Advanced Quantum Mechanics

Prof. Dr. J. Krug, Dr. J. Åberg

Exercise sheet 9 (Due: Monday December, 19th.)

9.1 P-representation of chaotic light

In the lecture you obtained a special mixed single-mode state, the "chaotic light", by maximizing the entropy for a fixed average photon number $\overline{n}$. (It is a good idea to look this up in the lecture notes.) In the lecture you also defined the P-representation of mixed states as

$$\rho = \int P(\beta)|\beta\rangle\langle\beta|\, d^2\beta \tag{1}$$

for a function P, where one should keep in mind that the integral $\int d^2\alpha$ should be interpreted as an ordinary integral over $\mathbb{R}^2$, i.e., as $\iint d\mathrm{Re}(\alpha)\, d\mathrm{Im}(\alpha)$.

Your task is to show that

$$P(\beta) = \frac{1}{\pi\overline{n}} e^{-\frac{|\beta|^2}{\overline{n}}}, \tag{2}$$

is the P-representation of a single-mode chaotic state with mean photon number $\overline{n}$. For this purpose, express the corresponding ρ obtained from (1) in the number basis, and compare to the expression that we obtained in the lecture.

Hint: "Express ρ in the number basis" means that we write $\rho = \sum_{nn'} \rho_{n,n'}|n\rangle\langle n'|$. Recall the expression for coherent states in the number basis, and apply this expansion on both $|\beta\rangle$ and $\langle\beta|$ in (1). By changing to polar coordinates $\beta = re^{i\theta}$ you get a separate integral over r, and another over θ. A change of variables $s = r^2$ can be useful, together with the fact that $\int_0^{+\infty} s^k e^{-qs} ds = \frac{k!}{q^{k+1}}$ for non-negative integers k and $q > 0$.

(4 points)

9.2 Two-level atoms and a photon mode

In the lecture we discussed a model of a single photon mode interacting with a collection of two-level atoms, where an excited atoms can emit a photon, and an atom in the ground state can absorb one. This process makes the probability P_n to find n photons in the mode to evolve in time. In the lecture we derived the following set of coupled differential equations

$$\frac{d}{dt}P_0(t) = \Gamma_0 N_1 P_1 - \Gamma_0 N_2 P_0,$$
$$\frac{d}{dt}P_n(t) = \Gamma_0 n N_2 P_{n-1}(t) + \Gamma_0 N_1 (n+1) P_{n+1}(t) - \Gamma_0 N_1 n P_n(t) - \Gamma_0 N_2 (n+1) P_n(t), \quad n \geq 1.$$

Here N_1 is the number of atoms in the ground state, and N_2 the number of atoms in the excited state. We assume that there is some external mechanism that keeps these numbers constant over time, and we also assume that there are more atoms in the ground state, than atoms in the excited state, i.e., $N_1 > N_2 \geq 0$. Moreover $\Gamma_0 > 0$.

(a) Show that the stationary distribution (i.e. the distribution that is such that $\frac{d}{dt}P_n = 0$ for $n = 0, 1, \ldots$) to this master equation has the form of a geometric distribution, and determine its mean $\overline{n}$ in terms of N_1 and N_2. Which type of light state is this?

Hint: Look at the resulting equation for $n = 0$ and compare with that for $n = 1$. Can one combine them to something simpler? Iterate this, and see if you can find a pattern.

(4 points)

(b) Suppose that all atoms are in the ground state, i.e., that $N_2 = 0$. Show that the equations are solved by a time dependent Poisson distribution

$$P_n(t) = e^{-\overline{n}(t)} \frac{\overline{n}(t)^n}{n!},$$

and determine $\overline{n}(t)$, assuming that $\overline{n}(0)$ is given. **(4 points)**

9.3 Second order coherence function for single-mode states

The second order coherence function is generally written

$$g^{(2)}(\vec{r}, t; \vec{r}', t') = \frac{\mathrm{Tr}\Big(E^{(-)}(\vec{r}, t)E^{(-)}(\vec{r}', t')E^{(+)}(\vec{r}', t')E^{(+)}(\vec{r}, t)\rho\Big)}{\mathrm{Tr}\Big(E^{(-)}(\vec{r}, t)E^{(+)}(\vec{r}, t)\rho\Big)\mathrm{Tr}\Big(E^{(-)}(\vec{r}', t')E^{(+)}(\vec{r}', t')\rho\Big)}. \tag{3}$$

For a state ρ that only has occupation on a single mode $\vec{k}$ (with fixed polarization) the operators $E^{(-)}(\vec{r}, t)$ and $E^{(+)}(\vec{r}, t)$ effectively simplify to

$$E^{(+)}(\vec{r}, t) = i\sqrt{\frac{2\pi\hbar\omega}{V}} e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} a, \quad E^{(-)}(\vec{r}, t) = -i\sqrt{\frac{2\pi\hbar\omega}{V}} e^{-i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)} a^\dagger, \tag{4}$$

where a and $a^\dagger$ are the annihilation operators associated to mode $\vec{k}$. (Recall that the operators in (4) actually are infinite sums over all modes. However, if ρ is a single-mode state, then only the single terms in (4) survive when we evaluate the expectations in (3).)

(a) Show that the second-order coherence function for any single-mode state ρ can be written

$$g^{(2)} = \frac{\mathrm{Tr}(a^\dagger a^\dagger a a \rho)}{\mathrm{Tr}(a^\dagger a \rho)^2} = \frac{\mathrm{Tr}(\hat{n}^2\rho) - \mathrm{Tr}(\hat{n}\rho)}{\mathrm{Tr}(\hat{n}\rho)^2}. \tag{5}$$

(2 points)

(b) Let $\rho_1 = \frac{1}{2}|\alpha\rangle\langle\alpha| + \frac{1}{2}|-\alpha\rangle\langle-\alpha|$, i.e., we have an equal weight mixture of the two (normalized) coherent states $|\alpha\rangle$ and $|-\alpha\rangle$. Determine $g^{(2)}$ for the state ρ_1. **(2 points)**

(c) Let $\rho_2 = |\psi\rangle\langle\psi|$, where

$$|\psi\rangle = \frac{|\alpha\rangle + e^{i\theta}|-\alpha\rangle}{\||\alpha\rangle + e^{i\theta}|-\alpha\rangle\|},$$

with $|\alpha| > 0$ and $\theta \in \mathbb{R}$. In other words, ρ_2 is an equal weight superposition of the two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$, with the relative phase factor $e^{i\theta}$. Determine $g^{(2)}$ for the state ρ_2 as a function of $|\alpha|$ and θ. Relate the cases $\theta = 0$ and $\theta = \pi$ to bunching and anti-bunching. What happens in the limit $|\alpha| \to \infty$?

(4 points)