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Advanced Statistical Mechanics (WS 2019/20)
Problem Set 2

Problem 2: Fully connected Ising model

In the lecture, we have discussed the mean field approximation for the general ferromagnetic Ising model with couplings $J_{\mathbf{r}\mathbf{r}'}$. In one important instance, the so-called *infinite-range* or fully connected Ising model, every spin is coupled with all the others with the same strength. As we will show here, the mean-field approximation becomes exact in this case.

(a) The Hamiltonian of N Ising spins $s_i \in \{\pm 1\}$ in the fully connected model has the form

$$\mathcal{H}(s_1, \dots, s_N) = -J \sum_{i,j=1}^{N} s_i s_j. \tag{1}$$

Since every spin is coupled with every other spin, we have dropped the positions $\mathbf{r}$. Show that $\mathcal{H}(\mathbf{s})$ can be written as a simple function of the magnetization density $m = \frac{1}{N}\sum_{i=1}^{N} s_i$ (simply called magnetization from now on). Do you notice any problem with the system-size dependence of the Hamiltonian? Define a new rescaled Hamiltonian $\mathcal{H}'$ in order to solve this problem.

(b) We now fix the magnetization of the system. Show that the number of states with magnetization m is

$$n(m) = \binom{N}{N\frac{1+m}{2}} \tag{2}$$

Hint: express the magnetization as a function of the number of up spins N_+ and down spins N_-. Now use Stirling approximation $\log(n!) \simeq n\log(n) - n$ and show that the number of states with magnetization m can be written as $n(m) = \mathrm{e}^{S(m)}$, with

$$S(m) = -Nk_b \left(\frac{1+m}{2} \log\left(\frac{1+m}{2}\right) + \frac{1-m}{2} \log\left(\frac{1-m}{2}\right) \right). \tag{3}$$

(c) Using the results of (a) and (b), and taking the continuum limit, show that the partition function $Z = \sum_{s_1=\pm1 \dots s_N=\pm1} \mathrm{e}^{-\beta\mathcal{H}'(\mathbf{s})}$ can be written as

$$Z = \int_{-1}^{1} \mathrm{d}m \, \mathrm{e}^{-\beta\mathcal{F}(m)}, \tag{4}$$

and express the free energy $\mathcal{F}(m)$ in terms of the energy and the entropy.

(d) To derive the saddle-point approximation for the integral (4), consider a function $f : \mathbb{R}^d \to \mathbb{R}$ that has a maximum at the position $x_0 \in \mathbb{R}^d$. By expanding $f(x)$ around x_0, show that

$$\lim_{N\to\infty} \frac{1}{N} \log\left(\int_{x\in\mathbb{R}^d} \mathrm{d}x \, \mathrm{e}^{Nf(x)} \right) = f(x_0). \tag{5}$$

Hint: try to obtain a Gaussian integral in Eq. (5). Inspired by this argument, discuss what happens to the system in the thermodynamic limit in terms of the free energy density $f = \mathcal{F}/N$. What would happen if we kept the unrescaled Hamiltonian $\mathcal{H}$ in Eq. (1)?

(e) While the internal energy and the entropy of the system are always functions of the magnetization, the fully connected model is the only case in which the mean field approximation is exact in the thermodynamic limit. Point out which step of the calculation would be different in the general case.

Note: The fully connected Ising model has an important application in statistical genetics. Here, m is a quantitative genetic trait, $i = 1, \ldots, N$ are the genomic sites encoding the trait, $s_1, \ldots, s_N$ are the sequence letters (alleles) at these sites, and $-\mathcal{H}(m)$ is the fitness.

To be discussed on: Mon, November 4

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`