Introduction to Renormalization

WITH APPLICATIONS

IN CONDENSED-MATTER AND HIGH-ENERGY PHYSICS

INSTITUTE FOR THEORETICAL PHYSICS, UNIVERSITY OF COLOGNE

LECTURE COURSE, WINTER TERM 2017/2018

Michael M. Scherer

February 3, 2018
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## 3 Wilson’s renormalization group
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I. Herbut, *A Modern Approach to Critical Phenomena*, Cambridge University Press. No-frills introduction to critical phenomena and basics of the renormalization group à la Wilson. Excellent first read to become acquainted with the physics and concepts. The start of this lecture follows this presentation, i.e. my chapters on the Ginzburg-Landau-Wilson theory and Wilson’s RG.

J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Oxford University Press (2nd edition). 1000+ pages of quantum field theory and statistical mechanics for enthusiasts with a lot of details, background and also explicit calculations. Also contains also some more formal and mathematical aspects. Can also be considered as a general reference work. Does not contain functional renormalization. My chapter on the functional integral approach to QFT is a compilation of Chaps. 5,6, 7 & 9 of this tome.


M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley. *The* introductory QFT textbook for particle physicists. After a canonical introduction to relativistic QFT in part I, the second part deals with functional integrals. Also contains a pretty nice chapter on critical phenomena which provides a complementary point of view. No functional RG.

Further useful reads

Quantum field theory provides a framework for the description of all fundamental interactions (strong, weak, electromagnetic, maybe gravity), phase transitions in particle physics, statistical mechanics and condensed-matter physics:

Quantum field theory is the framework for the discussion of systems with a large/infinite number of coupled degrees of freedom.

In quantum field theory and statistical mechanics renormalization is required to treat infinities which appear in calculated quantities, typically induced by effects of self-interactions. Moreover, even when no infinities occur in loop diagrams in QFT, renormalization of masses and fields appearing in the Lagrangian is needed.

Renormalization is a procedure to adjust the theoretical parameters describing a system in such a way as not to change the measurable properties on the length of time scales of interest.

Generally speaking, in quantum or statistical field theory, a system at one scale is described by a set of different parameters (particle masses, couplings, interactions,...). The renormalization group (RG) then describes how these parameters are changed when the system is considered at a different scale, i.e. the couplings of a system are running couplings, running with the scale. A variation of the scale loosely corresponds to changing the magnifying power of an appropriate microscope for viewing the system.

The renormalization group refers to a mathematical procedure that facilitates the systematic study of the changes of a physical system when viewed at different length or energy scales.

Upon changing the scale, it might turn out that a particular description of a system in terms of a chosen Lagrangian becomes inappropriate and new degrees of freedom arise. This can also be assessed by the RG.
In the following, I will first discuss some of the aspects of the RG to give a rough overview. These different aspects will then be worked out in more detail during the lecture.

1.1 Aspects of the renormalization group

- **phase transitions and critical phenomena in:**
  - statistical mechanics (liquid-vapor transitions, ferromagnetic transitions,...)
  - condensed-matter/solid-state physics (superfluid/superconducting transitions,...)
  - quantum chromodynamics (chiral symmetry-breaking, confinement-defconfinement transition,...)
  - Standard-Model-Higgs sector

- **perturbative quantum field theory**
  - divergencies and their removal
  - predictivity of QFTs
  - classification of (perturbatively) renormalizable QFTs

- **quantum field theories in the high-energy limit**
  - triviality problem
  - non-perturbatively renormalizable QFTs
  - UV completion
  - asymptotic freedom/safety

- **renormalization-group based definitions of quantum field theories**
Renormalization-group methods are relevant to a large diversity of fields
⇒ many (apparently) different implementations
⇒ sometimes hard to access
→ Functional RG provides unified formulation.

1.2 Phase transitions

• Sketchy phase diagram of water:

![Phase Diagram of Water]

- Definition:

A phase transition is a point in parameter space \((T, p, \mu, h, \ldots)\) where the thermodynamic potential becomes non-analytic.

- non-analyticity only possible in thermodynamic limit.
- in a finite system: partition function is sum of analytic functions of its parameters
  ⇒ analytic
- phase transitions may be continuous (2nd order) or discontinuous (1st order).
- transition lines in phase diagram of water: 1st order transitions
- at critical point in phase diagram of water: 2nd order transition

• continuous phase transition: change of phase of a macroscopic system in equilibrium is not accompanied by latent heat.

• discontinuous phase transition: change of phase of a macroscopic system in equilibrium is accompanied by latent heat.

• define physical quantity to distinguish phases: order parameter
1. Introduction

- examples for order parameters:
  - average density (liquid-gas)
  - resistance (liquid-solid)
  - magnetization (Fe, AFM: La$_2$Cu$_2$)
  - condensate (BEC: Li, Rb)
  - superfluid density (Al, Pb, YBa$_2$Cu$_3$O$_{6.97}$, $^4$He, $^3$He)

- examples for phase transitions: Ising ferromagnet (“Ising model”), superfluid transition, Bose-Einstein condensation, superconducting transition, Higgs mechanism, confinement-deconfinement transition (QCD, lattice spin models)

1.3 Critical phenomena

- Critical phenomena occur in continuous/2nd order phase transitions

- definition:

  Critical phenomena: non-analytic properties of systems near a continuous phase transition.

- definition:

  Critical point: point in the phase diagram where a continuous phase transition takes place.

- observation 1:
  - at phase transition, we see droplets in bubbles and bubbles in droplets in all sizes (strong fluctuations).
  - through microscope: same picture at any magnification.
    ⇒ system is scale invariant
  - a nice video which exhibits this behavior with an Ising model can be found here: https://www.youtube.com/watch?v=MxRddFrEnPc

- observation 2:
  - Completely different systems with many degrees of freedom and complex interactions show quantitatively identical behavior.
  - this can be described with a small set of variables and scaling relations
    → notion of universality
1.4 Perturbative quantum field theory

- Example: liquid-gas coexistence curves of fluids near the critical point:
  Here, we consider the condensation line, i.e. the first order transition line, where the gas coexist with the liquid, near the critical point. We show the temperature-density plane (see figure below) with a coexistence region below the curve.

  \[ \frac{\rho - \rho_c}{T - T_c} \]

  - for Ne, A, Kr, Xe, N\textsubscript{2}, O\textsubscript{2}, CO, CH\textsubscript{4}
  - curves are identical when rescaled with \( \rho_c, T_c \)
  - \( \Delta \rho \sim (T - T_c)^\beta \) with \( \beta \approx 0.33 \)
  - in some paramagnetic-ferromagnetic transitions \( m \sim (T_c - T)^\beta \) with \( \beta \approx 0.33 \) (MnF\textsubscript{2})

- Specific heat and susceptibility also follow simple power laws
- Power laws: \( C_V \sim |t|^{-\alpha} \) and \( \chi \sim |t|^{\gamma} \) with the reduced temperature: \( t = (T - T_c)/T_c \)
  - \( \alpha, \beta, \gamma \): critical exponents
  - Critical exponents can be non-rational, e.g., \( \beta \approx 0.326419(3) \) (best theoretical estimate)
  - Critical exponents often only depend on symmetries of a system and not on details

- Questions:
  - Where does the emergence of universality in different systems come from?
  - Can we calculate the critical exponents?

### 1.4 Perturbative quantum field theory

- Remark on relations between euclidean QFT and statistical mechanics
  - Euclidean QFT in \( d \)-dimensional spacetime \( \sim \) classical statistical mechanics in \( d \)-dimensional space
  - Euclidean QFT in \( (d + 1) \)-dimensional spacetime, \( 0 \leq \tau \leq \beta \sim \) quantum statistical mechanics in \( d \)-dimensional space
  - Euclidean QFT in \( d \)-dimensional spacetime \( \sim \) High-temperature quantum statistical mechanics in \( d \)-dimensional space

- Observation: perturbatively calculated correlation functions \( \langle \phi(x_1) \cdots \phi(x_n) \rangle \) are generally divergent, i.e. naïvely they are not well-defined.

- Example: \( \phi^4 \) theory in four dimensions
  - Action: \( S[\phi] = \int d^4x \left\{ \frac{1}{2} \phi(x)(-\partial^2 + m^2)\phi(x) + \frac{\lambda}{4!} \phi(x)^4 \right\} \)
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- one-loop two-point function gives contribution to mass term $\sim m^2$:

$$\langle \phi(p)\phi(-p) \rangle_{1\text{-loop}} = \int d^4xe^{ipx}\langle \phi(x)\phi(x) \rangle \sim \lambda \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} \sim \Lambda$$

- introduce high-momentum cutoff $\Lambda$:

$$\lambda \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} \rightarrow \lambda \int_{q^2 \leq \Lambda^2} \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 + m^2} = \Lambda \frac{\Omega_4}{2} (\Lambda^2 - m^2 \log(1 + \Lambda^2/m^2))$$

- cutoff regularization: quantitative control of divergence

**in summary:**

- tree level: $\langle \phi(p)\phi(-p) \rangle|_{p=0} \sim m^2$
- with 1-loop correction: $m^2 + \lambda(c_1\Lambda^2 + c_2m^2 \log(1 + \Lambda^2/m^2))$
- interpretation:
  * $m^2$: parameter of theory
  * $m^2 + \lambda(...)$: physical correlation function (should not be $\Lambda$-dependent!)
  
