum-relation for a particle with rest mass m and momentum p is

$$E_{\vec{p}} = \sqrt{c^2p^2 + m^2c^4}, \quad p = \|\vec{p}\|. \tag{1}$$

You may also recall that the apparent mass $\tilde{m}(v)$ and the momentum p of a particle from the point of view of an observer moving with the speed v relative to the particle are

$$\tilde{m}(v) = \gamma(v)m, \quad p = \tilde{m}(v)v = \gamma(v)mv. \tag{2}$$

- Since the speed is limited by c, does this mean that the magnitude p of the momentum also has to be bounded? What happens with the total energy $E_{\vec{p}}$ as v approaches c?
- Use (1) and (2) to show that $E_{\vec{p}} = \gamma(v)mc^2$.
- A Kaon K^0 decays into a π^+ and a π^- meson. The rest energy of the Kaon is $E^0_{K^0} \approx 498$MeV, and the rest energy of the pions are $E^0_{\pi^\pm} \approx 140$MeV. (The rest energy of a particle is $E^0 = mc^2$ where m is the rest mass. The π^+ and π^- mesons are anti-particles of each other, and have the same rest mass.) Approximately what fraction of the speed of light will the π-mesons have in the center of mass frame?

(3 points)

10.2 The Klein-Gordon equation for a free particle

In the lecture you showed that the Klein-Gordon equation can be decomposed into two coupled equations that are first-order in time

$$i\hbar\partial_t\phi = -\frac{\hbar^2}{2m}\nabla^2(\phi+\chi) + mc^2\phi, \qquad i\hbar\partial_t\chi = \frac{\hbar^2}{2m}\nabla^2(\phi+\chi) - mc^2\chi. \tag{3}$$

(a) Make an ansatz of the form

$$\begin{bmatrix}\phi(\vec{p},t)\\ \chi(\vec{p},t)\end{bmatrix} = e^{-\frac{i}{\hbar}(Et-\vec{p}\cdot\vec{r})}\begin{bmatrix}a\\ b\end{bmatrix}, \quad E\in\mathbb{R}, \quad a,b\in\mathbb{C},$$

in (3) and show that this leads to an eigenvalue problem of the form $M\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right] = E\left[\begin{smallmatrix}a\\ b\end{smallmatrix}\right]$ for a 2×2 matrix M. Determine M, and find its eigenvalues, and argue why we should expect to get these eigenvalues. **(4 points)**

(b) Determine the eigenvectors of M, and combine this with (a) to write down the corresponding solutions to (3) as

$$\Psi_{\pm}(\vec{p},t) = \begin{bmatrix}\phi(\vec{p},t)\\ \chi(\vec{p},t)\end{bmatrix}_{\pm} = \mathcal{N}e^{-\frac{i}{\hbar}(\pm E_{\vec{p}}t-\vec{p}\cdot\vec{r})}\begin{bmatrix}mc^2 \pm E_{\vec{p}}\\ mc^2 - \pm E_{\vec{p}}\end{bmatrix},$$

where $E_{\vec{p}}$ is as defined in (1), and $\mathcal{N}$ is a normalization factor that we do not bother to determine. **(2 points)**

(c) We can conclude from (a) and (b) that the free Klein-Gordon equation has two types of plane-wave solutions. One class where the energy is positive, and one where the energy is negative. Often these are (somewhat vaguely) associated to particles and anti-particles. A rather relevant question is how these solutions behave in the low energy limit, i.e., when speeds are not relativistic. In particular one can note that the two components of the vector $\left[\begin{smallmatrix}mc^2\pm E_{\vec{p}}\\ mc^2-\pm E_{\vec{p}}\end{smallmatrix}\right]$ determines the relative weight between ϕ and χ in the solutions $\Psi_{\pm}$.

- What are the weights of the two components ϕ and χ for the positive and negative plane-waves Ψ_+ and Ψ_- in the case when the momentum is zero?
- Expand $E_{\vec{p}}$ up to the first order in $\frac{p^2}{m^2c^2}$. You will get two energy terms. Interpret these two terms.
- What happens to $\begin{bmatrix}mc^2 \pm E_{\vec{p}}\\ mc^2 - \pm E_{\vec{p}}\end{bmatrix}$ in the limit of small $\frac{p^2}{m^2c^2}$. What does that mean for the relative weight of ϕ and χ in the solutions $\Psi_{\pm}$?
- What happens to the relative weight between ϕ and χ for very high speeds, i.e., when $E_{\vec{p}} >> mc^2$?

(4 points)

(d) Argue that the evolution of positive energy states in the non-relativistic regime (i.e. for small $\frac{p^2}{m^2c^2}$) is approximately governed by a Schrödinger equation. In other words, show that we in the non-relativistic limit regain what we are used to from standard non-relativistic quantum mechanics.

Hint: Consider the results in (c) for this regime. Which terms in (3) are going to be large, and which are going to be small? Be bold and only consider the equation for the dominant term, and put the small things to zero in that equation. Note that this problem to its very nature is rather hand-wavy, so we do not expect any particularly rigorous arguments. **(3 points)**