Lösungen Blatt 6

1.) Stammfunktion

a) $f(x) = 3x + 1$

$$F(x) = \int dx \; 3x + 1 = \frac{3}{2}x^2 + x + K \,, \; K \in \mathbb{R}$$

$$F(0) \overset{!}{=} 0 \Rightarrow K = 0 \, !$$

b) $F(x) = \frac{3}{2}x^2 + x + K \,, \; K \in \mathbb{R}$

$$F(1) = \frac{3}{2} + 1 + K \overset{!}{=} 2$$

$$\Rightarrow K = 2 - \frac{3}{2} - 1 = -\frac{1}{2}$$

c) $f(x) = \frac{1}{x}$

$$F(x) = \int dx \; \frac{1}{x} = \ln(x) + K \,, \; K \in \mathbb{R}$$

$$F(1) = \underbrace{\ln(1)}_{=0} + K \overset{!}{=} 2 \Rightarrow K = 2$$

2.) Integrale

a) $$\int dx \; (x-1)^{333} = \frac{1}{334}(x-1)^{334} + K \,, \; K \in \mathbb{R}$$

b) $$\int dx \; \frac{x^2}{\sqrt{1+x^3}} = \int dx \; \frac{2}{3} \frac{d}{dx}\left(\sqrt{1+x^3}\right)$$

$$= \frac{2}{3}\sqrt{1+x^3} + K \,, \; K \in \mathbb{R}$$

c) $$\int_0^x d\cancel{x}t \; e^{At} = \left[\frac{1}{A} e^{At}\right]_0^x = \frac{1}{A}\left(e^{Ax} - 1\right)$$

d) $$\int_0^x dt \sum_{i=0}^N a_i t^i = \sum_{i=0}^N a_i \int_0^x dt \; t^i = \sum_{i=0}^N a_i \left[\frac{1}{i+1} t^{i+1}\right]_0^x$$

$$= \sum_{i=0}^N \frac{a_i}{i+1} x^{i+1}$$

e) $\int dx \, \frac{4x+3}{2x^2+3x+5}$

$= \int dx \, \frac{d}{dx} \ln\left(2x^2+3x+5\right)$

$= \ln(2x^2+3x+5) + K \, , \; K \in \mathbb{R}$

3.) <u>Integrieren ohne Bestimmung der Stammfkt.</u>

a) $\int_{-\pi/3}^{\pi/3} dx \, \underbrace{(\sin x)^3}_{\text{ungerade}} = \int_{-\pi/3}^{0} dx \, (\sin x)^3 + \int_{0}^{\pi/3} dx \, (\sin x)^3$

$= -\int_{\pi/3}^{0} dx \, (\sin(-x))^3 + \; - \text{"} -$

$= -\int_{0}^{\pi/3} dx \, (\sin x)^3 + \int_{0}^{\pi/3} dx (\sin x)^3$

$= 0$

b) $\int_{0}^{2\pi} \sin x \, dx = \int_{0}^{\pi} \sin x \, dx + \int_{\pi}^{2\pi} \sin x \, dx$

$= \int_{0}^{\pi} \sin x \, dx + \int_{0}^{\pi} \sin(x+\pi) \, dx$

$= \int_{0}^{\pi} \sin x \, dx - \int_{0}^{\pi} \sin(x) \, dx = 0$

c) $\int_{-7}^{7} dx \; 7x^3 + 3x^7 = 0$ (wie bei a))

4.) Partielle Integration und Substitution

a) • $\int_0^{\pi/2} dx \; \sin x \cos x \, dx = \int_0^{\pi/2} \sin x \frac{d}{dx} \sin x \, dx$

$= \sin^2(x) \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \left(\frac{d}{dx} \sin x\right) \sin x \, dx$

$\Rightarrow \int_0^{\pi/2} dx \; \sin x \cos x \, dx = \frac{1}{2} \left[\sin^2(x)\right]_0^{\pi/2}$