  $\Rightarrow$ adjust $m = m(\Lambda)$, $\lambda = \lambda(\Lambda)$, $\phi = \phi(\Lambda)$ such that $\langle \phi(x_1) ... \phi(x_n) \rangle$ is $\Lambda$-independent and finite for $\Lambda \rightarrow \infty$.

  $\rightarrow$ perturbative renormalization

**questions:**

- is this strategy always possible (classification of theories)?
- are the predictions of the theory independent from the regularization scheme?
- what is the role/interpretation of the cutoff (effective theories)?
- why are so many theories realized in nature (for example the standard model of particle physics) renormalizable within perturbative QFT?

1.5 Renormalization-group based definition of QFTs

**observation:**

- systematic analysis of divergencies of quantum field theories is difficult beyond perturbation theory

**idea:** integration of fluctuations in one narrow momentum shell (= “RG step”) is finite

$\Rightarrow$ construct integration over all fluctuations by (infinite) sequence of RG steps

$\Rightarrow$ Exact/functional renormalization group, flow equations
1.6. QFTs in the high-energy limit

- RG flow of average effective action $\Gamma_k$
- flow equation describes continuous trajectory connecting the microscopic action and the full (quantum) effective action:
  \[ S = \Gamma_{k=\Lambda} \to \Gamma = \Gamma_{k=0} \]
  (see also QFT II)

1.6 QFTs in the high-energy limit

- for perturbatively renormalizable QFTs, the high-energy conditions (parameters) at cutoff $\Lambda$ can be chosen such that the low-energy predictions are independent of $\Lambda$.

- questions:
  - how “natural” are these initial conditions for different systems?
  - do we need to “fine tune” the conditions?
  - hierarchy problem in the standard model of particle physics
  - can cutoff of perturbatively renormalizable QFTs be removed (i.e. be sent to $\infty$)?
    * if yes: QFT is valid on all scales (*fundamental theory*)
    * if no: QFT predicts its own failure (e.g. triviality problem)
  - $\Rightarrow$ hint towards new physics

- example (for hint towards new physics):
  - Fermi’s theory: 4-Fermion interaction amplitude $\lambda$ for neutrino-neutrino scattering
  - dimensional analysis:
    \[ \mathcal{M} \sim \lambda + \lambda^2 \Lambda^2 \sim + \]
    $\Rightarrow$ Failure! Amplitude blows up
    $\rightarrow$ new physics can cure this problem (within perturbation theory):
    introduce vector boson

- mechanism can also be occur towards low energies, indicating appearance of an ordering transition, where the relevant degrees of freedom of the system change (e.g. in terms of condensation)

- are there QFTs that are perturbatively non-renormalizable and non-perturbatively renormalizable?
  - Weinberg’s Asymptotic Safety scenario
  - possibly applies to quantum gravity
  - reminiscent of asymptotic freedom in QCD
Chapter 2

Phase transitions and critical phenomena

2.1 Ising model

Before, we discuss the theory of critical phenomena from a general point of view, we reconsider some of the basic models which show (second order) phase transitions. The Ising model is a basic model to describe a simple ferromagnet or the disorder-order transition in binary alloys. Here, we will use it to introduce some basic notions, definitions and ideas.

- The **partition function** of the Ising model reads:
  \[ Z = \sum_{\{s_i=\pm1, i=1,\ldots,N\}} e^{-\beta H} \tag{2.1} \]
  where \( \beta = 1/(k_B T) \) is the inverse temperature multiplied by the Boltzmann constant. Further, we define the energy \( H \) of a magnetic dipole/spin configuration \( \{s_1, s_2, \ldots, s_N\} \):
  \[ H = -J \sum_{\langle i,j \rangle} s_i s_j - h \sum_i s_i \tag{2.2} \]

- model shall be defined on the sites of a lattice (chain, quadratic, cubic, hypercubic)
- \( \langle i, j \rangle \) indicates a sum over nearest neighbors
- \( J \) is the nearest-neighbor coupling, \( h \) is an external magnetic field
- for \( T \ll J \) (also if \( h = 0 \)): expect that all dipoles point in same direction
- for \( T \gg J \) (also if \( h = 0 \)): interaction between dipoles negligible \( \rightarrow \) random arrangement
- **magnetization** per dipole:
  \[ m = \langle s_j \rangle = \frac{1}{Z} \sum_{\{s_i=\pm1, i=1,\ldots,N\}} s_j e^{-\beta H} \tag{2.3} \]
  - magnetization \( m \) can serve as an **order parameter** for the phase transition at \( h = 0, T = T_c \)
  - **ferromagnetic** (FM) phase: \( m \neq 0 \)
2. Phase transitions and critical phenomena

- paramagnetic (PM) phase: \( m = 0 \)
- for \( h \neq 0 : m \neq \forall T \)
- equation (2.1) can be solved exactly in \( d \in \{1, 2\} \)

2.1.1 Remarks on solutions of the Ising model in \( d = 1 \) and \( d = 2 \)

In one dimension (\( d = 1 \), dipole chain of length \( N \to \infty \)) it is found that \( m = 0 \ \forall \ T > 0 \)
⇒ no phase transition.

Reasoning:
- assume all dipoles are oriented in one direction (i.e. system is in FM state)
- flip half of the dipoles ⇒ energy cost \( \Delta E = 2J \) (one pair points in opposite directions)
  ⇒ domain wall
- pair with dipoles in opposite directions can be chosen in \( N \) different ways
  ⇒ increase of entropy: \( \Delta S = k_B \log N \) (with \( N \) being the number of microstates realizing
  one domain wall)
  ⇒ free energy \( F = E - TS \) is always lowered in large system (\( N \gg 1 \)) by flipping half
  of the dipoles, even at infinitesimal \( T \)
  ⇒ ordered state is unstable ⇒ \( m \neq 0 \) only at \( T = 0 \).

Competition between \( E \) and \( S \) is different in \( d > 1 \):
- state with spontaneous magnetization at \( T > 0 \) is possible in \( d = 2 \) (R. Peierls)
- exact solution of the 2d-Ising model by L. Onsager:
  - system has continuous phase transition with \( k_B T_c/J = 2.269 \)
  - for \( T < T_c \) and near \( T_c \): \( m \sim (T_c - T)^{1/8} \) (magnetization) and \( C \sim -\log |T_c - T| \)
    (specific heat)
- no exact solution in \( d = 3 \). We know, however:
  - 3d-Ising model has critical point
  - characteristics known with great accuracy ⇒ we’ll calculate them in this lecture!

**Dimensionality** is crucial for critical behavior.
2.1.2 Symmetries of the Ising model

- at $h = 0$: global symmetry under transformation $s_i \rightarrow -s_i \forall i$

- this is called $Z_2$ (or Ising) symmetry, it is a *discrete* symmetry

- $Z_2$ symmetry is present in PM phase ($m = 0$)

- $Z_2$ symmetry is broken in FM phase:
  - there is nothing in the Hamiltonian, however, that explicitly breaks this symmetry
  - direction of magnetization depends on history of the system

  $\rightarrow$ phenomenon of *spontaneous symmetry breaking*

- **problem**: if partition function is calculated by summing over all configurations
  $\Rightarrow$ magnetization will vanish due to $Z_2$ symmetry

- to describe ordered phase (FM with $\neq 0$):
  - restrict space of configurations over which sum in $Z$ is performed, e.g., at $h \neq 0$
  - take limit $h \rightarrow 0$ after thermodynamic limit has been performed
  $\Rightarrow m \neq 0$ survives the $h \rightarrow 0$ limit

- actual physical process is different from mathematical procedure (as it is induced by the history of the system)

2.2 XY model and Heisenberg model

Allow dipoles to point arbitrarily:

- in the plane: *XY model*

- in space: *Heisenberg model*

These models then have *continuous symmetries*.

- partition function:

$$Z = \int \prod_{i=1}^{N} \left( \delta(|\vec{S}_i| - 1) d^d \vec{S}_i \right) e^{\beta(J \sum_{(i,j)} \vec{S}_i \cdot \vec{S}_j + h \sum_i \vec{S}_i)}$$ 

(2.4)

**symmetry** of the model will be crucial for critical behavior.
2. Phase transitions and critical phenomena

2.3 Universality and critical exponents

- some properties of a system near critical points are the same for completely different physical systems

- example: specific heat near liquid-gas critical point \( \equiv \) specific heat in PM-FM transition

\[ \Rightarrow \text{some macroscopic properties of systems near continuous phase transitions are independent from the microscopic interactions between particles} \]

\[ \rightarrow \text{these properties only depend on dimensionality & symmetry} \]

\[ \rightarrow \text{different physical systems exhibiting the same behavior near critical point} \]

\[ \rightarrow \text{universality or universal critical behavior} \]

- universal critical behavior can be expressed in terms of power laws, e.g., in liquid-gas transition with reduced temperature \( t \):

1. specific heat near \( T_c \):

\[ C_V = C_{\pm} |t|^{-\alpha}, \quad (2.5) \]

where \( C_{\pm} \) refers to \( t > 0 \) or \( t < 0 \).

