Trigonometrische und hyperbolische Funktionen

mittels (komplexen) Exponentialfunktion:

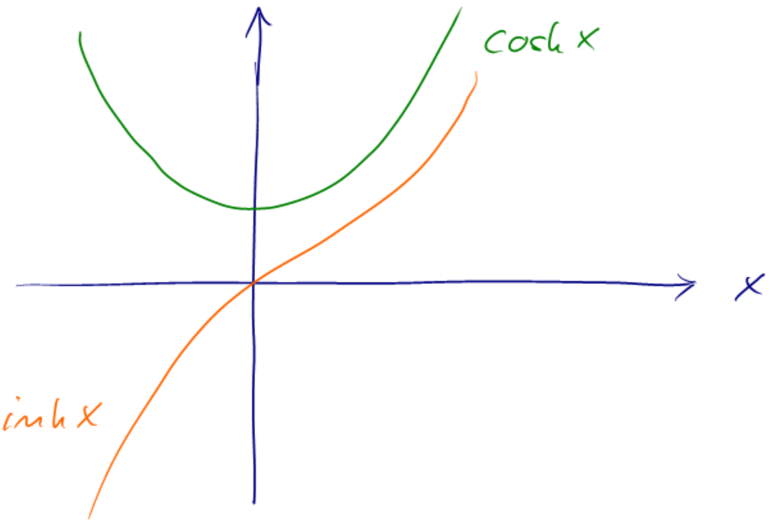
wegen Euler-Id. $e^{i\varphi} = \cos\varphi + i \sin\varphi$ offenbar:

$$\boxed{\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} = \operatorname{Re} e^{ix} \\ \sin x &= \frac{e^{ix} - e^{-ix}}{2i} = \operatorname{Im} e^{ix} \end{aligned}}$$

hyperbolische Funktionen:

Cosinus Hyperbolicus: $\cosh x := \dfrac{e^{x} + e^{-x}}{2}$ $(x \in \mathbb{C})$

Sinus Hyperbolicus: $\sinh x := \dfrac{e^{x} - e^{-x}}{2}$



Eigenschaften:

- $\cosh^2 x - \sinh^2 x = 1$
- $\cosh' x = \sinh x, \quad \sinh' x = \cosh x$

- $\cosh(ix) = \cos x$

  $\sinh(ix) = i \sin x$

- $$\cosh x = \sum_{\ell=0}^{\infty} \frac{x^{2\ell}}{(2\ell)!} = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \dots$$

  $$\sinh x = \sum_{\ell=0}^{\infty} \frac{x^{2\ell+1}}{(2\ell+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

<u>Tangens hyperbolicus</u>: $\tanh x := \frac{\sinh x}{\cosh x}$

<u>Cotangens hyperbolicus</u>: $\coth := \frac{\cosh x}{\sinh x}$

<u>Umkehrfunktionen</u>

$\cosh^{-1} x \equiv \text{arcosh}\, x$: Areacosinus hyperbolicus $(x \geq 1)$

$\sinh^{-1} x \equiv \text{arsinh}\, x$: Areasinus " " $(x \in \mathbb{R})$

$\tanh^{-1} x \equiv \text{artanh}\, x$: Areatangens " " $(|x| < 1)$

$\coth^{-1} x \equiv \text{arcoth}\, x$: Areacotangens " " $(|x| > 1)$

mit Ableitungen:

$$\text{arcosh}' x = \frac{1}{\sqrt{x^2 - 1}} \qquad (x > 1)$$

$$\text{arsinh}' x = \frac{1}{\sqrt{x^2 + 1}}$$

$$\operatorname{artanh}' x = \frac{1}{1-x^2} \qquad (|x| < 1)$$

$$\operatorname{arcoth}' x = \frac{1}{1-x^2} \qquad (|x| > 1)$$

(vgl. Übgen)

Komplexer Logarithmus

$$\boxed{\begin{aligned} \ln : \mathbb{C} \setminus \{0\} &\longrightarrow \mathbb{R} \times [0, 2\pi[\; \subset \mathbb{C} \\ z &\longmapsto \ln z := \ln|z| + i \arg z \end{aligned}}$$

