Quantized Scaling of Growing Surfaces

Michael Lässig

Max-Planck-Institut für Kolloid- und Grenzflächenforschung, Kantstrasse 55, 14513 Teltow, Germany
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The Kardar-Parisi-Zhang universality class of stochastic surface growth is studied by exact field-theoretic methods. From previous numerical results, a few qualitative assumptions are inferred. In particular, height correlations should satisfy an operator product expansion and, unlike the correlations in a turbulent fluid, exhibit no multiscaling. These properties impose a quantization condition on the roughness exponent $\chi$ and the dynamic exponent $z$. Hence the exact values $\chi = 2/5, z = 8/5$ for two-dimensional and $\chi = 2/7, z = 12/7$ for three-dimensional surfaces are derived. [S0031-9007(98)05491-X]

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Strongly driven dynamic systems offer some of the most intriguing realizations of statistical scale invariance. Hydrodynamic turbulence [1] or the growth of rough surfaces [2] are two classic examples, which turn out to be deeply connected from a theoretical point of view. In such systems, a stochastic force $\eta(r,t)$ generates long-ranged correlations of a fluctuating dynamic field—the local velocity $v(r,t)$ of a fluid or the height $h(r,t)$ of a surface. As typical differences $h(r_1,t) - h(r_2,t)$ or $v(r_1,t) - v(r_2,t)$ increase with the spatial separation $|r_1 - r_2|$, the scaling properties of these fields are generally more complex than those at a standard critical point. Indeed, the theoretical understanding of these universality classes far from equilibrium is still fragmentary.

The subject of this Letter is the simplest nonlinear model of stochastic surface growth, the famous Kardar-Parisi-Zhang equation

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta$$

for a $d$-dimensional surface [3]. The driving term $\eta(r,t)$, which describes the random adsorption of molecules onto the surface, is taken to be Gaussian distributed with correlations only over microscopic distances,

$$\langle \eta(r,t)\eta(r',t') \rangle = \sigma^2 \delta(t-t')\delta(r-r').$$

The relation of this model to the theory of turbulence is manifest: Eq. (1) is formally equivalent to Burgers equation

$$\partial_t v + (v \cdot \nabla)v = \nu \nabla^2 v + \nabla \eta$$

for the driven dynamics of a vortex-free velocity field $v(r,t) = \nabla h(r,t)$ (with $\lambda = -1$) [4]. In a fluid, however, the driving force is correlated over macroscopic spatial distances. This leads to important differences in the scaling behavior [5], which are discussed below.

A surface growing from a flat initial state $h(r,0) = 0$ develops height correlations with an increasing correlation length $\xi(t) - t^{1/\nu}$, which defines the dynamic exponent $z$ [6]. A self-similar growth pattern, characterized, e.g., by the height difference moments

$$\langle [h(r_1) - h(r_2)]^\nu \rangle \sim |r_{12}|^{-\nu z},$$

(with $r_{12} \equiv r_1 - r_2$), emerges on mesoscopic scales $\tilde{a} \ll |r_{12}| \ll \xi_t$. For $|r_{12}| \approx \tilde{a}$, the dissipation term $\nu \nabla^2 h$ in Eq. (1) breaks the asymptotic scale invariance. In the scaling regime (4), the height difference moments become stationary, i.e., independent of the correlation length $\xi_t$. They are characterized by a single critical index $\chi \equiv 0$, the roughness exponent of the surface. The scaling relation $\chi + z = 2$ follows from the Galilei invariance of Eq. (1) [7]. For $d = 1$, one can show Eq. (4) to be valid with the roughness exponent $\chi = 1/2$, equal to that of the linear theory ($\lambda = 0$) [2]. In higher dimensions, however, little is known analytically. For $d > 2$, the rough state of the surface exists only if the rescaled driving amplitude $\lambda_0^2 = \sigma^2 \xi_0^3/\nu^3$ exceeds a finite threshold value $\lambda_0^2$ [8]. Less rigorous theoretical arguments predict an upper critical dimension $d_{\ast} \approx 4$ beyond which Kardar-Parisi-Zhang surfaces are only logarithmically rough ($\chi = 0$) even in the strong-coupling regime $\lambda_0^2 > \lambda_0^2$ [9]. The numerical results presently available are consistent with Eq. (4). Extensive simulations yield $\chi = 0.39$ for $d = 2$, $\chi = 0.31$ for $d = 3$, and smaller positive values in higher dimensions, which are less reliable [10].