2. compressibility near \( T_c \):

\[ \kappa_T = \frac{1}{\rho} \frac{\partial \rho}{\partial p} = \kappa_{\pm} |t|^{-\gamma}. \quad (2.6) \]

- the analogue in a magnetic system is the susceptibility: \( \chi \propto |t|^{-\gamma} \).

3. difference between gas and liquid densities along coexistence curve near the critical point:

\[ \rho_L - \rho_G = \rho_c (-t)^\beta. \quad (2.7) \]

- the analogue in a magnetic system is the magnetization: \( m \propto (-t)^\beta \).

4. critical isotherm (\( T = T_c \)) near \( p = p_c \):

\[ \frac{p - p_c}{p_c} = \left| \frac{\rho_L - \rho_G}{\rho_c} \right|^\delta. \quad (2.8) \]

- analogue in a magnetic system: \( m \propto h^{1/\delta} \) at \( t = 0 \) and \( h \rightarrow 0 \).

- \( \alpha, \beta, \gamma, \delta \):

  - non-trivial and (not completely independent) real numbers
  - universal, i.e. they are the same for a whole class of various phase transitions
  - they are called the critical exponents
  - \( C_+/C_- \) and \( \kappa_+/\kappa_- \) are also universal (amplitude ratios)

Experimental evidence for universality of the critical exponents was found in 1930s. Microscopic understanding and calculation of critical exponents was achieved in 1970s with the help of the renormalization group.
2.4 Scaling hypothesis for the free energy

The power laws near a critical point can be derived from the assumption of scaling:

- consider a magnetic system near paramagnet-ferromagnet transition
- magnetization shall have preferred axis, i.e. can be assumed to be a scalar → uniaxial ferromagnet
  → same symmetry as Ising model → Ising universality class
- in vicinity of critical point \((T = T_c, h = 0)\): thermodynamic potential (free energy) becomes non-analytic/singular
- decompose Gibbs free energy per unit volume \(f_G(T, h)\) into singular and regular part:
  \[ f_G(T, h) = f(T, h) + f_{\text{reg}}(T, h). \] (2.9)
- sufficiently close to critical point the singular part \(f(T, h)\) is assumed to follow the scaling hypothesis/assumption:
  \[ f(T, h) = |t|^{1/w} \Psi_{\pm} \left( \frac{h}{|t|^{u/w}} \right), \] (2.10)

where \(\Psi_+(z)\) for \(t > 0\) and \(\Psi_-(z)\) for \(t < 0\) is a function of a single variable.

- Note:
  - the scaling assumption is very restrictive
  - once we assume it \(\Rightarrow\) power laws for \(C_V, m, \chi\) + two equations for \(\alpha, \beta, \gamma, \delta\).
2.4.1 Derivation of power laws from scaling hypothesis

- **magnetization** $m$ at $h = 0$: the magnetization is given as the negative first derivative of the free energy with respect to the magnetic field $h$ evaluated at $h = 0$, i.e.

$$ m = -\left. \frac{\partial f}{\partial h} \right|_{h=0} = -|t|^{1-u} \Psi_\pm'(0) \quad (2.11) $$

$$ \Rightarrow \text{ for } t > 0 : \ m = 0 \ \Rightarrow \ \Psi_+(0) = 0 \quad (2.12) $$

$$ \text{for } t < 0 : \ m \propto (-t)^\beta \ \Rightarrow \ \beta = \frac{1-u}{w} \quad (2.13) $$

- uniform magnetic susceptibility near $T_c$:

$$ \chi = -\left. \frac{\partial^2 f}{\partial h^2} \right|_{h=0} = -|t|^{1-2u/w} \Psi_\pm''(0) \quad (2.14) $$

and with $\chi \propto |t|^{-\gamma}$ \Rightarrow \[ \gamma = \frac{2u-1}{w} \quad (2.15) \]

- specific heat:

$$ C_V = -T \left. \frac{\partial^2 f}{\partial T^2} \right|_{h=0} = -\frac{\Psi_\pm(0)}{T_c} \frac{1}{w} \left( \frac{1}{w} - 1 \right) t^{\frac{1}{w}-2} \quad (2.16) $$

$$ \Rightarrow \quad \alpha = 2 - \frac{1}{w} \quad (2.17) $$

- rewriting of the scaling assumption: provides critical exponent $\delta$

$$ f(T, h) = h^{1/u} \tilde{\Psi}_\pm \left( \frac{h}{|t|^{u/w}} \right) \quad \text{where} \quad \tilde{\Psi}_\pm(z) = z^{-1/u} \Psi_\pm(z) \quad (2.18) $$

$$ \Rightarrow \ m = -\frac{1}{u} h^{\frac{1}{w}-1} \tilde{\Psi}_\pm(\infty) \quad \text{at} \quad t = 0, h \ll 1 \quad (2.19) $$

$$ \Rightarrow \quad \delta = \frac{u}{1-u} \quad (2.20) $$

From eqs. (2.13), (2.15), (2.17), (2.20), we can derive the scaling laws:

$$ \alpha + 2\beta + \gamma = 2 \quad (2.21) $$

$$ \alpha + \beta(\delta + 1) = 2 \quad (2.22) $$
2.4. Scaling hypothesis for the free energy

- remarks:
  - the most precisely measured value of any critical exponent is $\alpha$ in the superfluid transition in $^4$He in $d = 3$: $\alpha_{\text{exp}} \approx -0.0127 \pm 0.0003$
    - space shuttle experiment:
      [https://journals.aps.org/prb/abstract/10.1103/PhysRevB.68.174518](https://journals.aps.org/prb/abstract/10.1103/PhysRevB.68.174518)
  - the scaling form of the free energy, eq. (2.10), guarantees that the critical exponents are the same below and above the critical point
  - the concept of scaling rationalizes the appearance of power laws near the critical point and yields experimentally correct relations – the scaling laws

- questions:
  - why does the scaling assumption hold?
  - how to compute scaling functions?

2.4.2 First attempt to calculate critical exponents

We take the Ising model in Curie-Weiss mean-field approximation (probably like in your stat-mech course) and see what we get...

In mean-field approximation each dipole only feels the average local magnetic field (see problem set 1)

$\Rightarrow$ replace $J\sum_{\langle i,j \rangle} s_is_j$ by $J\sum_{\langle i,j \rangle} \langle s_i \rangle s_j$ in Ising model with an average magnetization per spin $m = \langle s_i \rangle$

$\Rightarrow$ $m = \frac{\sum_{s_1 = \pm 1} s_1 e^{\beta(zmJ+h)s_1}}{\sum_{s_1 = \pm 1} e^{\beta(zmJ+h)s_1}} = \tanh\left(\frac{zmJ+h}{k_BT}\right)$, \hspace{1cm} (2.23)

where $z = 2d$ counts the number of nearest neighbors on a $d$-dimensional hypercubic lattice

- at $h = 0$:
  - $m \neq 0$ for $T < T_c = zJ/k_B$
  - $m = 0$ for $T > T_c$

- expansion near $m = 0$ and at $h = 0$ $\Rightarrow m = \frac{T}{T_c} \sqrt{3(1 - T/T_c)}$ $\Rightarrow$ $\beta = 1/2$

- expansion at $T = T_c$ for small $h$ $\Rightarrow m = \left(\frac{3h}{k_BT_c}\right)^{1/3}$ $\Rightarrow$ $\delta = 3$

- Further, mean-field approach predicts $\gamma = 1$ and $\alpha = 0$ (see problem set 1)

- no dependence on $d$ within mean-field theory $\Rightarrow$ this deviates from experimental findings!

- experiment ($d = 3$): $\beta \approx 1/3$ and $\alpha$ small but finite (e.g. with Xe)
  - this cannot be attributed to material properties
  - need better theory!
2. Phase transitions and critical phenomena

2.5 Correlations and hyperscaling

Two-point correlation function or order parameter correlation function:

\[ G(\vec{r}, t) = \langle (m(\vec{r}) - m)(m(0) - m) \rangle = \langle m(\vec{r})m(0) \rangle - m^2 \tag{2.24} \]

Here, \( m(\vec{r}) \) is the local value of the magnetization, \( m \) is the average magnetization and \( \langle \ldots \rangle \) denotes the ensemble or thermal average. We have exploited translational invariance. Sometimes convenient to study the (spatial) Fourier transform of \( G(\vec{r}) \), i.e. work in wave-vector space

\[ G(\vec{q}, t) = \int d^d r e^{-i\vec{q}\cdot\vec{r}} G(\vec{r}, t) \tag{2.25} \]

- assume scaling form (for large distances \( r \)):

\[ G(r, t) = \frac{\Phi_{\pm}(\frac{r}{\xi(t)})}{r^{d-2+\eta}} \text{ where } \xi(t) \propto |t|^{-\nu} \tag{2.26} \]

and \( \xi \) is called the correlation length, \( d \) is the dimensionality.

