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Advanced Statistical Mechanics (WS 2019/20)
Problem Set 3

Problem 3: An introduction to the 1d Ising Model.

The Ising model is, perhaps, the most studied model in statistical physics. As introduced in the lecture, it consists of a discrete set of magnetic degrees of freedom (or spins) s_i. In this case, the system is uniaxial anisotropic, which means that the spins align along an unique axis (for example the z-axis). We say then that $s_i = \pm 1$. In the case of a 1d system (an spins chain), the system is completely solvable (Ising 1926) and this is what we will explore in this exercise. Consider a one-dimensional Ising model with nearest-neighbor interactions, in the absence of a magnetic field. The Hamiltonian of this system is

$$H = -J \sum_{i=1}^{N} s_i s_{i+1} \tag{1}$$

with $s_i = \pm 1$ and $J > 0$. A given spin state consists of *domains*, i.e., contiguous stretches of up or down spins. The domains are separated by boundaries called *domain walls*, which are marked by dots in Fig. 1. The system has two ground states, in which all spins are aligned and there are no domain walls.

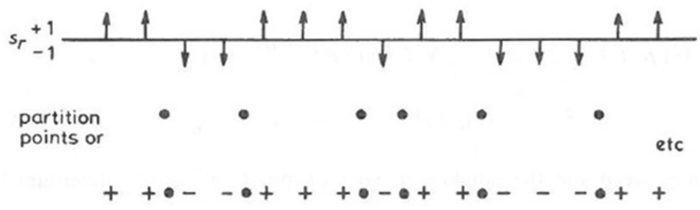

Figure 1: A configuration of spins (arrows) and of domain walls (dots) in the one-dimensional Ising model.

(a) Show that the energy of a spin state depends only on the number of domain walls, M, and compute the excitation energy $\Delta E_M = E_M - E_0$, where E_0 is the ground state energy. Compute the number of states with M domain walls (note that each configuration of domain walls corresponds to exactly two spin states).

Hint: The computation is easier if one assumes free boundary conditions for the spins. Alternatively one can assume periodic boundary conditions (positions $N+1$ and 1 are identical).This forces an even number of domain walls. The leading behavior in the thermodynamic limit ($N \gg 1$) will not depend on the boundary conditions.

(b) Compute the partition function Z at inverse temperature β as a sum over domain wall configurations.

(c) Compute the mean density of domain walls, $\rho = \langle M/N \rangle$, the mean energy density $u = U/N$, and the specific heat per unit length, c, at temperature β^{-1}. Evaluate the asymptotic behavior of these quantities in the low-temperature limit.

(d) The spin-spin correlation function is defined by

$$C(r) \equiv N^{-1}\sum_{i=1}^{N}\langle s_i s_{i+r}\rangle. \tag{2}$$

Using the domain wall representation, show that this correlation depends exponentially on the distance,

$$C(r) \sim \exp(-|r|/\xi), \tag{3}$$

and compute the correlation length as a function of the domain wall density, $\xi(\rho)$. Using the result of (c), then compute $\xi(\beta^{-1})$ and evaluate the asymptotic behavior in the low-temperature limit.

Hint: First convince yourself that the correlation function can be written as an inhomogeneous magnetisation with a constraint for one of the spins, $s_k = 1$:

$$C(r) = \langle s_{k+r}\rangle\Big|_{s_k=1} \equiv m(r). \tag{4}$$

Define the vector $\mathbf{p}_r$ with two components p_r^+ and $p_r^- = 1 - p_r^+$, which denote the probabilities of the two spin states at position r. Write down a recursion relation $\mathbf{p}_r = \mathcal{T}\mathbf{p}_{r-1}$ with a 2x2 matrix $\mathcal{T}$ and express the coefficients of this matrix in terms of the domain wall density ρ. Simplify to a recursion for the position-dependent magnetisation $m(r) = p_r^+ - p_r^-$ and solve this recursion.

