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Advanced Statistical Mechanics (WS 2019/20)
Problem Set 1

Problem 1: Scale covariance of correlation functions

(a) The power-law correlation function,

$$C(\mathbf{r}) = C_0 r^{-2x}, \tag{1}$$

with $r \equiv |\mathbf{r}|$ and a constant $C_0 > 0$, transforms covariantly under scale transformations,

$$\mathbf{r} \to b^{-1}\mathbf{r}, \qquad C \to b^{2x}C. \tag{2}$$

Show, by applying an infinitesimal scale transformation, that the transformation law (2) can equivalently be expressed by the flow equation

$$\left(r\frac{d}{dr} + 2x \right) C(r) = 0. \tag{3}$$

(b) Now consider a correlation function of the form

$$C(\mathbf{r}) = r^{-2x}\, \Phi\left(\frac{r}{a}, \frac{r}{\xi}\right), \tag{4}$$

which also depends on the molecular scale a and on the correlation length ξ. The function Φ expressing these dependencies is called a scaling function. Generalise the scale transformation law (2) to this case, by including the two additional length scales. Then apply again an infinitesimal scale transformation to derive the generalized form of the flow equation (3).

(c) The spin correlation function of the Ising model is of the form (4) with a so-called *canonical* scaling exponent,

$$C(\mathbf{r}) = r^{-2x_0}\, \Phi\left(\frac{r}{a}, \frac{r}{\xi}\right) \qquad \text{with } x_0 = \frac{d-2}{2}, \tag{5}$$

where d is the spatial dimension. This scaling exponent has been derived in the lecture from power counting. The scaling function has the form

$$\Phi\left(\frac{r}{a}, \frac{r}{\xi}\right) = \phi\left(\frac{r}{a}\right)\, \exp\left(-\frac{r}{\xi}\right). \tag{6}$$

Consider a measurement of the correlation function, $C(r) \sim r^{-2x}$, in the regime $a \ll r \ll \xi$. Assuming a singular dependence on the molecular scale, $\phi \sim a^{-\eta}$, compute the resulting "true" decay exponent x. Note: in the system to be measured, a has a fixed value.

(d) Recall from elementary statistical mechanics that the spin correlation function is related to the magnetic susceptibility,

$$\chi = \int C(\mathbf{r})\, d^d\mathbf{r}. \tag{7}$$

Show that the measurable scaling behaviour of the correlation function derived in (c) implies that the susceptibility grows as a power of the correlation length,

$$\chi \sim \xi^{\zeta}, \tag{8}$$

and compute the exponent ζ.

To be discussed on: Mon, October 28

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`