Advanced Quantum Mechanics

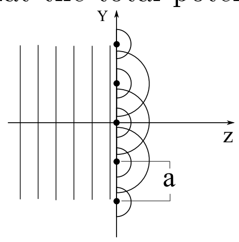
Prof. Dr. J. Krug, Dr. J. Åberg

Exercise sheet 14 (Due: Monday February, 6th.)

14.1 Scattering from a row of delta potentials

Here we consider a model for the elastic scattering of particles (e.g. of neutrons) by the nuclei of a linear molecule, or by a row of trapped ions or atoms. Suppose that an incoming particle of mass m, with wave-number k in the direction $\vec{e}_z$, is scattered by a row of delta-potentials, distributed along the y-axis in such a way that the total potential is

$$V(x,y,z) = \gamma \sum_{n=0}^{N-1} \delta(z)\delta(x)\delta\big(y - an\big),$$



where $a > 0$ and $\gamma > 0$.

(a) Show that the scattering amplitude in the first-order Born approximation is

$$f_B^{(1)}(\vec{q}) = -\frac{m\gamma}{2\pi\hbar^2} e^{-iaq_y(N-1)/2} \frac{\sin(aq_y N/2)}{\sin(aq_y/2)},$$

where q_y is the y-component of $\vec{q} = (q_x, q_y, q_z) = \vec{k}_{\text{out}} - \vec{k}_{\text{in}}$, with $\vec{k}_{\text{out}}$ being the wave-vector of the scattered wave.

Hint: The first-order Born approximation can be written $f_B^{(1)}(\vec{q}) = -\frac{m}{2\pi\hbar^2}\int d^3r V(\vec{r})e^{-i\vec{q}\cdot\vec{r}}$.

(2 points)

(b) Next, use the above result to find the differential cross section in the direction (θ, φ), where we use the spherical coordinates $(x, y, z) = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$.

(2 points)

(c) For the result in (b)

- If you let $N = 1$, what is the angular dependence of the differential cross section?
- In the general case, for arbitrary N, what is the differential cross-section in the limit of very small ka, i.e., for $ka \ll 1$?
- How does the $ka \ll 1$ limit depend on (θ, φ)?
- How does the scattering in the $ka \ll 1$ limit scale with N?

(4 points)

14.2 Zero energy scattering and the scattering length

In a spherically symmetric potential V the solutions to the time-independent Schrödinger equation decompose into an angular and a radial part. If the potential is $V(r) = 0$ for $r > R$, then the radial part of the wave function in the region $r > R$ takes the form

$$\varphi_l(r) = B_l[\cos(\delta_l)j_l(kr) - \sin(\delta_l)y_l(kr)], \quad \text{for } r > R, \tag{1}$$

where j_l and y_l are the spherical Bessel and Neumann functions, and δ_l is the scattering phase-shift, and B_l is an undetermined constant. Inside the region $r \leq R$, the form of the solutions will depend on the potential.

The so called "s-wave" corresponds to the spherical wave with zero angular momentum, i.e., $l = 0$. For $l = 0$ the spherical Bessel and Neumann functions take an especially simple form

$$j_0(x) = \frac{\sin(x)}{x}, \quad y_0(x) = -\frac{\cos(x)}{x}.$$

If we only scatter at low energies, then it can be a good approximation to only include the s-wave component of the scattering.

The scattering length a is defined in the limit of zero energy scattering as

$$\frac{1}{a} = -\lim_{k\to 0} \frac{k}{\tan \delta_0}. \tag{2}$$

The scattering length relates to the total cross section (in the zero energy limit) as $\sigma_{\text{tot}} = 4\pi a^2$. Hence, a can in some sense be regarded as the apparent radius of the scatterer (although it can be negative). In this exercise we investigate the scattering length of the potential well.

(a) Here we consider the spherically symmetric potential well

$$V(\vec{r}) = \begin{cases} -V_0 & |\vec{r}| \leq R, \\ 0, & |\vec{r}| > R, \end{cases} \tag{3}$$

where $V_0 > 0$. The solution inside the region $r \leq R$ can (for this particular potential) also be described by the spherical Bessel and Neumann functions, but they will *not* be combined as in equation (1).

- By demanding that the wave function should be non-singular at $r = 0$, argue that the radial solution must be of the form $A_0 j_0(k'r)$, for $r \leq R$, where A_0 is an undetermined constant.
- Argue that $k' = \sqrt{\frac{2M}{\hbar^2}V_0 + k^2}$.

(4 points)

(b) Use the continuity of the wave function and its first derivative in order to show that the s-wave phase-shift δ_0 satisfies

$$\frac{k}{k'}\tan(k'R) = \tan(kR + \delta_0). \tag{4}$$

Hint: Before employing the continuity, rewrite the solution in (1) by using the trigonometric identity $\sin(\phi + \chi) = \sin(\phi)\cos(\chi) + \cos(\phi)\sin(\chi)$.

(4 points)

(c) First show that

$$-\frac{k}{\tan(\delta_0)} = \frac{1}{R}\frac{1 + \frac{k}{k'}\tan(k'R)\tan(kR)}{\frac{\tan(kR)}{kR} - \frac{\tan(k'R)}{k'R}}.$$

then use this to show that the scattering length a is given by

$$a = R\left(1 - \frac{\tan(qR)}{qR}\right), \quad \text{where} \quad q = \sqrt{\frac{2M}{\hbar^2}V_0}.$$

Sketch (roughly) the behavior of $\frac{a}{R}$ as a function of qR.

Hint: $\tan(\phi + \chi) = \frac{\tan(\phi)+\tan(\chi)}{1-\tan(\phi)\tan(\chi)}$.

(4 points)