- the correlation length denotes the typical length scale of the regions where the degrees of freedom are strongly coupled

- \( \xi \) diverges at the critical point

- \( \nu \) and \( \eta \) are two new critical exponents
  - \( \nu \) is called the correlation length exponent
  - \( \eta \) is called the anomalous dimension

- relate \( \nu \) and \( \eta \) to the exponent \( \gamma \) by means of the non-local magnetic susceptibility:

\[ \chi(\vec{r} - \vec{r}') := \left. \frac{\partial m(\vec{r})}{\partial h(\vec{r}')} \right|_{h=0} \tag{2.27} \]

  - general theory of linear response: \( \chi(\vec{q}) = G(\vec{q}) \)
  - uniform susceptibility, eq. (3.3): \( \chi = \chi(\vec{q} = 0) = \int d^d \vec{r} G(\vec{r}, t) \)
  - use scaling assumption, eq. (2.26): \( \chi = C \cdot \xi^{2-\eta} \) with the constant \( C = \int d^d z \frac{\Phi_{\pm}(z)}{z^{d-2+\eta}} \)
  - For \( T > T_c \): values of magnetization at two distant points is uncorrelated
  - \( \Phi_{\pm}(z) \) should decay exponentially for large \( z \)
  - For \( T < T_c \): here \( m \neq 0 \). Deviations from finite magnetization also uncorrelated at large distances
  - \( \Phi_{\pm}(z) \) should decay exponentially for large \( z \)
  - From exponential decay of \( \Phi_{\pm} \) we conclude that integral \( C \) is finite in both cases
  - comparison with the definition of \( \gamma \)

\[ \gamma = \nu(2 - \eta) \] (Fisher’s scaling law)
2.5. Correlations and hyperscaling

- further assumption: the only relevant length scale near $T_c$ is provided by the correlation length $\xi$:

$$\frac{\text{free energy}}{\text{unit volume}} = f \propto \xi(t)^{-d} \quad (2.28)$$

  - this is an additional scaling assumption $\rightarrow$ hyperscaling
  - Differentiating $f$ twice w.r.t. the temperature, we obtain Josephson’s scaling law:

$$C \propto |t|^\nu d - 2 \quad \Rightarrow \quad \alpha = 2 - \nu d$$

  - Josephson’s scaling law involves the dimensionality $d$

**SUMMARY:**

- we have 6 critical exponents ($\alpha, \beta, \gamma, \delta, \nu, \eta$) to describe singular behavior of thermodynamic functions and of the correlation function near critical points

- we introduced scaling hypothesis for $f$ and $G$
  \Rightarrow 4 equations for the critical exponents
  \Rightarrow reduce number of independent exponents to 2
  \Rightarrow only have to compute $\nu$ and $\eta$ to know all 6 exponents (if hyperscaling holds)

- Example: assume Fourier transform of the correlation function to be $G^{-1}(q) = q^2 + t$ for $t > 0$ in $d = 3$

$$G(\vec{r}) = \int \frac{d^3 \vec{q}}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} = \frac{1}{4\pi} e^{-r\sqrt{t}} \quad \Rightarrow \quad \nu = 1/2, \ \eta = 0$$

\Rightarrow scaling laws are satisfied (check!) together with exponents from mean-field theory, except Josephson’s law (which is only satisfied in $d = 4$)
2.6 Ginzburg-Landau-Wilson theory

In this section, we study interacting bosons as relevant to the liquid-superfluid transition in $^4\text{He}$ ($\lambda$ transition, $T_c \approx 2K$). The construction can be performed analogously for other transitions. The goal of this section is to express the (grand-canonical) partition function $Z$ in terms of the so-called path integral or functional integral. For functional integrals many convenient tools have been developed which facilitate systematic approximate evaluations of the partition function. So, the general strategy is the following:

- We want to evaluate the partition function of an interacting many-particle system (for example $^4\text{He}$) to study thermodynamics, phase transitions, etc.

\[ Z = \text{Tr} e^{-\beta(\hat{H} - \mu \hat{N})} \]

\[ Z = \int D\Phi e^{-S[\Phi]}, \text{ with action } S[\Phi] \]

- learn about various tools to systematically evaluate $Z$ in its functional representation

- the Hamiltonian for interacting bosons reads in second quantization

\[ \hat{H} = \hat{H}(\hat{a}^\dagger, \hat{a}) = \sum_{\alpha,\beta} \epsilon_{\alpha\beta} \hat{a}^\dagger_\alpha \hat{a}_\beta + \sum_{\alpha,\beta,\gamma,\delta} V_{\alpha\beta\gamma\delta} \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta \hat{a}_\delta \hat{a}_\gamma \]  

(2.29)

- the particle number operator is

\[ \hat{N} = \sum_\alpha \hat{a}^\dagger_\alpha \hat{a}_\alpha \]  

(2.30)

2.6.1 Functional integral representation of the partition function

Coherent states (more details available in the book by Altland & Simons)

- standard orthogonal basis for quantum mechanical many-body states:

\[ |n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_N}\rangle = \prod_{i=1}^{N} \frac{(\hat{a}^\dagger_{\alpha_i})^{n_{\alpha_i}}}{\sqrt{n_{\alpha_i}}} |0\rangle \]  

(2.31)

- $\{\alpha_i\}$ label the states forming basis in single-particle Hilbert space of dimension $N$
- $|0\rangle$ is the vacuum
- $\hat{a}^\dagger_{\alpha_i}, \hat{a}_{\alpha_i}$ are bosonic creation/annihilation operators $\rightarrow$ single-particle state $|\alpha_i\rangle$
- standard commutation relations $[\hat{a}_{\alpha_i}, \hat{a}^\dagger_{\alpha_j}] = \delta_{\alpha_i\alpha_j}$
- any many-particle state is a linear combination:

\[ |\Phi\rangle = \sum_{n_{\alpha_1}=0}^{\infty} \cdots \sum_{n_{\alpha_N}=0}^{\infty} \Phi_{n_{\alpha_1},...,n_{\alpha_N}} |n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_N}\rangle \]  

(2.32)
2.6. Ginzburg-Landau-Wilson theory

- Definition: a *coherent state* is a common eigenstate of the annihilation operators:

\[ \hat{a}_{\alpha i} | \Phi \rangle = \Phi_{\alpha i} | \Phi \rangle , \quad (2.33) \]

with \( i = 1, 2, ..., N \) and complex eigenvalues \( \Phi_{\alpha i} \)

- for such a state we have (tutorial)

\[
\Phi_{n_{\alpha 1},...,n_{\alpha N}} = \prod_{i=1}^{N} \frac{\left( \Phi_{\alpha i} \right)^{n_{\alpha i}}}{\sqrt{n_{\alpha i}!}} \quad \Rightarrow \quad | \Phi \rangle = \sum_{n_{\alpha 1}=0}^{\infty} \cdots \sum_{n_{\alpha N}=0}^{\infty} \prod_{i=1}^{N} \frac{\left( \Phi_{\alpha i} \hat{a}_{\alpha i}^\dagger \right)^{n_{\alpha i}}}{n_{\alpha i}!} | 0 \rangle \\
= e^{\sum_{\alpha i} \Phi_{\alpha i} \hat{a}_{\alpha i}^\dagger} | 0 \rangle \quad (2.34) \]

- similarly: \( \langle \Phi | \hat{a}_{\alpha i}^\dagger = \langle \Phi | \Phi_{\alpha i}^\dagger \rightarrow \text{bra version}: \langle \Phi | = \langle 0 | e^{\sum_{\alpha i} \Phi_{\alpha i}^\dagger \hat{a}_{\alpha i}} \\
\Rightarrow \hat{a}_{\alpha i}^\dagger | \Phi \rangle = \frac{\partial}{\partial \Phi_{\alpha i}} | \Phi \rangle \) which can be shown by Taylor expansion of eq. (2.34)

- overlap of two coherent states: \( \langle \Phi | \Phi' \rangle = e^{\sum_{\alpha i} \Phi_{\alpha i}^\dagger \Phi_{\alpha i}'} \) (not orthogonal!)