It has remained a challenge for theorists to calculate the rough asymptotic state of Kardar-Parisi-Zhang surfaces for $d > 1$ exactly or in a controlled approximation. In particular, standard perturbative renormalization about the linear theory fails to produce a fixed point belonging to this regime [11]—a notorious difficulty familiar from the theory of turbulence. In this Letter, a quite different approach is taken. Guided by numerical and experimental results, I make a few qualitative assumptions, namely, the existence of an operator product expansion (8) and of a stationary state (10) that is directed (i.e., it has no up-down symmetry). These assumptions turn out to constrain severely the possible solutions of Eq.(1). In particular, they naturally lead to a quantization condition for the
by writing \( h(\mathbf{r}_i) - h(\mathbf{r}_j) = \int d\mathbf{s} \cdot \nabla h(\mathbf{s}) \), the same property follows for the height difference correlation functions \( \langle \prod_{i=1}^n [h(\mathbf{r}_i) - h(\mathbf{r}_j)] \rangle_i \), in particular, for the moments (4). This implies a feature familiar from simulations: one cannot recognize the value of \( \xi_i \) from snapshots of the surface in a region much smaller than \( \xi_i \).

By differentiating (8), one obtains an operator product expansion for the gradient field \( \mathbf{v} = \nabla h \) of the form

\[
\mathbf{v}(\mathbf{r}_1) \cdots \mathbf{v}(\mathbf{r}_k) = \sum_{O} \left[ \mathcal{O}(\mathbf{r}_1\ldots\mathbf{r}_k) \right] \mathcal{O}(\mathbf{r}_1), \tag{11}
\]

with new scaling functions \( \mathcal{C}_k^O \) and the dimension \( x_v = -\chi + 1 \). [Both sides of (11) are tensors of rank \( k \) whose indices are suppressed.] The fields \( \mathcal{O} \) on the right-hand side govern the time-dependent amplitudes \( \langle \mathbf{v}(\mathbf{r}_1) \cdots \mathbf{v}(\mathbf{r}_k) \rangle_i \sim \langle \mathcal{O} \rangle_i \sim \xi_i^{-x_v} \) in analogy to (9). Hence, the stationarity condition (10) allows in (11) only fields \( \mathcal{O} \) with a non-negative scaling dimension \( x_v \), such as \( I \) (the identity field), \( (\nabla h)^2 \) (\( r \), etc. This in turn restricts the possible terms in (8): (a) singular terms involving fields \( \mathcal{O}(\mathbf{r}) \) with \( x_0 \geq 0 \); (b) regular terms, where the coefficient \( \mathcal{C}_k^{\mathcal{O}} \) is a tensor of rank \( N \) in the differences \( \mathbf{r}_{ij} = (i = 2, \ldots, k) \). Such terms do not violate (10) since they have a vanishing coefficient \( \mathcal{C}_k^{\mathcal{O}} \) in (11) for \( N < k \). They can readily be associated with composite fields of dimensions

\[
x_{k,N} = -k\chi + N. \tag{12}
\]

The leading \( (N = 0) \) term involves the (normal-ordered) field \( \mathcal{O}_0(\mathbf{r}) = h^k(\mathbf{r}) \) and governs the asymptotic singularity (9); the higher terms correspond to fields with \( k \) factors \( h(\mathbf{r}) \) and \( N \) powers of \( \nabla \).