$\rightarrow$ • $e^{\ln z} = e^{\ln|z| + i\arg z} = |z| \, e^{i \arg z} = z$ ✓

• $\ln e^{x+iy} = \ln(e^x e^{iy}) = x + iy$ ✓

erlaubt Def. allg. Potenzen:

$$\boxed{z^w := e^{w \ln z}} \quad \text{für } w \in \mathbb{C}, \; z \in \mathbb{C} \setminus \{0\}$$

Beispiele:

• $\ln i = \ln|i| + i \arg i = \ln 1 + i \frac{\pi}{2} = i \frac{\pi}{2}$

kürzer: $i \overset{!}{=} e^{i\pi/2} \rightarrow \ln i = i\pi/2$

• $\ln(-1) = i\pi$

• $\ln(-i) = i \, 3\pi/2$

• $i^i = e^{i \ln i} = e^{i(i\pi/2)} = e^{-\pi/2}$

reell!

• $(-1)^i = e^{i \ln(-1)} = e^{i(i\pi)} = e^{-\pi}$

etc.

<u>Ableitung einer komplexwertigen Funktion</u> $f: \mathbb{R} \to \mathbb{C}$:

<u>genau</u> wie für reellwertige Fkt. $f: \mathbb{R} \to \mathbb{R}$ definiert:

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\stackrel{!}{=} (\operatorname{Re} f(x))' + i(\operatorname{Im} f(x))'$$

$\longrightarrow$ • Ableitungsregeln <u>genau</u> wie vorher!

• höhere Ableitungen analog

<u>Beispiele</u>:

• $$f(t) := a\, e^{i(\omega t + \varphi)}$$

$$\to \quad f'(t) = i \omega a\, e^{i(\omega t + \varphi)}$$

• $$\frac{d}{dx}\left(\frac{1}{x+i\varepsilon}\right) = -\frac{1}{(x+i\varepsilon)^2} = -\left(\frac{x-i\varepsilon}{x^2+\varepsilon^2}\right)^2$$

$$= \frac{\varepsilon^2 - x^2 + 2i\varepsilon}{(x^2+\varepsilon^2)^2}$$

Integral einer komplexwertigen Fkt $f : \mathbb{R} \to \mathbb{C}$:

$$\boxed{\int_a^b f(x)\,dx := \int_a^b \operatorname{Re} f(x)\,dx + i \int_a^b \operatorname{Im} f(x)\,dx}$$

- $F : \mathbb{R} \to \mathbb{C}$ Stammfkt zu $f : \mathbb{R} \to \mathbb{C}$ g.d.w.

$$F' = f$$

d.h. $\quad (\operatorname{Re} F)' = \operatorname{Re} f \quad , \quad (\operatorname{Im} F)' = \operatorname{Im} f$

$$\rightsquigarrow \quad \int_a^b f(x)\,dx = F(b) - F(a)$$

(wie zuvor für $f : \mathbb{R} \to \mathbb{R}$)

Beispiel:

- $$\int_a^b e^{ikx}\,dx = \frac{e^{ikx}}{ik}\Big|_a^b = \frac{1}{ik}\left(e^{ikb} - e^{ika}\right)$$

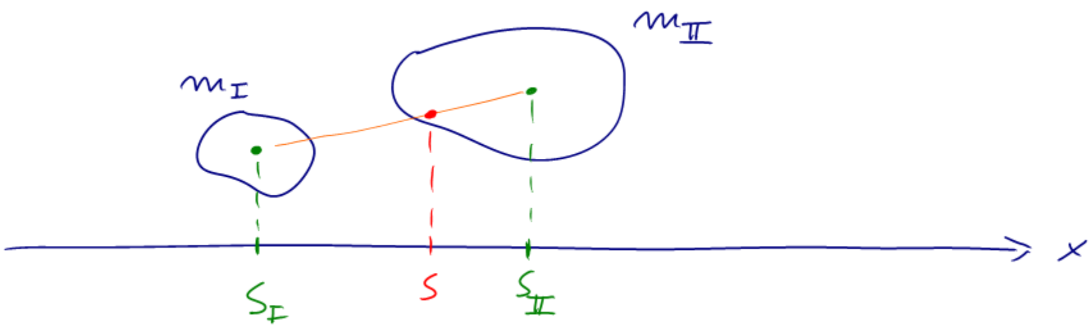
insbes. $$\int_{-b}^b e^{ikx} = \frac{1}{ik}\left(e^{ikb} - e^{-ikb}\right) = \frac{2}{k} \sin kb$$

