Advanced Quantum Mechanics

Prof. Dr. J. Krug, Dr. J. Åberg

Exercise sheet 11 (Due: Monday January, 16$^{\mathbf{th}}$.)

11.1 Equation of continuity for the Dirac equation

In the lecture we discussed the Dirac equation, which can be written in the form

$$i\hbar\frac{\partial}{\partial t}\Psi = -i\hbar c\alpha^j\partial_j\Psi + mc^2\beta\Psi,$$

where we apply the summation convention, and where $\alpha^1, \alpha^2, \alpha^3$, and β are 4×4 matrices as defined in the lecture, and where one should keep in mind that Ψ is a spinor (it is a column vector with four components). We define the density $\rho = \Psi^\dagger\Psi$ and the current $\vec{j} = c\Psi^\dagger\vec{\alpha}\Psi$, which means $(j^1, j^2, j^3) = (c\Psi^\dagger\alpha^1\Psi, c\Psi^\dagger\alpha^2\Psi, c\Psi^\dagger\alpha^3\Psi)$. Show that $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{j} = 0$. **(4 points)**

11.2 The algebra of the Pauli matrices

(a) Show that the Pauli-operators (see e.g. exercise 2.1) satisfy the following commutation relations

$$[\sigma^j, \sigma^k] = 2i\sum_l \epsilon_{jkl}\sigma^l \tag{1}$$

and the following anti-commutation relations

$$\{\sigma^j, \sigma^k\} = 2\delta_{jk}\hat{1}. \tag{2}$$

Hint: One can more or less simultaneously prove (1) and (2) by using the properties of the Pauli operators. What is $(\sigma^j)^2$, $\sigma^1\sigma^2$, and $\sigma^2\sigma^1$? These results can be combined to obtain all other products $\sigma^j\sigma^k$, which in turn yield the commutator and anti-commutator. Recall that the Levi-Civita symbol (or the completely anti-symmetric tensor) is defined such that $\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1$, $\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$, and is zero for all other values of the indices. **(4 points)**

(b) Show that

$$(\vec{a}\cdot\vec{\sigma})(\vec{b}\cdot\vec{\sigma}) = \vec{a}\cdot\vec{b}\mathbb{I}_2 + i\vec{\sigma}\cdot(\vec{a}\times\vec{b}). \tag{3}$$

Hint: Make use of (1) and (2). How can one express the "triple product" $\vec{c}\cdot(\vec{a}\times\vec{b})$ in terms of the Levi-Civita symbol? **(2 points)**

11.3 The algebra of the gamma matrices

In the lecture we defined the gamma matrices (or Dirac matrices) $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ in terms of the α and β matrices. (It seems advisable to look this up in the lecture notes.) The purpose of this exercise is to complement the derivations in the lecture. Show that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}_4, \tag{4}$$

where $\mathbb{I}_4$ is the 4×4 identity matrix, and where $g^{\mu\nu}$ is such that $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, and is zero otherwise. **(4 points)**

Remark: The relation in (4) means that the gamma-matrices forms an example of a Clifford algebra.

11.4 Angular momentum and the Dirac equation

The non-relativistic Hamiltonian of a free particle (i.e. $H = \vec{p}^2/(2m)$) commutes with the (orbital) angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$ (like it would for any rotationally symmetric system). Here we shall see that this is not the case for the Dirac Hamiltonian, and that we are more or less forced to include spin in order to regain conservation of angular momentum.

(a) Show that the orbital angular momentum operator $\vec{L} = \vec{r} \times \vec{p}$ does *not* commute with the Dirac Hamiltonian $H = c\vec{\alpha} \cdot \vec{p} + \beta mc^2$. (Recall that $\vec{L}$ can be written component-wise as $L_l = \sum_{j,k} \epsilon_{ljk} r_j p_k$. To show that $[H, \vec{L}] \neq 0$ you have to show that at least one of the components of $\vec{L}$ does not commute with H.) **(2 points)**

(b) Let us now introduce the operator $\vec{\Sigma} = \left[\begin{smallmatrix} \vec{\sigma} & \\ & \vec{\sigma} \end{smallmatrix}\right]$, with components

$$\Sigma^j = \begin{bmatrix} \sigma^j & \\ & \sigma^j \end{bmatrix}, \quad j = 1, 2, 3,$$

with σ^1, σ^2, and σ^3 being the Pauli matrices.

- Show that the Dirac Hamiltonian H *does* commute with $\vec{L} + \frac{\hbar}{2}\vec{\Sigma}$.
- What is the physical interpretation of the operator $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$, and of the operator $\vec{L} + \frac{\hbar}{2}\vec{\Sigma}$?

(4 points)