It is useful to introduce the (normal-ordered) vertex fields \( Z_{q_i}(\mathbf{r}) = \exp[q_i h(\mathbf{r})] \), which are the generating functions of the fields \( h^k(\mathbf{r}) \). Equation (8) is then consistent with the operator product expansion

\[
Z_{q_1}(\mathbf{r}_1) Z_{q_2}(\mathbf{r}_2) = \exp \left( \sum_{k,l} C_{k,l}^{q_1} w_{kl}^{q_1} w_{kl}^{q_1} Z_{q_1+q_2}(\mathbf{r}_1) \right) + O(C_{k,l}^{q_1+q_2}) \tag{13}
\]

where \( C_{k,l}^{q_1} \equiv C_{k+l}^{q_1}(0,\ldots,0;\mathbf{r}_{12}|\ldots,\mathbf{r}_{12}|\mathbf{r}_{12}) \) with the first \( k \) arguments equal to 0 and \( w_i \equiv q_i|\mathbf{r}_{12}|^{x_0} \) \( \chi \). Subleading singular terms (with positive-dimensional fields \( \mathcal{O} \)) and regular terms (with fields containing height gradients) are omitted. The vertex \( n \)-point functions \( \langle Z_{q_1}(\mathbf{r}_1) \cdots Z_{q_n}(\mathbf{r}_n) \rangle_i \) behave asymptotically as \( \exp[\xi_i^{z_0} \sum_{i=1}^n q_i] \). If \( \sum_i q_i = 0 \), they have a finite limit \( \langle Z_{q_1}(\mathbf{r}_1) \cdots Z_{-q_1}(\mathbf{r}_1) \rangle_i \). Since these are precisely the vertex correlators that generate the height difference correlation functions and since (13) is analytic in the \( q_i \), this leads back to the stationarity condition (10).

The operator product expansion (13) with the linear dimensions (12) is at the heart of the field theory for roughness exponent

\[
\chi = \frac{2}{k_0 + 2}, \tag{5}
\]

where \( k_0 \) is an odd integer for \( d \geq 2 \). Comparing with the above numerical estimates [12] and using the relation \( \chi + z = 2 \), then gives the main result of this Letter: the exact values \( \chi = 2/5, z = 8/5 \) for \( d = 2 \) and \( \chi = 2/7, z = 12/7 \) for \( d = 3 \).

The fundamental observables describing the equal-time surface configurations are the (connected) correlations

\[
\langle h(\mathbf{r}_1) \cdots h(\mathbf{r}_n) \rangle_i = \int D h h(\mathbf{r}_1) \cdots h(\mathbf{r}_n) P_t - \cdots, \tag{6}
\]

(the dots denoting the disconnected parts). The height probability distribution \( P_t([h]) \) obeys the functional Fokker-Planck equation

\[
\partial_t P_t = \left( \int d\mathbf{r} \left[ \sigma^2 \frac{\delta^2}{\delta h(\mathbf{r})^2} - \frac{\delta}{\delta h(\mathbf{r})} J(\mathbf{r}) \right] P_t \right), \tag{7}
\]

where \( J(\mathbf{r}) \equiv \nu \nabla^2 h(\mathbf{r}) + (\lambda/2) (\nabla h)^2(\mathbf{r}) \) is the deterministic part of the current.

In the scaling regime \( (\tilde{a} \ll |\mathbf{r}_{ij}| \ll \xi_i \) for \( i, j = 1, \ldots, n \) \), the correlation functions (6) will generically become singular as some of the points approach each other. For \( d < d_\omega \), these singularities are assumed to follow from an operator product expansion

\[
h(\mathbf{r}_1) \cdots h(\mathbf{r}_k) = \sum_{\mathcal{O}} |\mathbf{r}_{12}|^{-k\chi + x_0} \times \mathcal{C}_k^O \left( \frac{\mathbf{r}_{13}}{|\mathbf{r}_{12}|}, \ldots, \frac{\mathbf{r}_{ik}}{|\mathbf{r}_{12}|} \right) \mathcal{O} (\mathbf{r}_1), \tag{8}
\]