alternativ: $$\int_a^b e^{ikx} = \int_a^b (\cos kx + i \sin kx)\,dx$$

$$= \int_a^b \cos kx\,dx + i \int_a^b \sin kx\,dx = \dots$$

Hinweise zur Übung

1) Schwerpunkts-x-koordinate:



Schwerpkt-x-koordinate **beider** Körper I, II

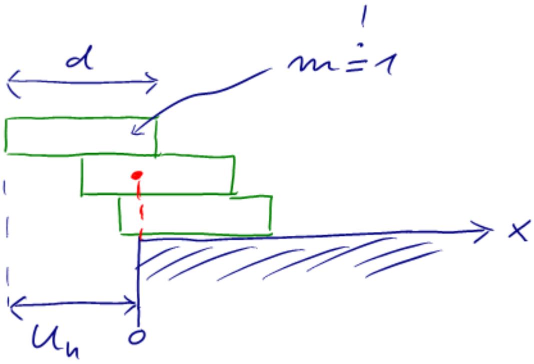
$$\boxed{S \overset{\nabla}{\dot=} \frac{m_{I}\, S_{I} + m_{II}\, S_{II}}{m_{I} + m_{II}}}$$

gewichtetes Mittel von S_{I} und S_{II}

2) Überhang

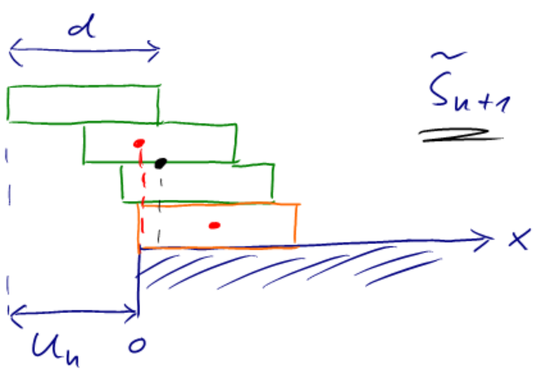
$(n = 3)$

a) n:



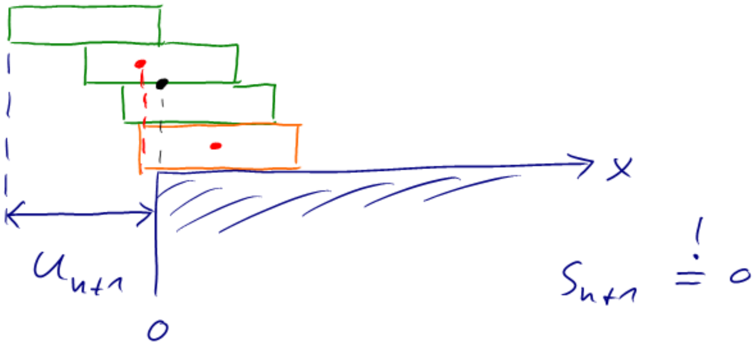
$S_n = 0$.

b) $n+1$



$$\underline{\underline{\tilde{S}_{n+1}}} \overset{1)}{=} \frac{n\, S_n + d/2}{n+1} \overset{a)}{=} \frac{d}{2(n+1)}$$