$= \frac{1}{2}$

• $\int_0^{\pi/2} dx \; \sin x \cos x \underset{\substack{\uparrow \\ x = \arcsin(y)}}{=} \int_0^1 dy \frac{1}{\sqrt{1-y^2}} \, y \sqrt{1-y^2}$

$= \left[\frac{1}{2} y^2\right]_0^1 = \frac{1}{2}$

b) • $\int_1^{2e} \frac{1}{x} \ln(x) \, dx = \int_1^{2e} \left[\frac{d}{dx} \ln(x)\right] \cdot \ln(x) \, dx$

$= \ln^2(x) \Big|_1^{2e} - \int_1^{2e} \ln(x) \frac{d}{dx} \ln(x) \, dx$

$\Rightarrow \int_1^{2e} \frac{1}{x} \ln(x) \, dx = \frac{1}{2} \left[\ln^2(2e)\right]$

$= \frac{1}{2} \left[\ln(2) + \ln(e)\right]^2$

$$= \frac{1}{2}\left(\ln^2(2) + 2\ln(2) + 1\right)$$

- $$\int_0^{2e} dx \ \frac{1}{x}\ln(x) \underset{\cancel{x}=e^y}{\overset{\uparrow}{=}} \int_0^{\ln(2e)} dy \ e^{-y}\, y \, e^y$$

$$= \int_0^{\ln(2e)} dy \ y$$

$$= \frac{1}{2}\left[y^2\right]_0^{\ln(2e)}$$

$$= \frac{1}{2}\left(\ln^2(2) + 2\ln(2) + 1\right)$$

c) • $$\int_0^{\pi} dx \ (\sin x)^2 \cos x \, dx$$

$$= \int_0^{\pi} dx \ (\sin x)^2 \frac{d}{dx}\sin x \, dx$$

$$= \sin^3 x\Big|_0^{\pi} - \int_0^{\pi} 2 \cdot \sin^2(x)\cos x \, dx$$

$$\Longrightarrow \int_0^{\pi} dx \ \sin^2 x \ \cos x \, dx = \frac{1}{3}\sin^3 x\Big|_0^{\pi}$$

$$= 0$$

- $$\int_0^{\pi} dx \ (\sin x)^2 \cos x \underset{y=\sin(x)}{=} \int_0^0 dy \ y^2 \frac{\sqrt{1-y^2}}{\sqrt{1-y^2}} = 0$$

5. Endlich oder unendlich?

5.1)
$$\int_0^\infty dt\, v(t) = \int_0^\infty dt \,\frac{10.000\,\text{kmh}}{(t+1\text{h})^2}$$

$$= 10.000\,\text{kmh}\left[-\frac{1}{t+1\text{h}} + k\right]_0^\infty$$

$$\underline{= 10.000\,\text{km}}$$

5.2)
$$\int_0^\infty \mu(t)\,dt = \int_0^\infty \frac{2\,\text{lh}}{(t+1\text{h})^2}\,dt$$

$$= 2\,\text{lh}\left[-\frac{1}{t+1\text{h}} + k\right]_0^\infty$$

$$\underline{= 2\,\text{l}}$$

Nehmen wir an alle Zeiteinheiten τ tritt genau ein Wassermolekül mit Volumen V_0 aus. Dann finden wir:

$$\mu(t) = \sum_{n=0}^{\infty} \delta(t-n\tau)\,\frac{V_0}{\tau}$$

Damit:

$$\int_0^\infty \mu(t)\,dt = \int_0^\infty \sum_{n=0}^{\infty} \delta(t-n\tau)\,\frac{V_0}{\tau}\,dt = \frac{V_0}{\tau}\sum_{n=0}^{\infty} 1 \to \infty$$

Alternativ: Für $t \to \infty$ kann $V(t)$, also das Austropfvolumen minimal auf V_0 sinken (Asymptote) $\Rightarrow$ Es tritt weiterhin konstant Wasser aus.

$$\Rightarrow \int^\infty \mu(t)\,dt \to \infty$$