Advanced Quantum Mechanics

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Exercise sheet 6 (Due: Monday November, 28$^{\mathbf{th}}$.)

6.1 The evolution of coherent states in a harmonic oscillator

One can define coherent states $|\alpha\rangle$ with respect to a bosonic annihilation operator a via the relation $a|\alpha\rangle = \alpha|\alpha\rangle$ for all $\alpha \in \mathbb{C}$, or alternatively as $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$. Here we shall investigate how coherent states evolve in a harmonic oscillator $H_{\text{oscillator}} = \hbar\omega a^\dagger a$.

- Suppose that a complex number α evolves in time according to the differential equation
$$i\hbar\frac{d}{dt}\alpha(t) = \hbar\omega\alpha(t). \tag{1}$$
Show that $\frac{d}{dt}|\alpha(t)|^2 = 0$.
- One can generate (normalized) coherent states from the vacuum state in the following way
$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}|0\rangle. \tag{2}$$
Show that if we put $\alpha = \alpha(t)$ in (2) where $\alpha(t)$ satisfies (1), then $\frac{d}{dt}|\alpha(t)\rangle = \frac{d\alpha}{dt}a^\dagger|\alpha(t)\rangle$.
- Finally, use the above results to show that $|\alpha(t)\rangle$ satisfies Schrödinger's equation $i\hbar\frac{d}{dt}|\alpha(t)\rangle = H_{\text{oscillator}}|\alpha(t)\rangle$.

(4 points)

Remark: Coherent states have the smallest possible simultaneous uncertainty in position and momentum, and are in this sense as close to classical points in phase space as one can get in quantum mechanics. By this exercise one can see that coherent state stay coherent in the harmonic oscillator. Moreover, (1) gives the evolution of the corresponding classical oscillator, if one interprets the real and complex part of $\alpha(t)$ as the position and momentum. Hence, these minimum uncertainty wave-packages follow the trajectory of the classical Harmonic oscillator.

6.2 The evolution of coherent states in a bosonic quantum field

In the previous exercise we investigated the evolution of coherent states in a single bosonic mode that evolves as a harmonic oscillator. Here we shall make an analogous construction for a (non-relativistic) bosonic quantum field. Let $\Psi(\vec{x})$ and $\Psi^\dagger(\vec{x})$ be the field annihilation and creation operators (with $\Psi(\vec{x}) = \sum_j \varphi_j(\vec{x})a_j$ for an orthonormal basis φ_j and annihilation operators a_j). We define coherent states $|\phi\rangle$ via the relation $\Psi(\vec{x})|\phi\rangle = \phi(\vec{x})|\phi\rangle$ for all $\vec{x} \in \mathbb{R}^3$, where $\phi(\vec{x})$ is a complex-valued function. (Each complex number α defines a coherent state on a single bosonic mode. Here we have infinitely many modes, and each complex-valued *function* ϕ defines a coherent field state.)

Here we shall investigate how coherent field states evolve under the Hamiltonian

$$H_{\text{field}} = \int \left(\frac{\hbar^2}{2m}\nabla\Psi^\dagger(\vec{x})\cdot\nabla\Psi(\vec{x}) + \mathcal{V}^{(1)}(\vec{x})\Psi^\dagger(\vec{x})\Psi(\vec{x})\right)d^3x, \tag{3}$$

where m is the mass of the particles.

(a) Suppose that the complex-valued function $\phi(\vec{x},t)$ evolves in time according to the partial differential equation

$$i\hbar\frac{\partial}{\partial t}\phi(\vec{x},t) = \big(-\frac{\hbar^2}{2m}\nabla^2 + \mathcal{V}^{(1)}(\vec{x})\big)\phi(\vec{x},t). \tag{4}$$

Show that $\frac{\partial}{\partial t}\int |\phi(\vec{x},t)|^2\, d^3x = 0$.

Hint: You can assume that $\phi(\vec{x},t)$ is such that
$\int \nabla^2\phi(\vec{x},t)\phi^*(\vec{x},t)\, d^3x = -\int \nabla\phi(\vec{x},t)\cdot\nabla\phi^*(\vec{x},t)\, d^3x = \int \phi(\vec{x},t)\nabla^2\phi^*(\vec{x},t)\, d^3x$. **(3 points)**

(b) One can generate (normalized) coherent field states from the vacuum state in the following way

$$|\phi\rangle = e^{-\frac{1}{2}\int|\phi(\vec{x})|^2\, d^3x} e^{\int \phi(\vec{x})\Psi^\dagger(\vec{x})\, d^3x}|0\rangle. \tag{5}$$

Define $|\phi(t)\rangle$ as in (5) with $\phi(\vec{x}) = \phi(\vec{x},t)$, where $\phi(\vec{x},t)$ satisfies (4). Show that $\frac{\partial}{\partial t}|\phi(t)\rangle = \int \frac{\partial\phi(\vec{x},t)}{\partial t}\Psi^\dagger(\vec{x})\, d^3x|\phi\rangle$.