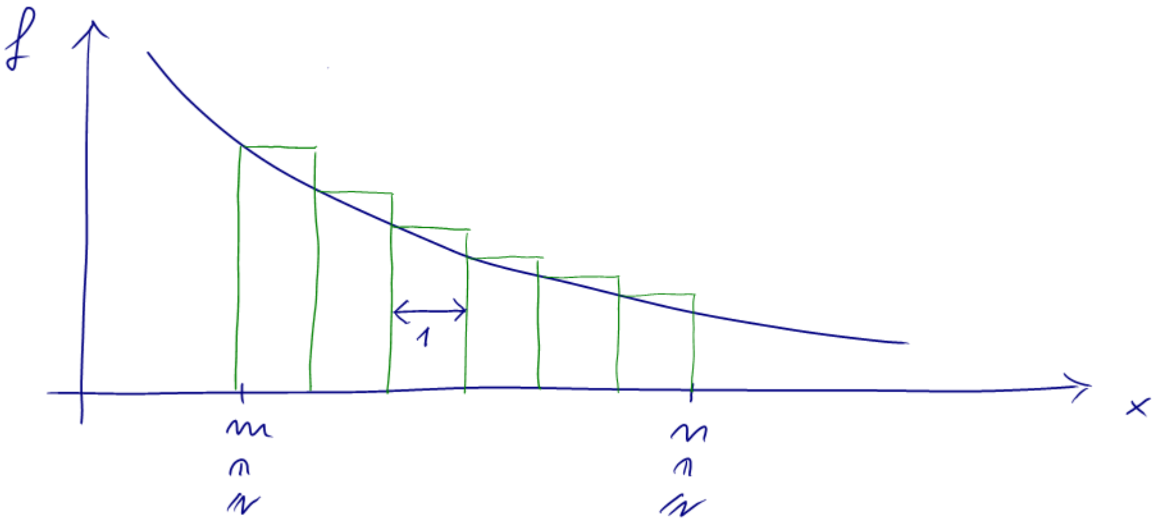
c)



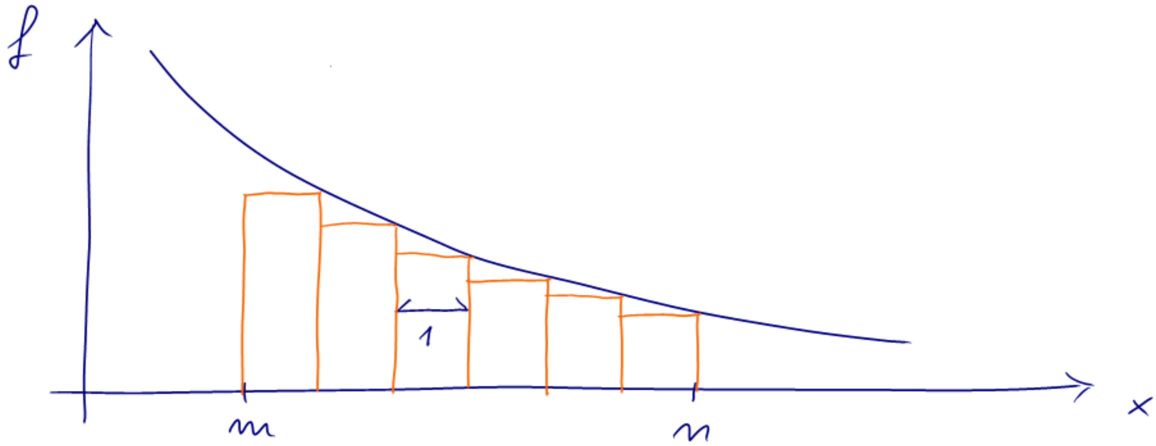
$$\rightarrow \quad \Delta u_{n+1} = u_{n+1} - u_n \overset{!}{=} \tilde{S}_{n+1} \overset{b)}{=} \frac{d}{2(n+1)}$$

3) <u>Abschätzung von Summen durch Integrale:</u>

$f: \mathbb{R} \to \mathbb{R}$ monoton fallende Fkt.



$$\rightarrow \quad \boxed{\sum_{\ell=m}^{n-1} f(\ell) \geq \int_m^n f(x)\, dx}$$



$$\sum_{\ell=m+1}^{n} f(\ell) \le \int_m^n f(x)\,dx$$