Advanced Quantum Mechanics

Prof. Dr. J. Krug, Dr. J. Åberg

Exercise sheet 5 (Due: Monday November, 21$^{\text{st}}$.)

5.1 Conservation of particle-number for one- and two-body Hamiltonians

We know from the lectures that a Hamiltonian for identical particles that contains single-particle terms, as well as two-particle terms, can be written on the form

$$H = \sum_{jk} T_{jk} a_j^\dagger a_k + \sum_{jkmn} F_{jk,mn} a_j^\dagger a_k^\dagger a_n a_m.$$

In this exercise we will show that Hamiltonians of this form always commute with the total number operator, in both the bosonic and fermionic case. This means that the particle number is conserved, i.e., the dynamics induced by H does not change the total number of particles in the system.

(a) If $a_j, a_j^\dagger$ are bosonic annihilation and creation operators, show that

$$\begin{aligned}
[a_j^\dagger a_k, a_l^\dagger a_l] =& a_j^\dagger a_l \delta_{k,l} - a_l^\dagger a_k \delta_{l,j},\\
[a_j^\dagger a_k^\dagger a_n a_m, a_l^\dagger a_l] =& a_j^\dagger a_k^\dagger a_n a_l \delta_{m,l} + a_j^\dagger a_k^\dagger a_m a_l \delta_{n,l} - a_l^\dagger a_j^\dagger a_n a_m \delta_{l,k} - a_l^\dagger a_k^\dagger a_n a_m \delta_{l,j}
\end{aligned}$$

Hint: Use the relation $[AB, C] = A[B, C] + [A, C]B$ from exercise 1.1(a) as well as the analogous relation $[A, BC] = B[A, C] + [A, B]C$, and then apply the bosonic commutation relations. **(3 points)**

(b) Show that the total number operator $N = \sum_l a_l^\dagger a_l$ commutes with H in the bosonic case. **(3 points)**

(c) If $a_j, a_j^\dagger$ are fermionic annihilation and creation operators, show that

$$\begin{aligned}
[a_j^\dagger a_k, a_l^\dagger a_l] =& a_j^\dagger a_l \delta_{k,l} - a_l^\dagger a_k \delta_{l,j},\\
[a_j^\dagger a_k^\dagger a_n a_m, a_l^\dagger a_l] =& a_j^\dagger a_k^\dagger a_n a_l \delta_{m,l} - a_j^\dagger a_k^\dagger a_m a_l \delta_{n,l} + a_l^\dagger a_j^\dagger a_n a_m \delta_{l,k} - a_l^\dagger a_k^\dagger a_n a_m \delta_{l,j}.
\end{aligned}$$

Hint: Use $[AB, C] = A[B, C] + [A, C]B$, $[A, BC] = B[A, C] + [A, B]C$, and $[AB, C] = A\{B, C\} - \{A, C\}B$ (see exercise 1.1(a)), as well as the fermionic commutation relations. **(3 points)**

(d) Show that the total number operator $N = \sum_l a_l^\dagger a_l$ commutes with H also in the fermionic case. **(3 points)**

5.2 Finding the spectrum of a Hamiltonian by transformation of annihilation and creation operators

Consider a single bosonic mode, with annihilation and creation operators $a, a^\dagger$, on which we define the Hamiltonian

$$H = \hbar\omega(a^\dagger a + \frac{1}{2}\hat{1}) + \frac{1}{2}\hbar(\Delta^* a^\dagger a^\dagger + \Delta aa),$$

where $\omega > |\Delta|$. We wish to find the spectrum of this Hamiltonian.

In exercise 4.2 we introduced the Bobogliubov transformation in the special case of a single pair of bosonic annihilation and creation operators $a, a^\dagger$. We found that the pair $b, b^\dagger$, defined by $b = Aa + Ba^\dagger$, are also bosonic annihilation and creation operators if and only if $|A|^2 - |B|^2 = 1$. Find such a transformation to bosonic $b, b^\dagger$ with real numbers $\tilde{\omega}$ and q, such that

$$H = \hbar\tilde{\omega} b^\dagger b + \hbar q \hat{1},$$

and use this to determine the spectrum of H. **(4 points)**

Remark: Hamiltonians of this type occur for example in quantum optics (related to something called "single mode squeezing") and we shall later see a generalization of this type of Hamiltonian (and a generalization of this type of transformation) when we discuss weakly interacting bosons.

5.3 Continuity equation for the particle density

Suppose that we have a Hamiltonian of the form

$$\begin{aligned} H = &\int \frac{\hbar^2}{2m} \nabla\Psi^\dagger(\vec{x}) \cdot \nabla\Psi(\vec{x})\, d^3x + \int \mathcal{V}^{(1)}(\vec{x})\Psi^\dagger(\vec{x})\Psi(\vec{x})\, d^3x \\ &+ \frac{1}{2}\iint \Psi^\dagger(\vec{x})\Psi^\dagger(\vec{x'})\mathcal{V}^{(2)}(\vec{x},\vec{x'})\Psi(\vec{x'})\Psi(\vec{x})\, d^3x\, d^3x', \end{aligned} \tag{1}$$

where the first term describes the kinetic energy, the second term the single-particle potential, and the third describes pairwise interactions. As we know from the lecture, one can define the particle density $n(\vec{x}) = \Psi^\dagger(\vec{x})\Psi(\vec{x})$, which is the operator that measures the density of particles at point $\vec{x}$. One can also define a *current* density operator by

$$\vec{j}(\vec{x}) = \frac{\hbar}{2im}[\Psi^\dagger(\vec{x})\nabla\Psi(\vec{x}) - \left(\nabla\Psi^\dagger(\vec{x})\right)\Psi(\vec{x})].$$

Show that systems governed by Hamiltonians of the form (1) satisfy the following continuity equation

$$\frac{d}{dt} n(\vec{x}, t) = -\nabla \cdot \vec{j}(\vec{x}, t),$$

where $n(\vec{x}, t)$ and $\vec{j}(\vec{x}, t)$ are the particle density and the current density operator in the Heisenberg picture. You can take the Heisenberg equation of motion of the field operator as the starting point of the proof. (The equation of motion of the field operator was derived in the lecture.) **(4 points)**

Remark: Just a reminder of some standard notation from vector calculus.

- For a function $f(\vec{x})$, the gradient is $\nabla f(\vec{x}) = \left(\frac{\partial f}{\partial x}(\vec{x}), \frac{\partial f}{\partial y}(\vec{x}), \frac{\partial f}{\partial z}(\vec{x})\right)$.
- For a vector-field $\vec{F}(\vec{x}) = \left(F_x(\vec{x}), F_y(\vec{x}), F_z(\vec{x})\right)$, the divergence is $\nabla \cdot \vec{F}(\vec{x}) = \frac{\partial F_x}{\partial x}(\vec{x}) + \frac{\partial F_y}{\partial y}(\vec{x}) + \frac{\partial F_z}{\partial z}(\vec{x})$.
- The Laplacian is $\nabla^2 f(\vec{x}) = \nabla \cdot \nabla f(\vec{x}) = \frac{\partial^2 f}{\partial x^2}(\vec{x}) + \frac{\partial^2 f}{\partial y^2}(\vec{x}) + \frac{\partial^2 f}{\partial z^2}(\vec{x})$.

As compared to "standard" vector calculus, the functions we are dealing with here are operator-valued (like $\Psi(\vec{x})$ and $\Psi^\dagger(\vec{x})$), and similarly the components of the vector-fields are operator-valued (such as $\vec{j}(\vec{x})$).