- coherent state form *overcomplete* set with resolution of unity (without proof):

\[
\int \prod_{\alpha} \frac{d\Phi_{\alpha}^* d\Phi_{\alpha}}{2\pi i} e^{-\sum_{\alpha} \Phi_{\alpha}^* \Phi_{\alpha}} | \Phi \rangle \langle \Phi | = 1 \quad (2.35) \]

where we omitted the index \( i \) for simplicity

- eq. (2.35) can be shown by proof for \( N = 1 \) and then directly generalized to \( N > 1 \)

- we will use the coherent states to re-express the Hamiltonian operator (which in turn is expressed in terms of creation and annihilation operators) in terms of eigenstates
Path-integral representation of the grand-canonical partition function

- grand-canonical partition function $Z = \text{Tr} e^{-\beta(\hat{H} - \mu \hat{N})} = \sum_n \langle n | e^{-\beta(\hat{H} - \mu \hat{N})} | n \rangle$
  where the sum extends over a complete set of Fock states $\{|n\rangle\}$

- together with eq. (2.35), we get:

$$Z = \int \prod_\alpha \frac{d\Phi_\alpha^* d\Phi_\alpha}{2\pi i} e^{-\sum_\alpha \Phi_\alpha^* \Phi_\alpha} \langle \Phi | e^{-\beta(\hat{H} - \mu \hat{N})} | \Phi \rangle$$

(2.36)

- divide imaginary “time interval” $\beta$ into $M$ pieces and insert unity operator $\left(M-1\right)$ times

$$\Rightarrow Z = \int \prod_{k=0}^{M-1} \prod_\alpha \frac{d\Phi_\alpha^* d\Phi_\alpha}{2\pi i} e^{-\sum_{k=0}^{M-1} \sum_\alpha \Phi_\alpha^* \Phi_\alpha} \prod_{k=1}^{M} \langle \Phi_{k-1} | e^{-\Delta \beta(\hat{H} - \mu \hat{N})} | \Phi_k \rangle$$

(2.37)

where we have the boundary condition $\Phi_0 = \Phi_M = \Phi$ and $\Delta \beta = \beta/M$

- for $\Delta \beta \ll 1$:

$$\langle \Phi_{k-1} | e^{-\Delta \beta(\hat{H} - \mu \hat{N})} | \Phi_k \rangle = \langle \Phi_{k-1} | \Phi_k \rangle e^{-\Delta \beta(\langle \Phi_{k-1}|(\hat{H} - \mu \hat{N})|\Phi_k \rangle + \mathcal{O}(\Delta \beta^2) \rangle}$$

(2.38)

- in Hamiltonian and number operator all creation operators are to the left of annihilation operators $\rightarrow$ normal ordering

- for arbitrary normally ordered function $A$, the definition of coherent states implies:

$$\langle \Phi | A(\hat{a}_\alpha^+, \hat{a}_\alpha) | \Phi' \rangle = A(\phi_\alpha^*, \phi'_\alpha) e^{\sum_\alpha \Phi_\alpha^* \Phi'_\alpha}$$

(2.39)

$$\Rightarrow \text{ for } M \rightarrow \infty :$$

(2.40)

$$Z = \lim_{M \rightarrow \infty} \int \prod_{k=0}^{M-1} \prod_\alpha \frac{d\Phi_\alpha^* d\Phi_\alpha}{2\pi i} e^{-\sum_{k=0}^{M-1} \sum_\alpha \Phi_\alpha^* \Phi_\alpha (\Phi_{\alpha,k-1} - \Phi_{\alpha,k+1})}$$

$$\times e^{-\sum_{k=0}^{M-1} \Delta \beta \left(H(\Phi_{\alpha,k}, \Phi_{\alpha,k+1}) - \mu \sum_\alpha \Phi_\alpha^* \Phi_{\alpha,k} \Phi_{\alpha,k+1}\right)}$$

(2.41)

where $k$ labels moments in imaginary time.
Continuum limit

- the continuum limit of eq. (2.41) gives the functional integral:

\[
Z = \int_{\phi_\alpha(0) = \phi_\alpha(\beta)} D\phi_\alpha^*(\tau) D\phi_\alpha(\tau) e^{-S[\phi_\alpha^*(\tau), \phi_\alpha(\tau)]} 
\]  

(2.42)

with the action

\[
S = \int_0^\beta d\tau \left[ \sum_\alpha \phi_\alpha^*(\tau)(-\partial_\tau - \mu)\phi_\alpha(\tau) + H[\phi_\alpha^*(\tau), \phi_\alpha(\tau)] \right] 
\]  

(2.43)

- where \( \alpha \) can be: momentum, position \( \vec{r} \), lattice site,...

- interacting bosons: choose continuous position coordinate \( \alpha = \vec{r} \)

- partition function of a system of interaction bosons =

  "sum over all possible complex functions \( \Phi(\vec{r}, \tau) \) of space and imaginary time which are periodic in imaginary time."

- we can decompose the periodic function (periodicity due to boundary condition!):

\[
\Phi(\vec{r}, \tau) = \frac{1}{\beta} \int \frac{d\vec{k}}{(2\pi)^d} \sum_{\omega_n} \Phi(\vec{k}, \omega_n) e^{i\vec{k} \cdot \vec{r} + i\omega_n \tau} 
\]  

(2.44)

with the bosonic Matsubara frequencies \( \omega_n = 2\pi n / \beta = 2\pi nk_B T, \, n \in \mathbb{Z} \)

- remarks:
  - in the case of fermions \( \rightarrow \) antiperiodic, anticommuting Grassmann numbers, \( \omega_n,F = (2n + 1)\pi / \beta \)
  - example: non-interacting bosonic system

\[
Z = Z_0 = \prod_{\vec{k}, \omega_n} \int \frac{d\phi^{*}(\vec{k}, \omega_n) d\phi(\vec{k}, \omega_n)}{2\pi i} e^{-\beta^{-1} \sum_\omega \left( -i\omega_n + \frac{\hbar^2 \vec{k}^2}{2m} - \mu \right) |\phi(\vec{k}, \omega_n)|^2} 
\]  

(2.45)

\[
= \prod_{\vec{k}, \omega_n} \frac{\beta}{-i\omega_n + \frac{\hbar^2 \vec{k}^2}{2m} - \mu} 
\]  

(2.46)

\( \rightarrow \) recover Bose-Einstein condensation from \( F_0 = -k_B T \log Z_0 \) and \( N = -\frac{\partial F_0}{\partial \mu} \ldots \)

- at \( T = 0 \), the Matsubara frequencies form a continuum
2. Phase transitions and critical phenomena

Path-integral representation at finite temperature

- at finite $T$: Fourier mode $\omega_0 = 0$ separated by finite amount $\propto T$ from the action for the other modes ($\omega_n \neq 0$)

$$S = \beta^{-1} \int \frac{d\vec{k}}{(2\pi)^d} \sum_n (-i\omega_n - \mu) |\Phi(\vec{k}, \omega_n)|^2 + ...$$ (2.47)

- $\omega_n$ modes ($|n| \geq 1$) only yield analytic contribution to free energy
  $\Rightarrow$ these modes are non-critical and do not affect universal properties

- for finite temperature transition, only retain the $\omega_0$-modes in the partition function
  $\Rightarrow$ fields independent of imaginary time!
  $\Rightarrow$ partition function

$$Z = \int \mathcal{D}\Phi^*(\vec{r}) \mathcal{D}\Phi(\vec{r}) e^{-S}$$ (2.48)

- example: non-relativistic bosons of mass $m$

  $\rightarrow$ energy dispersion $\epsilon_k = \frac{\hbar^2 k^2}{2m}$, i.e. non-interacting part of Hamiltonian, eq. (2.43):

$$H_0[\Phi^*(\vec{r}), \Phi(\vec{r})] = \beta \int d\vec{r} \left[ \Phi^*(\vec{r}) \left( -\frac{\hbar^2 \nabla^2}{2m} \right) \Phi(\vec{r}) \right]$$ (2.49)

  $\rightarrow$ contact interaction $V(\vec{r}_1 - \vec{r}_2) = \lambda \delta(\vec{r}_1 - \vec{r}_2)$, cf. eq. (2.29), gives the action:

$$S[\Phi] = \frac{1}{k_BT} \int d\vec{r} \left[ \frac{\hbar^2}{2m} |\nabla \Phi(\vec{r})|^2 - \mu |\Phi(\vec{r})|^2 + \lambda |\Phi(\vec{r})|^4 \right]$$ (2.50)

- together with $Z$ this defines Ginzburg-Landau-Wilson theory
- for superfluid transition (in $^4$He), $\Phi(\vec{r})$ is complex: two real components ($N = 2$)
- contact interaction is sufficient to understand critical behavior
- more general: field $\Phi$ with $N$ real components and same action as in eq. (2.50)
- for $N = 1$: symmetry is $Z_2$ ($\Phi(\vec{r}) \rightarrow -\Phi(\vec{r})$)
  $\Rightarrow$ same symmetry as Ising model
  $\Rightarrow$ eq. (2.50) with $N = 1$ describes uniaxial FM transition
- for $N = 3$: same symmetry as Heisenberg model

Ginzburg-Landau-Wilson (GLW) partition function with general $N$ serves as meta-model to which specific models like Ising, XY, or Heisenberg model reduce in the critical region.