This identity is nothing but a consistency relation for the height correlations. Inserted in (6), it expresses any \( x_h \) \( j \)-point correlation functions are assumed to have a finite limit

\[
\lim_{\xi_i \to 0} \langle \nabla h(\mathbf{r}_1) \cdots \nabla h(\mathbf{r}_n) \rangle_i = \langle \nabla h(\mathbf{r}_1) \cdots \nabla h(\mathbf{r}_n) \rangle. \tag{10}
\]
Kardar-Parisi-Zhang systems. It is instructive to compare this theory with models of turbulence. Burgers equation (3) with velocity correlations

$$\mathbf{v}(\mathbf{r}, t)v(\mathbf{r}^\prime, t^\prime) = \sigma^2 R^2 \delta(t - t^\prime) \Delta(|\mathbf{r} - \mathbf{r}^\prime|/R) \tag{14}$$

over large distances $R$ develops multiscaling: for example, the longitudinal velocity difference moments

$$\langle |v_1(\mathbf{r}_1) - v_1(\mathbf{r}_2)|^k \rangle \sim |\mathbf{r}_{12}|^{-k_1 + 2} R^{-k_2} \tag{15}$$

have a $k$-dependent singular dependence on $|\mathbf{r}_{12}|$ and $R$ in the inertial scaling regime $\bar{a} \ll |\mathbf{r}_{12}| \ll R$ [5,19]. Similar multiscaling is present in Navier-Stokes turbulence. Kolmogorov’s famous argument predicts the exact scaling dimension of the velocity field, $x_v = -1/3$, from dimensional analysis [20]. This determines the scaling of the third moment in (15) since $\bar{x}_3 = 0$. The higher exponents $\bar{x}_4, \bar{x}_5, \ldots < 0$ cannot be obtained from dimensional analysis. Assuming the existence of an operator product expansion (11), the term (15) is generated by the lowest-dimensional field $\tilde{O}_k$ with a singular coefficient [21]. Multiscaling thus implies the existence of a (presumably infinite) number of composite fields with anomalous negative dimensions. For the velocity vertex fields $\exp[\mathbf{q}v(\mathbf{r})]$ of Burgers turbulence in one dimension, Polyakov has conjectured an operator product expansion similar to (13) and consistent with multiscaling [14]. The distinguishing feature of Kardar-Parisi-Zhang surfaces is the absence of multiscaling [22]. Notice that the resulting properties (12) and (13) have been derived solely from the assumptions (8) and (10) without using Eq. (1) explicitly.

To establish the consistency of the operator product expansion with the underlying dynamic equation, one has to construct correlation functions that remain finite in the continuum limit $\bar{a} \to 0$. With the probability distribution (7), the height correlations (6) develop singularities dictated by their normalization in the linear regime ($|\mathbf{r}_{ij}| \ll \bar{a}$). The existence of a well-defined asymptotic scaling regime for $|\mathbf{r}_{ij}| \gg \bar{a}$ implies that these singularities can be absorbed by a change of variables

$$h(\mathbf{r}) \to Z_h(\bar{h}/r_0)h(\mathbf{r}), \quad r \to Z_r(\bar{a}/r_0)r, \tag{16}$$

such that the “renormalized” correlations (6) satisfy normalization conditions independently of $\bar{a}$ at some mesoscopic scale $r_0$ [18,23]. The $Z_h$ have the asymptotic behavior $Z_h \sim (\bar{a}/r_0)^{\chi_h - \lambda_0}$ and $Z_r \sim (\bar{a}/r_0)^{-z_0}$ as $\bar{a}/r_0 \to 0$, where $\chi_0 = (2 - d)/2$ and $z_0 = 2$ are the exponents in the linear regime. Of course, I do not assume perturbative renormalizability (i.e., that the $Z$ factors are analytic functions of $\lambda_0$). Since the scaling dimensions (12) are linear in $k$, the renormalization (16) also removes the singularities from correlations of the fields $h^k(\mathbf{r})$ and $Z_i(\mathbf{r})$, ensuring a finite limit of the coefficients $C$ in (8), (11), and (13) and of the amplitudes $\langle \tilde{O}_k \rangle$ in (9). The substitution (16) also leads to new coefficients in (1) and (7):

$$\nu(\bar{a}/r_0) \sim Z_{\nu}^{-1} = \nu^* \times (\bar{a}/r_0)^{\gamma},$$

$$\sigma^2(\bar{a}/r_0) \sim Z_{\sigma}^2 Z_{\nu}^{-1} = \sigma^{\nu^*} \times (\bar{a}/r_0)^{\gamma - 2 + 3\chi}, \tag{17}$$