Hint: If $A(t)$ is an operator-valued function and if $[\frac{dA}{dt}(t), A(t)] = 0$, then $\frac{d}{dt}e^{A(t)} = \frac{dA}{dt}e^{A(t)} = e^{A(t)}\frac{dA}{dt}$. **(3 points)**

(c) Define $|\phi(t)\rangle$ as in (5), where $\phi(\vec{x}) = \phi(\vec{x},t)$ satisfies (4). Show that $|\phi(t)\rangle$ satisfies the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle = H_{\text{field}}|\phi(t)\rangle$, with H_{field} as in (3).

Hint: You can assume that $|\phi\rangle$ is such that
$\int \nabla\Psi^\dagger(\vec{x})\cdot\nabla\Psi(\vec{x})\, d^3x|\phi(t)\rangle = -\int \Psi^\dagger(\vec{x})\nabla^2\Psi(\vec{x})\, d^3x|\phi(t)\rangle$. **(3 points)**

Remark: The function $\phi(\vec{x},t)$ can be regarded as a classical field that evolves according to the field equation (4). (One can alternatively regard (4) as the single-particle Schrödinger equation, but where the wave-function $\phi(\vec{x},t)$ is not necessarily normalized to 1.) Analogous to coherent states in Harmonic oscillators, the coherent field states follow the trajectory of the classical field $\phi(\vec{x},t)$.

6.2 Single-body correlation functions

In statistical physics and condensed matter physics it is often useful to describe properties of systems in terms of correlation functions. Here we shall take a look at one of these, namely the single-body (or single-particle) correlation function $\langle\phi|\Psi^\dagger(\vec{x'})\Psi(\vec{x})|\phi\rangle$, which is some sense measures how much a state $|\phi\rangle$ changes if we remove a particle from one position $\vec{x}$ and put it at another position $\vec{x'}$. Here we investigate the correlation function of the ground state of gases of non-interacting particles. For the sake of simplicity we shall not consider $\mathbb{R}^3$ but assume that everything happens in $\mathbb{R}$ (and thus we write $\Psi(x)$ and $\Psi^\dagger(x)$ for $x \in \mathbb{R}$). For a single particle of mass m with Hamiltonian $H = P^2/(2m)$, which moves on an interval of length L with periodic boundary conditions, the eigenstates are $\varphi_k(x) = e^{ikx}/\sqrt{L}$, with $k = 2\pi\frac{l}{L}$, $l \in \mathbb{Z}$.

(a) For a collection of identical particles (bosons or fermions) let $|\phi\rangle$ be such that each single-particle state φ_k is occupied by $\tilde{n}_k$ quanta. (Hence, the state can be written $|\phi\rangle = \Pi_k \frac{a^{\dagger\tilde{n}_k}}{\sqrt{\tilde{n}_k!}}|0\rangle$.) Show that the single-body correlation function is $\langle\phi|\Psi^\dagger(x')\Psi(x)|\phi\rangle = \frac{1}{L}\sum_k e^{-ik(x'-x)}\tilde{n}_k$. **(2 points)**

(b) For N non-interacting bosons, what would the correlation function $\langle\phi|\Psi^\dagger(x')\Psi(x)|\phi\rangle$ be if $|\phi\rangle$ is the ground state? **(2 points)**

(c) If $|\phi\rangle$ is the ground state for N non-interacting (spinless) fermions, show that the correlation function becomes $\langle\phi|\Psi^\dagger(x')\Psi(x)|\phi\rangle = \frac{1}{L}\frac{\sin(\frac{N\pi}{L}(x'-x))}{\sin(\frac{\pi}{L}(x'-x))}$. It is enough that you do the cases where N is odd. (The cases of even N get a bit annoying due to a degeneracy.) Moreover, fix the average density $\rho = \frac{N}{L}$ and take the limit $L \to \infty$. What is the resulting correlation function? **(3 points)**

Remark: Note that these correlations are only functions of the difference $x' - x$, which is due to the fact that these systems are translation invariant.