$\rightarrow$ only universal critical behavior may be expected to be given correctly by GLW theory (not thermodynamics outside the critical region)
2.6.2 Saddle-point approximation (SPA)

SPA: evaluation of GLW partition function, eq. (4.1), with value of integrand at its maximum:
\[ F = k_B T \cdot S[\Phi_0] \quad (2.51) \]

- here \( \Phi_0 \) is the configuration that minimizes the action: \( \frac{\delta S}{\delta \Phi}|_{\Phi_0} = 0 \)
- \( \Phi_0 \) will be independent of the coordinate and be (1) \( \Phi_0 = 0 \), or (2) \( |\Phi_0|^2 = \frac{\mu}{2\lambda} \)
- take \( \mu \) as a tuning parameter:
  (a) for \( \mu < 0 \): non-trivial solution (2) is impossible \( \Rightarrow \Phi_0 = 0 \)
  (b) for \( \mu > 0 \): \( S[\Phi_0] = -\frac{\mu^2 V}{4\lambda k_B T} < S[0] = 0 \) \( \Rightarrow \Phi_0 \) is finite
  \[ \Phi_0 = 0 \] becomes a local maximum
  \( \Rightarrow \) action has \textit{mexican hat} shape with a continuum of minima

- a finite value of \( \Phi_0 \) signals the ordered phase \( \Rightarrow \) take \( \Phi_0 \) as \textit{order parameter}
- in general the order parameter is defined as \( \langle \Phi(\vec{r}) \rangle \) which in the SPA coincides with \( \Phi_0 \)

\textbf{In terms of temperature} \( T \):
- transition in SPA takes place at \( T_c \) where \( \mu(T) \) changes sign
  \( \Rightarrow -\mu \) is coefficient of the quadratic term of \( S \), which changes sign near the transition
  \[ \mu \sim (T_c - T) + \mathcal{O}\left((T_c - T)^2\right) \quad (2.52) \]

\textbf{Spontaneous symmetry breaking}:
- only absolute value of \( \Phi_0 \) is fixed by \( |\Phi_0|^2 = \frac{\mu}{2\lambda} \), the phase is left arbitrary
- any choice of phase breaks global \( U(1) \) invariance \( (\Phi(\vec{r}) \rightarrow e^{i\phi}\Phi(\vec{r})) \) of the action (2.50)
  \( \Rightarrow \) \( U(1) \) symmetry is spontaneously broken in the ordered/superfluid phase!
- remarks:
  - \( U(1) \) symmetry is analogue of rotational symmetry that is broken by the direction of magnetization, e.g., in FM transition
  - consequence of the breaking of continuous symmetries \( \rightarrow \text{Goldstone bosons} \) appear
  - spontaneous breaking of \( Z_2 \): no Goldstone bosons
2. Phase transitions and critical phenomena

2.6.3 Critical exponents in saddle-point approximation

- since $\mu \sim T_c - T$ in GLW theory \(\rightarrow\) differentiating \(S[\Phi_0]\) twice w.r.t \(T\):
  \(\rightarrow\) specific heat only has a finite discontinuity at the transition
  \(\Rightarrow\) critical exponent $\alpha = 0$

- define superfluid susceptibility:
  \[
  \chi(\vec{r} - \vec{r}') := \delta^2 \log Z[J]\left|_{J=0}\right.
  \]
  where
  \[
  S \to S + \int d\vec{r} [\Phi(\vec{r}) J(\vec{r}) + \Phi^*(\vec{r}) J^*(\vec{r})]
  \]
  defines $Z[J]$
  \[
  \Rightarrow \chi(\vec{r} - \vec{r}') = \langle \Phi^*(\vec{r}) \Phi(\vec{r}') \rangle - |\langle \Phi(\vec{r}) \rangle|^2
  \]
  – note the similarity to eq. (2.24)
  – also note that the source field $J(\vec{r})$ directly couples to $\Phi$ in the action in the same way as the real magnetic field would enter the partition function of magnetic systems, see eq. (2.27).

- to compute $\chi$ in saddle-point approximation consider small fluctuations of $\Phi(\vec{r})$ around the saddle point (for $\mu < 0$, i.e. expand the action around trivial saddle point)
  \(\rightarrow\) then neglect $\lambda \Rightarrow$ action is quadratic in fluctuating fields
  \(\Rightarrow Z[J]\) can be computed by completing the square:
  \[
  \log Z[J] = \int d\vec{r} d\vec{r'} J^*(\vec{r}) \chi_0(\vec{r} - \vec{r'}) J(\vec{r'}) + \text{const.}
  \]
  where
  \[
  \chi_0(\vec{r}) = k_B T \int \frac{d\vec{k}}{(2\pi)^d} \frac{e^{i\vec{k} \cdot \vec{r}}}{\sqrt{2m^2 + \mu}}
  \]
  near $T_c$ where $\mu(T_c) = 0$ rewrite $\chi_0$ as:
  \[
  \chi_0(\vec{r}) = \frac{2mk_B T_c}{\hbar^2} \cdot \frac{F(r/\xi)}{r^{d-2}}
  \]
  with the scaling function $F(z) = z^{d-2} \int \frac{d\vec{q}}{(2\pi)^d} \frac{e^{i\vec{q} \cdot \vec{r}}}{q^2 + 1}$
  and the correlation length $\xi = \hbar / \sqrt{2m|\mu|}$

- this result for $\chi_0$ can be cast into the form in eq. (2.26)
  – since $\mu \sim T_c - T$ near $T_c$:
    correlation length exponent $\nu = 1/2$ and anomalous dimension $\eta = 0$
  \(\Rightarrow\) saddle-point approximation leads to the mean-field values of the critical exponents and is independent of dimension $d$ and number of field components $N \rightarrow$ not satisfactory!
2.7 Kosterlitz-Thouless phase transition

- no SSB of continuous symmetry in 2 dimensions ⇒ no obvious order parameter
- can there still be a phase transition?
- Kosterlitz & Thouless (1973, Nobel prize 2016):
  Yes, phase transition by unbinding of topological defects (vortices)

2.7.1 XY model in two dimensions

- classical statistical model on 2-dimensional square lattice with action:
  \[ S[\phi_i] \equiv \beta H[\phi_i] = -J \sum_{(i,j)} \cos (\phi_i - \phi_j) \]  \hspace{1cm} (2.59)

- \( \phi_i \in [0, 2\pi] \) is an angle variable attached to each point
- the action has a global (continuous) U(1) symmetry: \( \phi_i \rightarrow \phi_i + \alpha \)
- physical picture/interpretation:
  - spins with “easy plane” (free rotation in \( x - y \)-plane, frozen in \( z \)-direction)
  - lattice regularization of complex boson theory at finite temperature \( T \) and at frequencies below \( T \):
    \[ H = \int d^2x \left[ \frac{\kappa}{2} |\nabla \Psi|^2 + \frac{t}{2} |\Psi|^2 + u |\Psi|^4 \right] \]  \hspace{1cm} (2.60)
  - where \( \Psi = \Psi_1 + i \Psi_2 = r e^{i\phi} \)
  - more precisely:
    * in-plane amplitude – phase only comes with \( (\nabla \phi)^2 \) (gapless mode).
    * the above model, eq. (2.59) is a regularized (discrete) version of this part of the complex boson theory and reproduces it at long wavelength (see below).
    * the amplitude fluctuations are gapped and do not contribute to the long-wavelength physics.
- agenda:
  - we first argue that the low- and high-\( T \) phases of the XY model are qualitatively distinct (algebraic vs. exponential decay of correlation functions)
  - then we will explain this following Kosterlitz and Thouless.
2. Phase transitions and critical phenomena

2.7.2 Limiting cases at low and high temperatures

Low-temperature phase

- At $T = 0$, SSB can take place and we have an ordered phase profile with $\phi_i = \phi_0$.
- At low $T$, we therefore expect an almost ordered phase profile $\phi_i \approx \text{const.} = \phi_0$, and only small local fluctuations of the phase.
- thus we decompose $\phi_i = \phi_0 + \delta \phi_i$, where $\delta \phi_i$ are slowly varying small fluctuations

$$\Rightarrow \quad \beta H[\phi_i] = -\frac{J}{T} \sum_{\langle i,j \rangle} \cos (\delta \phi_i - \delta \phi_j) \approx \text{const.} + \frac{J}{2T} \sum_{\langle i,j \rangle} (\delta \phi_i - \delta \phi_j)^2 \quad (2.61)$$

$\quad \text{continuum limit} \quad \overset{\text{continuum limit}}{\Rightarrow} \quad \text{const.} + \frac{a^2 J}{2T} \int d^2 r (\vec{\nabla} \phi)^2 = \frac{\bar{J}}{2T} \int d^2 q q^2 (\phi_q)^2 \quad (2.62)$

- where $\bar{J} = a^2 J$, $a$ is the lattice constant and we have dropped the constant term.
- this reproduces the continuum model with the long wavelength limit.
- the asymptotic (long-wavelength) behavior of the correlation function is:

$$\langle e^{i(\phi(\vec{r}) - \phi(\vec{r}'))} \rangle \approx e^{-\frac{1}{2} \langle (\phi(\vec{r}) - \phi(\vec{r}'))^2 \rangle} = e^{-\frac{T}{\pi J} \log \frac{|\vec{r} - \vec{r}'|}{a}} = \left( \frac{|\vec{r} - \vec{r}'|}{a} \right)^{-\frac{T}{2\pi J}} \quad (2.63)$$

- this is an algebraic decay, quasi long-range order replacing SSB due to smooth long-wavelength fluctuations.