$$r_0^{\chi_0} \lambda(\bar{a}, r_0) \sim Z_r^{-1} Z_{\sigma}^{-1} = \bar{a}^\gamma.$$
in nonzero odd moments \( \langle (\nabla h)^2 (r_1) [h(r_1) - h(r_2)] \rangle \). However, this is consistent with Eqs. (20) and (21) only for odd values of \( k_0 \), where

\[
\langle Z_q (r_1) Z_{-q} (r_2) \rangle - \langle Z_{-q} (r_1) Z_q (r_2) \rangle = - (\nu^* / g^*) c_{1,k_0} q^{k_0} \langle Z_{-q} (r_1) Z_{-q} (r_2) \rangle .
\]

(22)

and, hence, for odd values of \( k \geq k_0 \),

\[
\langle (\nabla h)^2 (r_1) [h(r_1) - h(r_2)] \rangle
= - (\nu^* / g^*) c_{1,k_0} \langle [h(r_1) - h(r_2)] \rangle^{k_0} .
\]

(23)

The directedness of the stationary growth pattern thus requires a nonzero anomaly \( c_{1,k_0} \) with an odd integer \( k_0 \). The roughness exponent is then determined by Eq. (5). The values \( k_0 = 3 \) for \( d = 2 \) and \( k_0 = 5 \) for \( d = 3 \) give the exponents quoted above, in reasonable agreement with the numerical results [10,12].

In summary, the scaling of growing surfaces has been determined by requiring consistency of the effective large-distance field theory subject to a few phenomenological constraints. The Galilei symmetry of the dynamic equation conspires with these constraints to allow only discrete values of the roughness exponent in two and three dimensions. The underlying solutions of Eq. (1) are distinguished by a dynamical anomaly in the strong-coupling regime: The dissipation term contributes a finite part to the effective equation of motion (20) despite being formally irrelevant. The anomaly manifests itself in identities such as (23) between stationary correlation functions. The quantization rule (5) is analogous to the exact Kolmogorov scaling of the third velocity difference moment in Navier-Stokes turbulence. The deeper reason for this rigidity is yet to be explained.

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[6] In a system of size \( L \), the correlation length will eventually saturate to a value \( \xi \sim L \). Hence, the scaling regime (4) exists only for \( L \gg |r_{12}| \).
[11] This has been found to second order by E. Frey and U. C. Täuber, Phys. Rev. E 50, 1024 (1994); Phys. Rev. E 51, 6319 (1995); see also the criticism by K. J. Wiese, Phys. Rev. E 56, 5013 (1997). In a different representation, Lässig [23] has obtained the same result to all orders.
[12] The accuracy of the numerical data may be affected by corrections to scaling not taken into account systematically, see also M. Lässig and H. Kinzelbach, Phys. Rev. Lett. 80, 889 (1998). The corrections are expected to vary with \( d_s \) since the short-distance regime \(|r_{12}| \ll a\) is governed by the linear theory (\( \lambda_0 = 0 \)) for \( d_s \leq 2 \) and by the critical point \( \lambda_0 = \lambda_s^c \) for \( d_s > 2 \).
[16] The average local height has the nonuniversal part \( \langle h \rangle = \sum_\ell \xi_\ell \) independent of \( \xi \). It is eliminated by normal ordering Eq. (1): \( \langle (\nabla h)^2 \rangle \rightarrow \langle (\nabla h)^2 \rangle - \langle (\nabla l)^2 \rangle \). The higher connected correlations (6) remain invariant.
[17] Equation (13) is a generalization of Wick’s theorem. In an expansion of the exponential, each term \( C_{1,l} w_{1,l}^1 w_{2,l}^2 \) represents an elementary contraction of \( k + l \) fields \( h \).
[18] For details, see M. Lässig (to be published).
[22] Certain growth models with additional conservation laws, however, seem to show multiscale; see J. Krug, Phys. Rev. Lett. 72, 2907 (1994).