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Advanced Statistical Mechanics (WS 2019/20)
Problem Set 9

Problem 9: Detailed balance in a field theory

The dynamics of a field $\phi(\mathbf{r},t)$ is given by

$$\frac{\partial}{\partial t}\phi(\mathbf{r},t) = -\Gamma \frac{\delta H(\underline{\phi})}{\delta \phi(\mathbf{r},t)} + \zeta(\mathbf{r},t), \tag{1}$$

where $\underline{\phi} : \mathbf{r} \to \phi(\mathbf{r},t)$ denotes field configurations at a given time, $H(\underline{\phi})$ is the Hamiltonian, and the noise $\zeta(\mathbf{r},t)$ is Gaussian with moments

$$\langle \zeta(\mathbf{r},t) \rangle = 0, \qquad \langle \zeta(\mathbf{r},t)\zeta(\mathbf{r}',t') \rangle = D\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t'). \tag{2}$$

This dynamics leads to an equilibrium of the form

$$P_{\mathrm{eq}}(\underline{\phi}) = \frac{1}{Z}\exp[-\beta H(\underline{\phi})], \qquad \beta = \frac{2\Gamma}{D}. \tag{3}$$

(a) For a field Hamiltonian of your choice, compute the propagator $G_\tau(\underline{\phi}'|\underline{\phi})$, where τ is a short time interval and $\underline{\phi}', \underline{\phi}$ are close field configurations (in a sense to be made more precise).

(b) Establish the detailed balance relation

$$G_\tau(\underline{\phi}'|\underline{\phi})\, P_{\mathrm{eq}}(\underline{\phi}) = G_\tau(\underline{\phi}|\underline{\phi}')\, P_{\mathrm{eq}}(\underline{\phi}'). \tag{4}$$

Problem 10: Dynamic scaling

Consider a dynamical field theory with variables $\phi(\mathbf{r},t)$ that is invariant under scale transformations

$$\mathbf{r} \to b^{-1}\mathbf{r}, \qquad t \to b^{-z}t, \qquad \phi \to b^{x}\phi, \tag{5}$$

where x is the scaling dimension of the field and z denotes the dynamical exponent.

(a) Establish the differential (Callan-Symanzik) form of the scale transformation for the two-point correlation function

$$C(\mathbf{r}_1-\mathbf{r}_2, t_1-t_2) \equiv \langle \phi(\mathbf{r}_1,t_1)\phi(\mathbf{r}_2,t_2) \rangle, \tag{6}$$

assuming translation invariance in space and time.

(b) Show that the solution of the Callan-Symanzik equation takes the form

$$C(\mathbf{r}_1-\mathbf{r}_2, t_1-t_2) = |\mathbf{r}_1-\mathbf{r}_2|^{-2x}\,\mathcal{G}\left(\frac{|t_1-t_2|}{|\mathbf{r}_1-\mathbf{r}_2|^{z}}\right) \tag{7}$$

with a scale-invariant, dimensonless function $\mathcal{G}$ (called a scaling function).

(c) From the solution (7), derive the scaling of temporal correlations,

$$C(0, t_1 - t_2) = |t_1 - t_2|^{-2x/z}. \tag{8}$$

(d) Now identify the field variable with the local height of a surface, $h(\mathbf{r}, t)$, with a so-called roughness exponent $\zeta \equiv -x > 0$. The *width* of the surface, $\Delta(\mathbf{r}, t)$, describes local height fluctuations,

$$\Delta^2(\mathbf{r}, t) \equiv \langle (h(\mathbf{r}, t) - \langle h(\mathbf{r}, t) \rangle)^2 \rangle. \tag{9}$$

Assuming scale invariance of the form (5), compute the scaling of the width for the following two processes:
(i) surface dynamics in an infinite system with a flat initial configuration $h(\mathbf{r}, t\!=\!0) = 0$, generating a time-dependent width

$$\Delta(t) \sim t^?; \tag{10}$$

(ii) stationary surface dynamics in a finite system of size R, generating a size-dependent width

$$\Delta(R) \sim R^?. \tag{11}$$

To be discussed on: Mon, January 20th

Course information: `http://www.thp.uni-koeln.de/~lassig/teaching.html`