High-temperature phase

High-temperature phase...
3.1 General strategy

Observation: In a finite system the correlation length is bound by the system size
⇒ singular thermodynamic behavior (critical behavior) comes from thermodynamic limit
→ i.e. from modes with arbitrarily low energies in “∞”-large system:

\[ \text{infrared singularity} \]

Consider the action in Eq. (2.50):

\[ S[\Phi] = \frac{1}{k_B T} \int d\vec{r} \left[ \frac{\hbar^2}{2m} |\nabla \Phi(\vec{r})|^2 - \mu |\Phi(\vec{r})|^2 + \lambda |\Phi(\vec{r})|^4 \right] \]  

(3.1)

• \( m \) provides energy scale ⇒ rescale into chemical potential and interaction:

\[ 2m\mu/\hbar^2 \rightarrow \mu, \quad 2m\lambda/\hbar^2 \rightarrow \lambda \]  

(3.2)

• near critical point \((T \approx T_c)\): rescale action \( 2mk_B T_c S/\hbar^2 \rightarrow S = \int d\vec{r} |\nabla \Phi|^2 - \mu |\Phi|^2 + \lambda |\Phi|^4 \)

⇒ superfluid susceptibility is function of wavevector, chemical potential and interaction only:

\[ \chi(\vec{k}) = F(k, \mu, \lambda) \]  

(3.3)

This expression implicitly depends on an

\[ \text{ultraviolet cutoff } \Lambda \sim \frac{1}{a}, \]  

(3.4)

restricting all wavevectors to \(|\vec{k}| < \Lambda|.

• \( a \) defines the shortest length scale in the system (lattice spacing, atom size,...)

• we will find that its exact value does not affect the critical behavior
**Strategy:** low-energy modes (*slow modes*) induce singular behavior.
⇒ integrate only over shell of high-energy modes (large \( k \), *fast modes*):
\[
\frac{\Lambda}{b} < k < \Lambda, \quad \text{where} \quad b \approx 1
\]
\[\text{(3.5)}\]
⇒ recast partition function for remaining slow modes (\( k < \frac{\Lambda}{b} \)) into form before integration
⇒ values of \( \mu \) and \( \lambda \) have to be changed:
\[
\mu \longrightarrow \mu(b), \quad \lambda \longrightarrow \lambda(b)
\]
\[\text{(3.6)}\]
⇒ susceptibility for low \( k \):
\[
\chi(\vec{k}) = b^x F(bk, \mu(b), \lambda(b))
\]
\[\text{(3.7)}\]
- argument “\( bk \)” due to change of shortest length scale \( \Lambda \to \Lambda/b \)
- factor \( b^x \): in units of length scale, \( \chi \) has dimension \( x \), i.e. \( \chi \sim \Lambda^{-x} \)
- Note: this strategy closely resembles the real-space RG discussed on your first problem set [http://www.thp.uni-koeln.de/~scherer/set01.pdf](http://www.thp.uni-koeln.de/~scherer/set01.pdf) only in momentum space!

**Question:** What happens for \( b \to \infty \) (more and more modes integrated out)?
- suppose \( \lim_{b \to \infty} \lambda(b) = \lambda^* \)
\[
\Rightarrow \chi(\vec{k}) = b^x F(bk, \mu(b), \lambda^*)
\]
\[\text{for large } b.\]
- if \( \lambda^* \) is small ⇒ use perturbation theory in \( \lambda^* \)
- i.e. determine effective *renormalized* coupling for low-energy modes first

**Critical behavior:**
- assume \( \lambda^* \) exists and is finite
- further, assume for \( b \gg 1 \) and small \( \mu < 0 \): \( \mu(b) \approx \mu b^y \)
- now, choose \( b \) such that \( \mu b^y = \mu_0 \) with constant \( \mu_0 \)
\[
\Rightarrow \chi(\vec{k}) = \left| \frac{\mu_0}{\mu} \right|^{x/y} F \left( k, \left| \frac{\mu_0}{\mu} \right|^{1/y}, \mu_0, \lambda^* \right)
\]
\[\text{(3.9)}\]
- then, the Fourier transform \( \chi(\vec{r}) \) exhibits the desired scaling form
- the correlation length exponent and the susceptibility exponent are accordingly:
\[
\nu = 1/y, \quad \gamma = x/y.
\]
\[\text{(3.10)}\]
- we still need the functions \( \lambda(b) \) and \( \mu(b) \) to get \( x \) and \( y \) (next section!).
Possible complications: Success of strategy is not guaranteed:

- as $b$ increases: new terms could appear in action for slow modes

$$\lambda_6(b)|\Phi|^6, \quad \lambda_{2,2}(b)|\nabla \Phi|^2|\Phi|^2$$

(3.11)

- generated terms must still be invariant under symmetries of original action
- as $b \to \infty$: a coupling is called
  
  (1) irrelevant if limit is zero (i.e. it was correct to neglect it)
  
  (2) relevant if limit is finite (i.e. coupling needs to be included)

- theory with finite number of relevant couplings is called renormalizable (next chapter!)
3. Wilson’s renormalization group

3.2 Momentum-shell transformation

Implement above strategy for the partition function of eq. (2.50)
→ therefore divide the field \( \Phi \) into slow and fast modes:

\[
\Phi(\vec{r}) = \Phi_<(\vec{r}) + \Phi_>(\vec{r})
\]  

(3.12)

where

• \( \Phi_< \) are the modes with \( k < \Lambda/b \)
• \( \Phi_> \) are the modes with \( \Lambda/b < k < \Lambda \)

⇒ partition function:

\[
Z = \int \prod_{k<\Lambda} \frac{d\Phi^*(\vec{k})d\Phi(\vec{k})}{2\pi i} e^{-(S_0>+S_{\text{int}})}
\]  

(3.13)

\[
= \int \prod_{k<\Lambda} \frac{d\Phi^*(\vec{k})d\Phi(\vec{k})}{2\pi i} \int \prod_{\frac{\Lambda}{b}<k<\Lambda} \frac{d\Phi^*(\vec{k})d\Phi(\vec{k})}{2\pi i} e^{-(S_0>+S_{\text{int}})},
\]  

(3.14)

with

\[
S_{0<} = \int_0^{\Lambda/b} \frac{d\vec{k}}{(2\pi)^d} (\vec{k}^2 - \mu)|\Phi(\vec{k})|^2,
\]  

(3.15)

and

\[
S_{0>} = \int_{\Lambda/b}^{\Lambda} \frac{d\vec{k}}{(2\pi)^d} (\vec{k}^2 - \mu)|\Phi(\vec{k})|^2,
\]  

(3.16)

and

\[
S_{\text{int}} = \lambda \int_0^{\Lambda} \frac{d\vec{k}_1...d\vec{k}_4}{(2\pi)^{3d}} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4)\Phi^*(\vec{k}_4)\Phi^*(\vec{k}_3)\Phi(\vec{k}_2)\Phi(\vec{k}_1).
\]  

(3.17)

• the interaction term in Fourier space can be represented pictorially
• each field \( \Phi \) is represented by a line with an arrow pointing into the vertex
• a field \( \Phi^* \) is represented by a line with an arrow pointing out of the vertex
• we also show the momentum labels of the fields:
3.2. Momentum-shell transformation

3.2.1 Perturbation theory

Zeroth order in $\lambda$:

$Z = Z_{0>}$ \Rightarrow $Z$ factorizes

\rightarrow in $Z_{0>}$ all the modes have large $k$

\Rightarrow $Z_{0>}$ only contributes to regular, analytic part of free energy

\begin{itemize}
  \item now, bring $Z_{0<}$ into exact old form by first rescaling $b\vec{k} \rightarrow \vec{k}$:

$$S_{0<} = \frac{1}{b^d} \int_0^\Lambda \frac{d\vec{k}}{(2\pi)^d} \left( \frac{\vec{k}^2}{b^2} - \mu \right) |\Phi(\vec{k})|^2$$ (3.18)

  \item define: $\mu(b) = \mu b^2$ and $\frac{\Phi(\vec{k})}{b^{d+2/2}} \rightarrow \Phi(\vec{k})$

  \Rightarrow $S_{0<}$ now precisely takes the old form in terms of rescaled Fourier components

  \item variables $\Phi(\vec{k})$ are to be integrated over

  \Rightarrow critical behavior is not affected by their rescaling (also in the integration measure)

  \rightarrow only adds regular part to free energy.