(e) More generally, the recursive formalism covers the entire statistics of the system. Consider the partition function $Z_N^{\sigma,\sigma'}$ in a system of size N with constrained boundary spins, $s_1 = \sigma$ and $s_{N+1} = \sigma'$. Compute the 2x2 matrix

$$\mathbf{T} = \begin{pmatrix} Z_1^{++} & Z_1^{+-} \\ Z_1^{-+} & Z_1^{--} \end{pmatrix}, \tag{5}$$

which is called the *transfer matrix* of the system (each of these partition functions consists of a single term). By complete induction, show that the partition functions in a system of size N with constrained boundary spins are simply related to the transfer matrix,

$$\begin{pmatrix} Z_N^{++} & Z_N^{+-} \\ Z_N^{-+} & Z_N^{--} \end{pmatrix} = \mathbf{T}^N \tag{6}$$

(it is useful to write out the matrix product $\mathbf{T}^N$ explicitly for, say, $N = 2, 3$). In particular, the partition function with periodic boundary conditions takes the form $Z_N = \mathrm{Tr}\,\mathbf{T}^N$. Compute Z_N using the asymptotic identity $\mathrm{Tr}\,\mathbf{A}^N \simeq \lambda_1^N$, where λ_1 is the largest eigenvalue of $\mathbf{A}$, and recover the result of (b).

Problem 4: Mean Field Theory of the XY−model

The Ising model discussed above is a particular case of a broader type of systems with many magnetic degrees of freedom, in which the spins tend to align along a particular axis, e.g the $z-$axis, such that $s^z = \pm 1$. Alternatively, unit-length spins may prefer to lay down on a plane, e.g. the $xy-$plane, from which follows that $(s^x)^2 + (s^y)^2 = 1$. This is known as the planar or $XY-$model, for which we discuss the mean field theory in this exercise.

(a) Starting with the Hamiltonian

$$H = -\sum_{\vec{r},\vec{r}'} J(\vec{r} - \vec{r}')\vec{s}_{\vec{r}} \cdot \vec{s}_{\vec{r}'} - \vec{h}\cdot\sum_{\vec{r}} \vec{s}_{\vec{r}} \tag{7}$$

and defining the magnetisation vector $\vec{M} = (M^x, M^y)$, expand the spin vector functions to first order around the magnetisation and rewrite the mean field hamiltonian H_{MF} in terms of the norm

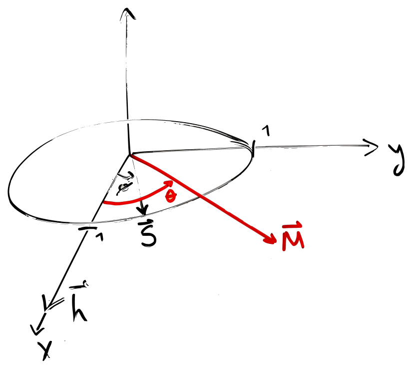

Figure 2: At each site in position $\vec{r}$ there is a spin $\vec{s}_{\vec{r}}$ that lies down on the $xy-$plane. In this figure, we define the angle between $\vec{s}_{\vec{r}}$ and the $x-$axis as $\phi_{\vec{r}}$. On the same plane lies the magnetisation vector $\vec{M}$. We define the angle between $\vec{M}$ and the $x-$axis as θ. For convenience, the external magnetic field $\vec{h}$ lies along the $x-$axis.

of the magnetisation M and the corresponding angles θ and $\phi_{\vec{r}}$ (See the figure above).
`Hint`: As has been done before, define $J \equiv \sum_{\vec{r}} J(\vec{r}-\vec{r}')$. As shown in the picture, for convenience, one can define the coordinates such that the external magnetic field lies down along the $x-$axis, $\vec{h} = h\hat{x}$.

(b) Write the mean field partition function Z_{MF}. In particular, think about how to properly define the sum over all the configurations in this case.

(c) Using the following integral

$$\int_0^{2\pi} e^{a\cos(\theta)+b\sin(\theta)} d\theta = 2\pi I_0(\sqrt{a^2+b^2}) \tag{8}$$

with $I_0(x)$ the modified Bessel function of the first class, calculate the free energy of the system F_{MF}.

(d) Argue that the mean field values for the critical exponents of the $XY-$model are the same as those for the Ising model that were calculated in the lecture.

To be discussed on: Mon, November 11

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`