Universität zu Köln
Institut für Biologische Physik

Prof. Michael Lässig
Dr. Fernanda Pinheiro
Denny Trimcev
Matthijs Meijers

Advanced Statistical Mechanics (WS 2019/20)
Problem Set 5

Problem 6: Dilute Ising model, tricritical behavior, and ϕ^6 field theory

In this problem, we study an Ising model with spin variables $s(\mathbf{r})$ taking the values $s = -1, 0, +1$ at each point of the lattice. The Hamiltonian takes the form

$$H(\mathbf{s}) = -\frac{1}{2}\sum_{\mathbf{r},\mathbf{r}'} J(\mathbf{r}-\mathbf{r}')s(\mathbf{r})s(\mathbf{r}') + \Delta \sum_{\mathbf{r}} s^2(\mathbf{r}) - h\sum_{\mathbf{r}} s(\mathbf{r}). \tag{1}$$

This model describes, for example, a ferromagnet with vacancies.

(a) Discuss the zero-temperature physics of this model, by evaluating the energy of the homogeneous spin configurations $s = -1, 0, 1$ (use the coupling constant $J \equiv \sum_{\mathbf{q}} J(\mathbf{q})$ as in the standard Ising model). Depending on the parameters J, Δ, and h, what are the possible degeneracies and first-order transitions at low temperature? For what choice of Δ does the system reduce to the standard Ising model?

(b) Convince yourself that the lowest-order polynomial field theory consistent with symmetry and ground states of this spin model takes the form

$$H(\phi) = \int \left[\frac{1}{2}(\nabla\phi)^2(\mathbf{r}) + \tau\phi^2(\mathbf{r}) + \delta\phi^4(\mathbf{r}) + \lambda\phi^6(\mathbf{r}) - h\phi(\mathbf{r})\right] d\mathbf{r}. \tag{2}$$

Consider now the continuum model (2). Let us put ourselves close to the so-called tricritical point $\tau = \delta = h = 0, \lambda > 0$. Play around with the perturbations of τ, δ and discuss the change in the shape of the energy landscape and the number of minima.

Convince yourself that:

- Turning on τ generates a paramagnetic (as in the standard Ising model) ($\tau > 0$) or an ordered phase ($\tau < 0$).
- Turning on $\delta > 0$ brings us back to the usual ϕ^4-theory and decouples the spin state $s = 0$ but preserves the Ising critical point of the spin states $s = \pm 1$. What does this tell you about the relation of δ and Δ?

(c) Review the scaling dimensions of the local fields ϕ^k and their conjugate couplings λ_k ($k = 1, 2, \dots$) at the Gaussian fixed point as functions of the dimension d. Use scaling arguments (as in problem 1) to compute the singularities of the correlation length, the magnetic susceptibility, and the specific heat at the tricritical point in $d = 3$,

$$\xi \sim |\lambda_k|^{-\nu_k}, \qquad \chi \sim |\lambda_k|^{-\gamma_k}, \qquad c \sim |\lambda_k|^{-\alpha_k} \qquad (k = 2, 4), \tag{3}$$

where $\lambda_2 \equiv \tau$ and $\lambda_4 \equiv \delta$.

(d) Show that the upper critical dimension corresponding to tricritical behavior is $d = 3$. Assuming that the field ϕ^6 has a term $\phi^6(\mathbf{r})\phi^6(\mathbf{r}') \sim C|\mathbf{r}-\mathbf{r}'|^{-x_6}\,\phi^6(\mathbf{r}_+)$ in its operator product expansion, compute the perturbative fixed point describing tricritical behavior to first order in $(3 - d)$.

To be discussed on: Mon, December 2nd

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`