Advanced Quantum Mechanics

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Exercise sheet 3 (Due: Monday November, 7$^{\text{th}}$.)

3.1 Invariant subspaces and para-statistics

Imagine three indistinguishable particles occupying three orthonormal single-particle states $|\alpha\rangle$, $|\beta\rangle$, and $|\gamma\rangle$. We have learned that this should be associated with the totally symmetric state $|\psi_s\rangle$ (for bosons) or the totally anti-symmetric state $|\psi_a\rangle$ (for fermions)

$$
\begin{aligned}
|\psi_s\rangle =& \frac{1}{\sqrt{6}}(|\alpha,\beta,\gamma\rangle + |\beta,\alpha,\gamma\rangle + |\alpha,\gamma,\beta\rangle + |\gamma,\beta,\alpha\rangle + |\gamma,\alpha,\beta\rangle + |\beta,\gamma,\alpha\rangle),\\
|\psi_a\rangle =& \frac{1}{\sqrt{6}}(|\alpha,\beta,\gamma\rangle - |\beta,\alpha,\gamma\rangle - |\alpha,\gamma,\beta\rangle - |\gamma,\beta,\alpha\rangle + |\gamma,\alpha,\beta\rangle + |\beta,\gamma,\alpha\rangle),
\end{aligned}
$$

where we use the short-hand notation $|\alpha,\beta,\gamma\rangle = |\alpha\rangle_1|\beta\rangle_2|\gamma\rangle_3$. Could one imagine something other than this? In this exercise we explore a theoretical alternative, sometimes referred to as para-statistics. It is useful to introduce the six possible permutations of the three particles $S_3 = \{1, P_{12}, P_{23}, P_{31}, P_{123}, P_{132}\}$ (the symmetric group over three objects). Here P_{123} means that particle 1 is relabelled to 2, 2 is relabelled to 3, and 3 to 1, i.e. $P_{123}|\alpha\rangle_1|\beta\rangle_2|\gamma\rangle_3 = |\alpha\rangle_2|\beta\rangle_3|\gamma\rangle_1 = |\gamma\rangle_1|\alpha\rangle_2|\beta\rangle_3$. When these six permutations act on the state $|\alpha,\beta,\gamma\rangle$, we obtain six orthonormal states that thus span a six-dimensional space. If the particles are indistinguishable, no experiment would be able to detect any difference between the elements of this space. The states $|\psi_s\rangle$ and $|\psi_a\rangle$ each span a one-dimensional subspace that is invariant under the action of S_3, since $P|\psi_s\rangle = |\psi_s\rangle$ for all $P \in S_3$, and since $P|\psi_a\rangle = |\psi_a\rangle$ for even permutations P, and $P|\psi_a\rangle = -|\psi_a\rangle$ for odd permutations. Moreover, these two subspaces are orthogonal to each other. Hence, the symmetric and anti-symmetric subspace only use two out of the six dimensions available, so there is indeed room for something more.

(a) Show that all elements of S_3 can be written as products of P_{12} and P_{31}. (The technical term is that P_{12} and P_{31} are generators of S_3.) **(2 points)**

(b) Consider the two orthonormal vectors

$$
\begin{aligned}
|\psi_1\rangle =& \frac{1}{\sqrt{12}}(2|\alpha,\beta,\gamma\rangle + 2|\beta,\alpha,\gamma\rangle - |\alpha,\gamma,\beta\rangle - |\gamma,\beta,\alpha\rangle - |\gamma,\alpha,\beta\rangle - |\beta,\gamma,\alpha\rangle),\\
|\psi_2\rangle =& \frac{1}{2}(-|\alpha,\gamma,\beta\rangle + |\gamma,\beta,\alpha\rangle + |\gamma,\alpha,\beta\rangle - |\beta,\gamma,\alpha\rangle).
\end{aligned}
$$

The action of P_{12} on these vectors is given by

$$
P_{12}|\psi_1\rangle = |\psi_1\rangle, \quad P_{12}|\psi_2\rangle = -|\psi_2\rangle.
$$

Argue that this means that $P_{12}|v\rangle \in \mathcal{V}$ for all $|v\rangle \in \mathcal{V} = \mathrm{Sp}\{|\psi_1\rangle, |\psi_2\rangle\}$, i.e., the space spanned by $|\psi_1\rangle, |\psi_2\rangle$ is left invariant by P_{12}. Show that $\mathcal{V}$ also is invariant under P_{31}. Combine these results with 3.1(a) to prove that $\mathcal{V}$ is invariant under all of S_3. **(4 points)**

Remark: This shows that there exist additional invariant subspaces, apart from the symmetric and anti-symmetric ones. One could imagine that states of identical particles would be elements of such subspaces, thus obeying para-statistics, rather than Bose- or Fermi-statistics. However, it seems that nature does not favour such alternatives.

