Lässig and Kinzelbach Reply: Kardar-Parisi-Zhang directed growth in high dimensions has been controversial for some time due to an apparent discrepancy between numerical results and theoretical claims of a finite upper critical dimension $d_>$. The issue has become more disturbing since we have found recently [1] that $d_>$ is ≤ 4, without using any approximation scheme for the strong-coupling phase. In recent Comments, Ala-Nissila [2] and Kim [3] report improved numerical results obtained from a restricted solid-on-solid growth model and from the finite-temperature transfer matrix of the equivalent directed polymer system, respectively. As we show here, the data still have a very short scaling regime and give no evidence of nontrivial strong-coupling exponents in $d \le 4$. They are compatible with our findings if *corrections to scaling* are taken into account.

A surface governed by the strong-coupling fixed point of the Kardar-Parisi-Zhang equation has a finite roughness exponent χ for $d < d_>$, and is expected to be logarithmically rough ($\chi = 0$) for $d \ge d_>$. Consider the width of the surface, $w^2(t) \equiv \langle [h(\mathbf{r},t) - \langle h(\mathbf{r},t)\rangle]^2\rangle$, and the height difference correlation $\frac{1}{2}\langle [h(\mathbf{r}_1,t) - h(\mathbf{r}_2,t)]^2\rangle \equiv C|\mathbf{r}_1 - \mathbf{r}_2|, t)$, which has a stationary limit $C(r) \equiv \lim_{t\to\infty} C(r,t)$. In an infinite system, this should be of the form

$$C_\infty(r) = \begin{cases} (r/r_0)^{2\chi}, & (d < d_>), \\ 1 + c\ln(r/r_0) + \dots, & (d \ge d_>), \end{cases} \qquad (1)$$

with a constant $c > 0$ and the dots denoting subleading terms. For finite t and $r \gtrsim r_\infty(t)$, $C(r,t)$ saturates to the square width $w_\infty^2(t)$. For the initial condition $h(\mathbf{r},0) = 0$, the saturation value is

$$w_\infty^2(t) = \begin{cases} (t/t_0)^{2\chi/z}, & (d < d_>), \\ 1 + (c/z)\ln(t/t_0) + \dots, & (d \ge d_>), \end{cases} \qquad (2)$$

with the dynamical exponent $z = 2 - \chi$. The saturation length $r_\infty(t)$, defined by $C_\infty[r_\infty(t)] = w_\infty^2(t)$, also measures the asymptotic displacement fluctuations of the directed polymer, $\overline{\langle r^2(t)\rangle}_\infty = r_\infty^2(t)$. We expect the form

$$\frac{r_\infty(t)}{r_0} = \begin{cases} (t/t_0)^{1/z}, & (d < d_>), \\ (t/t_0)(1 + c'\ln(t/t_0) + \dots), & (d \ge d_>). \end{cases} \qquad (3)$$

In any real system, these scaling laws are broken on large scales since the system is of finite size L *and* on small scales since $\mathbf{r}$, t, and h are discrete variables (taken to be integer valued). These lattice corrections become significant for $r \sim r_0$ in (1) and $t \sim t_0$ in (2) and (3). The scales r_0 and t_0 characterize the *onset of roughness* [4]. We write $w^2(t) = w_\infty^2(t)\mathcal{W}[r_\infty(t)/L, r_0/r_\infty(t)]$ and $\overline{\langle r^2(t)\rangle} = r_\infty^2(t)\mathcal{R}[r_\infty(t)/L, r_0/r_\infty(t)]$. The asymptotic scaling characterized by Eqs. (1)–(3) emerges only if all arguments of the crossover functions $\mathcal{W}$ and $\mathcal{R}$ are small. Numerical simulations in low dimensions show clear evidence of a scaling regime (of size $L/r_0 \sim 10^3$ in $d = 1$ for the data of Ref. [5] at $L = 3000$). With increasing dimension, however, the available system sizes decrease and for the solid-on-solid model, r_0 increases. The data of Refs. [2] and [3] for $d = 4$ have a "scaling regime" of *less than half a decade* ($L/r_0 \sim 10^{0.4}$ at $L = 100$ [2,6] and $r_{\max}/r_0 \sim 10^{0.5}$ even if $r_0 \sim 1$ at low temperatures [3]). For $d > 4$ [2], not even the onset of scaling is reached ($L < r_0$). Hence, these data are insensitive to the alternative of Eqs. (2) and (3).

The leading lattice correction to scaling for the solid-on-solid model can be obtained from a continuum equation

$$\partial_t h = \nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta + \mu\sin h, \qquad (4)$$

with an extra term that breaks the invariance under translations of h. For $d \ge d_>$, where $h(\mathbf{r},t)$ has logarithmic correlations, we expect the vertex operators $\exp[\pm i h(\mathbf{r},t)]$ to become scaling fields, just as for Gaussian surfaces ($\lambda = 0$) in $d = 2$. We denote by y the scaling dimension of the conjugate coupling μ. As long as $y < 0$ (which is the case for the model of Ref. [2]), we then have a power-law correction $\mathcal{W} \sim 1 + O((t/t_0)^{y/z})$ for $t_0 \lesssim t \ll t_0(L/r_0)^z$ to the logarithmic scaling of Eq. (2) which may indeed explain the upward curvature in the double-logarithmic plot of $w^2(t)$ in [2]. We stress again that these lattice effects persist in the limit $L \to \infty$, unlike the initial-time oscillations of $w^2(t)$ [2]. The periodic driving force may even turn relevant (i.e., $y > 0$) for different model parameters. In that case, the lattice model has a roughening transition at zero temperature, which does not exist for the continuum system at $\mu = 0$.

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