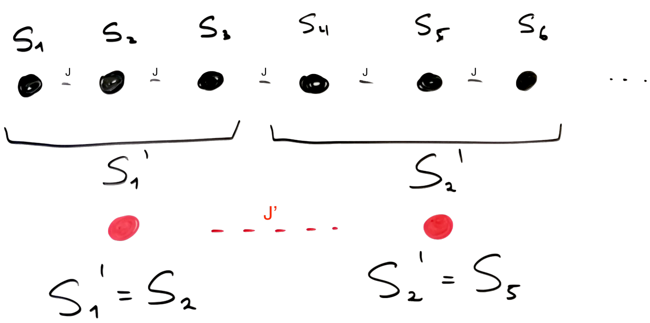
Universität zu Köln
Institut für Biologische Physik

Prof. Michael Lässig
Dr. Fernanda Pinheiro
Denny Trimcev
Matthijs Meijers

Advanced Statistical Mechanics (WS 2019/20)
Problem Set 4

Problem 5: Block spin renormalization of the 1d Ising model.

One of the main ideas of the renormalization group is to express the parameters of a coarse-grained Hamiltonian in terms of the parameters of the original Hamiltonian for the same system. This coarse-graining process is performed such that the long-range physics is left unaltered. In this problem, we coarse-grain the 1d Ising chain in the absence of an external magnetic field. We choose a specific procedure with a coarse-graining factor $b = 3$, i.e., we coarse-grain groups of three neighboring spins, $\{s_{i-1}, s_i, s_{i+1}\}$, to a single new spin $s'_j = s_i$ at the midpoint position (see figure).



(a) Write the partition function of the system as a product of Boltzmann factors $e^{Js_is_{i+1}}$, where J is the reduced coupling constant (measured in units of k_BT). Note there are three factors, containing the spin products s_2s_3, s_3s_4, and s_4s_5, between the two midpoint spins $s'_1 \equiv s_2$ and $s'_2 = s_5$.

(b) Using the coarse-graining rule defined above, write this triple product in terms of the corresponding new spins (s'_1, s'_2) by performing the sum over the original spins in between, $s_3, s_4 = \pm 1$. *Hint*: Use $e^{ks_is_j} = \cosh(k(1 + s_is_j \tanh k))$.

(c) Generalize the result of (b) to write the entire partition function of the coarse-grained system, $Z = \sum_{\{s'\}} e^{-H'(\{s'\})}$ with a new Hamiltonian $H'\{s'\}$ depending only on the coarse-grained spins $\{s'\}$,

$$H'(s') = Ng(J, J') - J' \sum_i s'_i s'_{i+1}. \tag{1}$$

Compute the new coupling constant $J'(J)$ and the function $g(J, J')$. Give an interpretation of g.

(d) Defining $x \equiv \tanh(J)$ and $x' \equiv \tanh(J')$, find the relation between x and x'. This relation tells us how the nearest-neighbour coupling constant of the Hamiltonian changes under the coarse-graining procedure. What is the physical interpretation of the limits $x \to 0^+$ and $x \to 1^-$?

(e) Suppose we iterate the procedure: $x \to x' \to x''$..., etc. What are the two fixed points of this dynamics? Are they stable or unstable? Give a physical interpretation.

To be discussed on: Mon, November 25th

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`