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Advanced Statistical Mechanics (WS 2019/20)
Problem Set 6

Problem 6: A special kind of phase transition

For this problem, we return to the XY-model as introduced in problem set 3: unit-length spins lay down on a 2-dimensional lattice, from which follows that $(S^x)^2 + (S^y)^2 = 1$.

The hamiltonian is given as:

$$\mathcal{H} = -\frac{K}{2} \sum_{\langle rr' \rangle} \mathbf{S}_r \cdot \mathbf{S}_{r'}, \tag{1}$$

where the sum is over the nearest-neighbours.

(a) Show that the Hamiltonian in the continuum limit is given as $\mathcal{H} = \frac{1}{2} K \int d\mathbf{r} (\nabla \phi(\mathbf{r}))^2$.
Hint: Assume that the spins are slowly varying differentiable functions.

Consider the following configurations of spins, in which there exists a vortex around which the spins revolve in a system of size L and lattice constant a. The winding number of this vortex is $n = -1$.

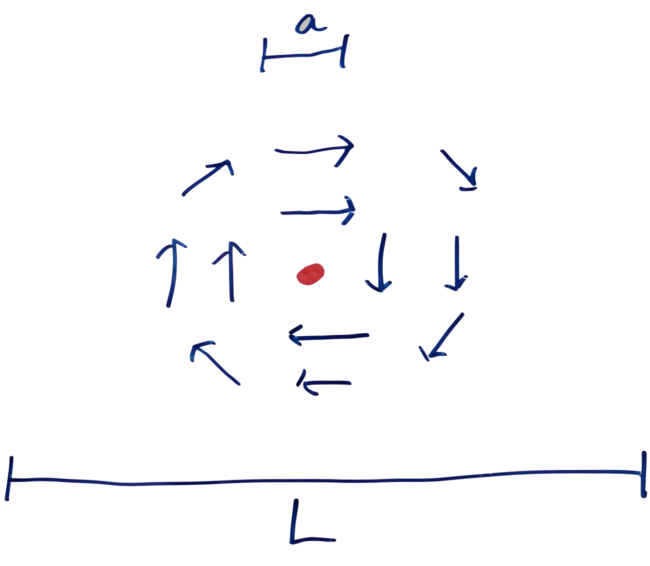

Figure 1: A vortex (red point) in the XY-model

(b) Given that the change in ϕ along any curve encircling a vortex with winding number n has to be $2\pi n$, what is the energy E of the vortex? Can a single vortex exist in the thermodynamic limit?

The situation is different when we consider a pair of vortices with opposite winding number. The two vortices cancel each other out at distances r that are large compared to the separation R between the two vortices.

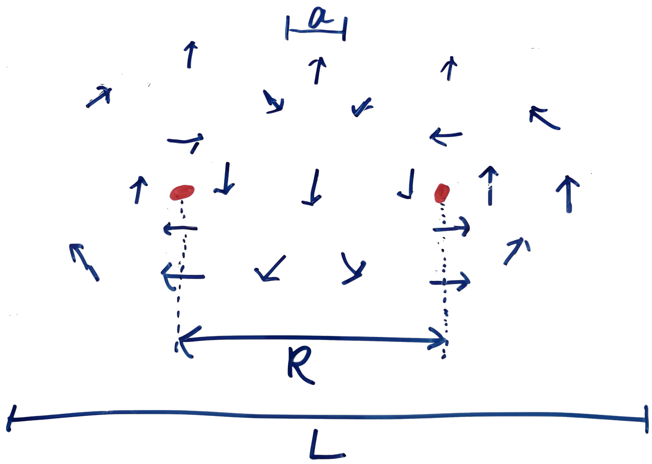

Figure 2: The red points are the two vortices at distance R from each other.

(c) What is the energy E for such a vortex pair?

(d) Consider the vortices as point particles in the XY-plane. What is the entropy of the vortex-pair? Write down the free energy for a vortex-pair at distance R. Consider now an effective force $\mathcal{F} = -\frac{\partial F}{\partial R}$. When is it attractive, and when is it repulsive?

(e) Analyse the free energy and consider the different phases that can exist. What is the critical temperature? How is this phase transition different from the phase transition in the normal Ising system?

To be discussed on: Mon, December 9th

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`