Advanced Quantum Mechanics

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Exercise sheet 8 (Due: Monday December, 12$^{\text{th}}$.)

8.1 Gauge invariance

The starting point in the lecture for the quantization of the electromagnetic field was the classical electromagnetic field expressed in the Coulomb gauge. Here we recall some facts about gauge symmetry of the classical electromagnetic field. The electric and magnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ depend on the scalar and vector potentials $\phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ via the equations

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \nabla\times\vec{A}. \tag{1}$$

(a) Show that (the classical) fields $\vec{E}$ and $\vec{B}$ as in (1) stay invariant under the gauge transformation

$$\phi' = \phi - \frac{1}{c}\frac{\partial f}{\partial t}, \quad \vec{A}' = \vec{A} + \nabla f, \tag{2}$$

where $f(\vec{x},t)$ is a function. **(2 points)**

(b) In a semiclassical model of a charged particle interacting with the electromagnetic field, we treat the particle as a quantum object, but still treat the potential ϕ and the vector potential $\vec{A}$ as classical (i.e., just being real and vector-valued functions of position $\vec{x}$ and time t). A standard choice of Hamiltonian is $H = \frac{1}{2m}\big(\vec{P} - \frac{e}{c}\vec{A}(\vec{X},t)\big)^2 + e\phi(\vec{X},t)$, and the time-dependent Schrödinger equation does in this case become

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \frac{1}{2m}\big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)^2\psi(\vec{x},t) + e\phi(\vec{x},t)\psi(\vec{x},t). \tag{3}$$

Suppose that $\psi(\vec{x},t)$ is a solution to (3). Show that $\psi'(\vec{x},t) = \mathrm{e}^{i\frac{e}{\hbar c}f(\vec{x},t)}\psi(\vec{x},t)$ is a solution to

$$i\hbar\frac{\partial}{\partial t}\psi'(\vec{x},t) = \frac{1}{2m}\big(-i\hbar\nabla - \frac{e}{c}\vec{A}'(\vec{x},t)\big)^2\psi'(\vec{x},t) + e\phi'(\vec{x},t)\psi'(\vec{x},t), \tag{4}$$

where ϕ' and $\vec{A}'$ are as in (2). Hence, when doing a gauge transformation of the fields, the wave-function of the particle also has to be transformed.

Hint: Keep in mind that $\big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)^2\psi(\vec{x},t) = \big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)\cdot\big(-i\hbar\nabla - \frac{e}{c}\vec{A}(\vec{x},t)\big)\psi(\vec{x},t)$, and that one should be careful to let the ∇ operate on everything they should operate on. **(4 points)**

8.2 Momentum of a single mode of the quantized electromagnetic field

In the previous exercise the electromagnetic field was treated classically. Here we shall investigate the quantized field. In the lecture we have considered free space (no charges and no currents), and used the usual trick of enclosing the field in a square box of volume V with periodic boundary conditions. Within the Coulomb gauge we found that the quantized vector potential $\vec{A}(\vec{x},t)$ becomes $\vec{A}(\vec{r},t) = \sum_{\vec{k},\lambda} \vec{A}_{\vec{k},\lambda}(\vec{r},t)$, where the contribution from the mode with wave vector $\vec{k}$ and polarization λ (which can take two values) is

$$\vec{A}_{\vec{k},\lambda}(\vec{r},t) = \sqrt{\frac{2\pi\hbar c}{V\|\vec{k}\|}}\vec{e}_{\vec{k},\lambda}\left(\mathrm{e}^{i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}a_{\vec{k},\lambda} + \mathrm{e}^{-i(\vec{k}\cdot\vec{r}-\omega_{\vec{k}}t)}a^{\dagger}_{\vec{k},\lambda}\right). \tag{5}$$

Classically, the momentum carried by an electromagnetic wave is given by

$$\vec{P} = \frac{1}{4\pi c}\int_V \vec{E}\times\vec{B}\, d^3r. \tag{6}$$

Use the expression (6) to show that the contribution to the momentum from mode $\vec{k},\lambda$ in the quantum case can be written

$$\vec{P}_{\vec{k},\lambda} = \hbar\vec{k}(a^{\dagger}_{\vec{k},\lambda}a_{\vec{k},\lambda} + \frac{1}{2}\hat{1}). \tag{7}$$

Hint: Apply (1) to (5) with $\phi(\vec{r},t) = 0$ (there are no charges) to obtain the operators $\vec{E}_{\vec{k},\lambda}(\vec{x},t)$ and $\vec{B}_{\vec{k},\lambda}(\vec{x},t)$. Next, insert the result into (6), and keep in mind the following relations: If $f(\vec{r})$ is a scalar function, and $\vec{v}$ is a vector that is independent of $\vec{r}$, then $\nabla\times f(\vec{r})\vec{v} = [\nabla f]\times\vec{v}$. Also keep in mind the relation $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$. Note also that the polarization vector $\vec{e}_{\vec{k},\lambda}$ is normalized, and orthogonal to $\vec{k}$. **(6 points)**

8.3 Phase operators

A classical oscillation (think e.g. of a standing wave) has an amplitude (the strength of the oscillation) and a phase (where in the oscillatory cycle it is). Since this is a very basic concept, it would be nice to have some corresponding notion in the quantum case. In the lecture you defined the phase operators $\widehat{\mathrm{e}^{i\phi}}$ and $\widehat{\mathrm{e}^{-i\phi}}$, as well as the observables $\widehat{\cos\phi}$ and $\widehat{\sin\phi}$ for a single bosonic mode, and in this exercise we are going to explore the properties of these. Before starting, it is probably a good idea to read about them in the lecture notes, for example to find out their definitions. (For the sake of avoiding confusion we here put a ˆ on the operators, including the number operator $\hat{n}$.)

(a) Show the following commutation relations

$$[\hat{n}, \widehat{\cos\phi}] = -i\widehat{\sin\phi}, \quad [\hat{n}, \widehat{\sin\phi}] = i\widehat{\cos\phi}, \tag{8}$$

where $\hat{n}$ is the number operator.

Hint: Make use of the relations that you proved in 7.1(a). **(4 points)**

(b)

1. With respect to the number basis $\{|l\rangle\}_{l=0}^{\infty}$, show that the only non-zero matrix element of the commutator $[\widehat{\mathrm{e}^{i\phi}}, \widehat{\mathrm{e}^{-i\phi}}]$ is $\langle 0|[\widehat{\mathrm{e}^{i\phi}}, \widehat{\mathrm{e}^{-i\phi}}]|0\rangle = 1$.

2. Show that $[\widehat{\mathrm{e}^{i\phi}}, \widehat{\mathrm{e}^{-i\phi}}] = |0\rangle\langle 0|$, and $\widehat{\mathrm{e}^{-i\phi}}\widehat{\mathrm{e}^{i\phi}} = \hat{1} - |0\rangle\langle 0|$

3. Show that $[\widehat{\cos\phi}, \widehat{\sin\phi}] = \frac{i}{2}|0\rangle\langle 0|$.

4. Find a simple expression for the operator $(\widehat{\cos\phi})^2 + (\widehat{\sin\phi})^2$.

(4 points)