3.2 Energy spectrum for identical particles

Suppose that we have two identical particles of mass m on a line, and that these interact via a harmonic potential, such that the Hamiltonian is $H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2} + \frac{k}{2}(x_1 - x_2)^2$. As you may recall, it is useful to change to the center of mass coordinate $R = (mx_1 + mx_2)/(2m) = \frac{1}{2}(x_1 + x_2)$ and the relative coordinate $r = x_1 - x_2$. By separation of variables, the eigenfunctions of H can be written $\psi_{p,n}(R, r) = e^{ipR}\phi_n(r)$, where ϕ_n for $n = 0, 1, 2, \ldots$ are the solutions to the harmonic oscillator $H' = -\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{k}{2}r^2$, with the reduced mass $\mu = m/2$.

(a) How does r and R transform under the exchange of particles? What can we conclude concerning the symmetry or anti-symmetry of the factor e^{ipR}? Argue that it is only $\phi_n(r)$ that determines whether $\psi_{p,n}(R, r)$ is symmetric or anti-symmetric under particle exchange. **(2 points)**

(b) For which $n = 0, 1, 2, \ldots$ does ϕ_n correspond to a solution that is symmetric under permutation of particles 1 and 2, and for which n are they anti-symmetric? If these particles have no additional degrees of freedom, what would be the spectrum be for two identical bosons, and what would it be for two identical fermions? Ignore the center of mass motion, and express the spectrum in terms of m and k. **(2 points)**

(c) Suppose now that the two particles in addition have a spin degree of freedom, and more precisely that they are spin-half fermions. What is the spectrum, and what are the degeneracies? (Do again ignore the center of mass motion.) What is the lowest energy that the system can have if the total spin is restricted to be in a spin-singlet state? What is the lowest energy if it is restricted to a spin triplet? **(4 points)**

Remark: The Hamiltonian in this example has no explicit dependence on the spin. Nevertheless, the total energy depends on the spin-state. This may at first sight seem perplexing, but is due to the restriction of the state space imposed by the anti-symmetrization (or symmetrization). A striking example of this is the hydrogen molecule, H_2. The proton is a spin-half fermion, and thus the two protons can either be in a nuclear spin singlet state (parahydrogen) or in a nuclear spin triplet (orthohydrogen). Apart from the nuclear spin, the two protons can also orbit each other, and much analogous to this exercise, it turns out that the nuclear spin state affects the allowed orbital states, which leads to a different ground state energy for ortho- and parahydrogen. In the production of liquid hydrogen one often uses a catalyst to speed the conversion to the lowest energy state.

3.3 The social life of fermions and bosons

Suppose that $|\psi_a\rangle$ and $|\psi_b\rangle$ are two orthonormal single-particle states. One possible two-particle state for two *distinguishable* particles would be $|\psi_{\text{dist}}\rangle = |\psi_a\rangle_1|\psi_b\rangle_2$, meaning that particle 1 is in state ψ_a and particle 2 is in state ψ_b. Let X_1 and X_2 be the position operator of particle 1 and 2, respectively. The operator $(X_1 - X_2)^2$ (or $(X_1 \otimes \hat{1}_2 - \hat{1}_1 \otimes X_2)^2$ in a more elaborate notation) would thus measure the (square) distance between the two particles. Let $|\psi_{\text{Bose}}\rangle$ be the two-particle symmetrized state based on $|\psi_a\rangle$ and $|\psi_b\rangle$, and let $|\psi_{\text{Fermi}}\rangle$ be the two-particle anti-symmetrized state.

(a) Write $\langle\psi_{\text{dist}}|(X_1 - X_2)^2|\psi_{\text{dist}}\rangle$ in terms of the single-particle expectations $\langle\psi_a|(X - \langle\psi_a|X|\psi_a\rangle)^2|\psi_a\rangle$, $\langle\psi_b|(X - \langle\psi_b|X|\psi_b\rangle)^2|\psi_b\rangle$, and $(\langle\psi_a|X|\psi_a\rangle - \langle\psi_b|X|\psi_b\rangle)^2$. **(2 points)**

(b) Do the same for $\langle\psi_{\text{Bose}}|(X_1 - X_2)^2|\psi_{\text{Bose}}\rangle$.

Hint: Compared to (a) a new term appears. **(2 points)**

(c) Finally determine $\langle\psi_{\text{Fermi}}|(X_1 - X_2)^2|\psi_{\text{Fermi}}\rangle$. It is often stated that bosons are very 'social' and tend to cluster, while fermions tend to avoid each other. How do the above results fit with these claims? **(2 points)**