Advanced Quantum Mechanics

Prof. Dr. J. Krug, Dr. J. Åberg

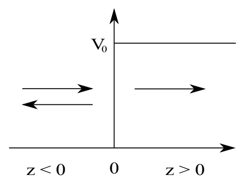
Exercise sheet 12 (Due: Monday January, 23rd.)

12.1 The Klein paradox

When a non-relativistic quantum particle impinges on a potential step it gets reflected if the kinetic energy is lower than the height of the potential, although the wave function has an exponentially decaying tail that extends into the classically forbidden region. Similarly, a Schrödinger particle can tunnel through a potential *barrier*, but the tunneling rate diminishes very rapidly with increasing height and width of the barrier. Oddly enough, the Dirac equation appears to have solutions where the transmission remains non-zero even though the barrier height goes to infinity, and this is often referred to as the Klein paradox. In this exercise we are going to see a hint of this for the solutions of the Dirac equation for a potential step.

We consider a potential $V(z) = 0$ for $z < 0$ and $V(z) = V_0$ for $z > 0$. For a particle of rest mass m we wish to find stationary solutions with a well defined total energy E. As an ansatz we divide the wave-function into $\psi_{z<0}$ for the region $z < 0$ and $\psi_{z>0}$ for $z > 0$, and for $\vec{p} = (0, 0, p)$, $\vec{q} = (0, 0, q)$ and $p > 0$ we let

$$\begin{aligned}\psi_{z<0}(z) &= e^{ipz}U_+^{(1)}(p) + Ae^{-ipz}U_+^{(1)}(-p),\\ \psi_{z>0}(z) &= Be^{iqz}U_+^{(1)}(q),\end{aligned} \qquad (1)$$



where $U_+^{(1)}$ is one (of the four) spinors associated to the solutions of the free Dirac particle that we determined in the lecture, and where A and B are yet undetermined coefficients.

(a)

- Interpret $e^{ipz}U_+^{(1)}(p)$ and $e^{-ipz}U_+^{(1)}(-p)$ in physical terms (in terms of direction of motion, positive or negative energy, helicity).
- Argue that it must be the case that $q^2 = \frac{1}{c^2}(E - V_0)^2 - m^2c^2$.
- For which values of the energy E is q a real number, and for which values is it imaginary?
- Relate the cases of real and imaginary q to plane-waves and decaying solutions in the region $z > 0$. (In the imaginary case, we need to choose a suitable sign.) For a fixed total energy E, argue that the solution becomes a plane-wave in the region $z > 0$ for all sufficiently large V_0.

(4 points)

(b) Our guess in (1) contains the undetermined coefficients A and B. We determine these by demanding that the solution is continuous at $z = 0$, i.e., we assume that $\psi_{z<0}(0) = \psi_{z>0}(0)$. Show that

$$A = \frac{1-\eta}{1+\eta}, \quad B = \frac{2}{1+\eta}, \quad \text{where} \quad \eta = \frac{q}{p}\frac{E + mc^2}{E - V_0 + mc^2}.$$

(2 points)

(c) From exercise 11.1 we can recall the definition of the current density $\vec{j} = (j^1, j^2, j^3)$. Here we are interested in the z-component of the current of the incoming, reflected, and transmitted components of the above solution. Show that for both imaginary and real q it is the case that these currents balance each other, such that $j^3_{\text{in}} + j^3_{\text{refl}} = j^3_{\text{trans}}$, where one should keep in mind to keep the signs of the j^3s. **(3 points)**

(d) From (c) we see that the currents balance as one would expect. However, things are still odd. Show that if $V_0 > E + mc^2$ and $q > 0$, then $|j^3_{\text{refl}}| > |j^3_{\text{in}}|$, and $j^3_{\text{trans}} < 0$. In other words, show that the reflected current is larger than the incoming current, and the transmitted current is negative. This lead Klein to choose $q < 0$. In this case, show that

$$\lim_{V_0 \to +\infty} \frac{|j^3_{\text{refl}}|}{|j^3_{\text{in}}|} = \frac{E - cp}{E + cp}. \tag{2}$$

(3 points)

12.2 Symmetries of the free Dirac-particle

In the lectures we showed that the Dirac equation for a free particle is invariant under charge conjugation, parity, and time-reversal. Here we shall see this in action on specific solutions of the Dirac equation.

(a) With the definition of the gamma-matrices given in the lecture as the starting point, show that

$$\gamma^0 = \begin{bmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{bmatrix}, \quad \gamma^j = \begin{bmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{bmatrix}, \quad j = 1, 2, 3,$$

where $\mathbb{I}_2$ is the 2×2 identity matrix, and $\sigma^1 \equiv \sigma_x, \sigma^2 \equiv \sigma_y, \sigma^3 \equiv \sigma_z$ are the Pauli-operators.

Moreover, we define the gamma-matrix $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Show that

$$\gamma^5 = \begin{bmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{bmatrix}.$$

(3 points)

(b) We define the action of the charge conjugation $\mathcal{C}$, parity $\mathcal{P}$, and time-reversal $\mathcal{T}$ on a spinor $\Psi(\vec{r}, t)$ by

$$\mathcal{C}[\Psi](\vec{r}, t) = i\gamma^2\Psi^*(\vec{r}, t), \quad \mathcal{P}[\Psi](\vec{r}, t) = \gamma^0\Psi(-\vec{r}, t), \quad \mathcal{T}[\Psi](\vec{r}, t) = \gamma^1\gamma^3\Psi^*(\vec{r}, -t). \tag{3}$$

The free-particle Dirac-equation is invariant under all these three operations. This means that if Ψ is a solution, then $\mathcal{C}[\Psi]$, $\mathcal{P}[\Psi]$, and $\mathcal{T}[\Psi]$ are also solutions.

In the lecture we determined the positive-helicity, positive-energy, free particle Dirac spinor, for a plane-wave moving in the z-direction. Here we denote this by

$$\psi_{\uparrow+}(p, z, t) = e^{-\frac{i}{\hbar}(E_{\vec{p}}t - pz)} U^{(1)}_+(\vec{p}) = e^{-\frac{i}{\hbar}(E_{\vec{p}}t - pz)} \begin{bmatrix} 1 \\ 0 \\ \frac{cp}{E_{\vec{p}} + mc^2} \\ 0 \end{bmatrix}. \tag{4}$$

Calculate $\mathcal{C}[\psi_{\uparrow+}](p, z, t)$, $\mathcal{P}[\psi_{\uparrow+}](p, z, t)$, and $\mathcal{T}[\psi_{\uparrow+}](p, z, t)$, and express the results in terms of the other plane-wave solutions, where you choose the appropriate positive or negative helicity, energy, and momentum. **(5 points)**

Remark: In the lecture we did strictly speaking define the time-reversal as $\gamma^1\gamma^3\Psi^*(\vec{r}, t)$, which maps a solution of the Dirac-equation, to a solution of the *time-reversed* Dirac equation. However, in (3) we have incorporated the coordinate transformation $t \mapsto -t$, and as a result $\gamma^1\gamma^3\Psi^*(\vec{r}, -t)$ is a solution of the *original* equation.