Advanced Quantum Mechanics

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Exercise sheet 13 (Due: Monday January, 30$^{\text{th}}$.)

13.1 The optical theorem

Suppose that a plane wave propagating in the z-direction with wave-number k impinges on a spherically symmetric scatterer. Far from the scatterer we can approximate the resulting scattering state as a superposition of the incoming plane-wave ψ_{in} and an outgoing spherical wave ψ_{s},

$$\psi(\vec{r}) = \psi_{\text{in}}(\vec{r}) + \psi_{\text{s}}(\vec{r}), \quad \psi_{\text{in}}(\vec{r}) = e^{ikz}, \quad \psi_{\text{s}}(\vec{r}) = f(\theta)\frac{e^{ikr}}{r}, \tag{1}$$

where $\vec{r} = (x, y, z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$ and $r = \|\vec{r}\|$. The function f does not depend on φ since we assume a spherically symmetric scatterer. From the total current $\vec{J}_\psi = \frac{\hbar}{m}\text{Im}[\psi^*\nabla\psi]$, we subtract the current $\vec{J}_{\psi_{\text{in}}}$ of the incoming wave ψ_{in}, as well as the current $\vec{J}_{\psi_{\text{s}}}$ of the outgoing spherical wave ψ_{s}, resulting in the "leftover" $\vec{J}_{\text{int}} = \vec{J}_\psi - \vec{J}_{\psi_{\text{in}}} - \vec{J}_{\psi_{\text{s}}}$. The current $\vec{J}_{\text{int}}$ is due to interference between the incoming plane-wave ψ_{in} and the outgoing spherical wave ψ_{s}. As we shall see, this interference plays a crucial role for the optical theorem.

(a) Let S_r denote a spherical surface with radius r, centered around the scatterer.

- Argue that

$$\int_{S_r} \vec{J}_{\psi_{\text{in}}} \cdot \vec{e}_r dS = 0 \quad \text{and} \quad \int_{S_r} \vec{J}_{\psi_{\text{s}}} \cdot \vec{e}_r dS = -\int_{S_r} \vec{J}_{\text{int}} \cdot \vec{e}_r dS, \tag{2}$$

  where these are surface integrals over the spherical surface S_r, and $\vec{e}_r$ is the unit radial vector.

- Use the right hand side expression in (2) to show that

$$\sigma_{\text{tot}} = -2\pi\frac{m}{\hbar k}\int_0^\pi \vec{e}_r \cdot \vec{J}_{\text{int}} r^2 \sin\theta d\theta, \tag{3}$$

  where σ_{tot} is the total cross section.

Hint: Due to conservation of probability, it follows that the total flow through any closed surface has to be zero. Keep in mind that $\sigma_{\text{tot}} = \frac{m}{\hbar k}\int_{S_r} \vec{J}_{\psi_{\text{s}}} \cdot \vec{e}_r dS$. **(4 points)**

(b) Show that

$$\vec{e}_r \cdot \vec{J}_{\text{int}} = \frac{\hbar k}{mr}(1+\cos\theta)\text{Re}[f(\theta)e^{ikr(1-\cos\theta)}] - \frac{\hbar}{mr^2}\text{Im}[f(\theta)e^{ikr(1-\cos\theta)}]. \tag{4}$$

Hint: In polar coordinates $\nabla = \vec{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\vec{e}_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\vec{e}_\varphi\frac{\partial}{\partial\varphi}$. Note also that $\vec{e}_r \cdot \vec{e}_z = \cos\theta$. **(4 points)**

(c) Let $g(x)$ be a (sufficiently smooth) function. Show that

$$\lim_{r\to+\infty}\int_0^2 g(x)e^{ikrx}dx = 0, \qquad \lim_{r\to+\infty} r\int_0^2 (2-x)g(x)e^{ikrx}dx = \frac{2i}{k}g(0). \tag{5}$$

Hint: Use partial integration. **(4 points)**

(d) Show that

$$\lim_{r\to+\infty}\int_0^\pi r^2\sin\theta\,\vec{e}_r\cdot\vec{J}_{\text{int}}d\theta = \frac{2i}{k}f(0),$$

and prove the optical theorem

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}f(0).$$

Hint: By a change of variables one can make use of (5). **(2 points)**

13.2 Scattering from an oscillatory potential

Suppose that we have a spherically symmetric potential with the radial dependence $V(r) = \sin(ar)e^{-br}$, for $a > 0$ and $b > 0$. In other words, the potential oscillates, but the oscillations go to zero exponentially fast as r approaches infinity.

(a) Show that the scattering amplitude in the first order Born approximation is

$$f_B^{(1)}(q) = \frac{m}{\hbar^2 q}\left[\frac{b^2-(q+a)^2}{[b^2+(q+a)^2]^2} - \frac{b^2-(q-a)^2}{[b^2+(q-a)^2]^2}\right], \tag{6}$$

where $q = \|\vec{k}_{\text{out}} - \vec{k}_{\text{in}}\|$. Note that we here use the simplified expression for the first order Born scattering that is valid for spherically symmetric potentials.

Hint: Keep in mind the relation $\sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$. It can be convenient to use

$$\int_0^{+\infty} xe^{\gamma x}dx = \frac{1}{\gamma^2}, \quad \text{for} \quad \text{Re}(\gamma) < 0.$$

(4 points)

(b) If we let $b \to 0$ in (6), i.e., if we take the limit of an infinitely slow decay of the potential with the distance, one finds that the scattering amplitude goes to infinity for a certain value of q (apart from $q = 0$). What value is that? **(2 points)**