  \Rightarrow we find $\mu(b) = \mu b^2 + O(\lambda)$

  \Rightarrow $\nu = 1/y = 1/2$ \rightarrow mean-field result!

\end{itemize}

First order in $\lambda$:

avoiding slow modes \rightarrow can be done without encountering the singularity

$$Z = Z_{0>} \int \prod_{k<\Lambda/b} \frac{d\Phi^*(\vec{k})d\Phi(\vec{k})}{2\pi i} e^{-S_{0<}} [1$$

$$-\lambda \int_0^\Lambda \frac{d\vec{k}_1...d\vec{k}_4}{(2\pi)^{4d}} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \langle \Phi^*(\vec{k}_4)\Phi^*(\vec{k}_3)\Phi(\vec{k}_2)\Phi(\vec{k}_1) \rangle_{0>} + O(\lambda^2) \right]$$ (3.19)

where we have introduced the fast-mode average

$$\langle A \rangle_{0>} = \frac{1}{Z_{0>}} \int \prod_{\Lambda/b<k<\Lambda} \frac{d\Phi^*(\vec{k})d\Phi(\vec{k})}{2\pi i} e^{-S_{0>}} A.$$ (3.20)

As $S_0$ is quadratic in $\Phi$ and diagonal in $\vec{k}$ we find from Gaussian integration (problem set 3):

$$\langle \Phi^*(\vec{k}_1)\Phi(\vec{k}_2) \rangle_0 = \frac{(2\pi)^d}{k_1^2 - \mu} \delta(\vec{k}_1 - \vec{k}_2),$$ (3.21)

$$\langle \Phi^*(\vec{k}_4)\Phi^*(\vec{k}_3)\Phi(\vec{k}_2)\Phi(\vec{k}_1) \rangle_0 = \langle \Phi^*(\vec{k}_4)\Phi(\vec{k}_2) \rangle_0 \langle \Phi^*(\vec{k}_3)\Phi(\vec{k}_2) \rangle_0 + (\vec{k}_1 \leftrightarrow \vec{k}_2),$$ (3.22)

where the second equation is a variant of Wick's theorem:

- average factorizes into product of all possible averages of pairs of $\Phi^*$ and $\Phi$. 

3. Wilson’s renormalization group

So, next we want to calculate the following part of eq. (3.19):

\[ -\lambda \int_0^\Lambda \frac{d^d \vec{k}_1 \ldots d^d \vec{k}_4}{(2\pi)^{3d}} \delta_{1234} \frac{1}{Z_0^>} \int \prod_{\Lambda/b < k < \Lambda} \frac{d\Phi^*(\vec{k})d\Phi(\vec{k})}{2\pi i} \Phi^*(\vec{k}_4)\Phi^*(\vec{k}_3)\Phi(\vec{k}_2)\Phi(\vec{k}_1)e^{-S_0^>} \]  

(3.23)

where we have introduced the short-hand notation \( \delta_{1234} = \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \).

### 3.2.2 Combinations in eq. (3.23) that contribute in eq. (3.19)

...by giving a finite average:

1. \( \{ \vec{k}_4 = \vec{k}_2 \text{ and } \Lambda/b < k_2 < \Lambda \text{ and } \vec{k}_3 = \vec{k}_1 \text{ and } k_1 < \Lambda/b \} \)
   - then one part of eq. (3.23) reads

\[ -\lambda \int_0^\Lambda \frac{d^d \vec{k}_1 \ldots d^d \vec{k}_4}{(2\pi)^{3d}} \delta_{1234} \Phi^*(\vec{k}_3)\Phi(\vec{k}_1) \left( \frac{1}{Z_0^>} \int \prod_{\Lambda/b < k < \Lambda} \frac{d\Phi^*(\vec{k})d\Phi(\vec{k})}{2\pi i} \Phi^*(\vec{k}_4)\Phi^*(\vec{k}_3)\Phi(\vec{k}_2)\Phi(\vec{k}_1)e^{-S_0^>} \right) \]

\[ = -\lambda \int_0^\Lambda \frac{d^d \vec{k}_1 \ldots d^d \vec{k}_4}{(2\pi)^{3d}} \delta_{1234} \Phi^*(\vec{k}_3)\Phi(\vec{k}_1) \left( \frac{(2\pi)^d}{\vec{k}_2^2 - \mu} \delta(\vec{k}_2 - \vec{k}_4) \right) \]  

(3.24)

momenta \( \vec{k}_4 \) as chosen above

\[ = -\lambda \int_0^{\Lambda/b} \frac{d\vec{k}_1 d\vec{k}_3}{(2\pi)^d} \delta(\vec{k}_1 - \vec{k}_3)\Phi^*(\vec{k}_3)\Phi(\vec{k}_1) \int_{\Lambda/b}^\Lambda \frac{d\vec{q}}{(2\pi)^d} \frac{1}{\vec{q}^2 - \mu} \]  

(3.25)

or the same with \( \{ \vec{k}_4 = \vec{k}_1 \text{ and } \Lambda/b < k_1 < \Lambda \text{ and } \vec{k}_3 = \vec{k}_2 \text{ and } k_2 < \Lambda/b \} \)

2. same with \( \vec{k}_4 \) and \( \vec{k}_3 \) exchanged

\[ \rightarrow \text{combinations 1. and 2. give 4 different ways} \]

\[ \exists \text{ fast modes} \]

\[ \text{slow modes} \]

3. when all \( k < \Lambda/b \)

4. when all \( \Lambda/b < k < \Lambda \rightarrow \text{only contributes to prefactor in eq. (3.19), i.e. to analytic part of free energy!} \)

\[ \rightarrow \text{omit factor from fast modes as it only contributes to analytic part.} \]
3.2. Momentum-shell transformation

⇒ $Z \propto Z_\prec$ with action in $Z_\prec$:

$$S_\prec = \int_0^{\Lambda/b} \frac{d\vec{k}}{(2\pi)^d} \left( k^2 - \mu + 4\lambda \int_{\Lambda/b}^{\Lambda} \frac{d\vec{q}}{(2\pi)^d} \frac{1}{q^2 - \mu} \right) |\Phi(\vec{k})|^2$$

$$+ \lambda \int_0^{\Lambda/b} \frac{d\vec{k}_1...d\vec{k}_4}{(2\pi)^{4d}} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \Phi^*(\vec{k}_4) \Phi^*(\vec{k}_3) \Phi(\vec{k}_2) \Phi(\vec{k}_1) + \mathcal{O}(\lambda^2)$$

(3.27)

→ rescale to bring cutoff back to $\Lambda$ and $\Phi(\vec{k})/b^{(d+2)/2} \to \Phi(\vec{k})$

$$\mu(b) = b^2 \left( \mu - 4\lambda \int_{\Lambda/b}^{\Lambda} \frac{d\vec{q}}{(2\pi)^d} \frac{1}{q^2 - \mu} + \mathcal{O}(\lambda^2) \right)$$

(3.28)

$$\lambda(b) = b^{4-d} + \mathcal{O}(\lambda^2)$$

(3.29)

→ eq. (3.28): interaction shifts critical value of $\mu$ to

$$\mu_c(b) = 4\lambda \int_{\Lambda/b}^{\Lambda} \frac{d\vec{q}}{(2\pi)^d} \frac{1}{q^2 - \mu} + \mathcal{O}(\lambda^2)$$

(3.30)

• here, we use a series expansion for small $c$: $\frac{1}{q^2 - c} = \frac{1}{q^2} + \frac{c}{q^4} + \mathcal{O}(c^2)$

• location of the critical point is not a universal quantity (depends on $\Lambda$)!

• consider deviation of chemical potential from its critical value at current value of $b$:

$$\tilde{\mu} = \mu - \mu_c(b) \Rightarrow \tilde{\mu}(b) = b^2 \left[ \tilde{\mu} - 4\lambda \int_{\Lambda/b}^{\Lambda} \frac{d\vec{q}}{(2\pi)^d} \left( \frac{1}{q^2 - \mu} - \frac{1}{q^2} \right) + \mathcal{O}(\lambda^2) \right]$$

(3.31)

• therefore, in the critical region where $|\tilde{\mu}| \ll \Lambda^2$:

$$\tilde{\mu}(b) = b^2 \tilde{\mu} \left[ 1 - 4\lambda \frac{\Lambda^{d-4} S_d}{(2\pi)^d} \int_{\Lambda/b}^{1} \frac{x^{d-1}}{x^4} dx + \mathcal{O}(\lambda^2) \right]$$

(3.32)

• the integral in eq. (3.32) can be evaluated approximately close to four dimensions, i.e. we assume that $|d - 4| \ll 1$:

$$\int_{\Lambda/b}^{1} \frac{x^{d-1}}{x^4} dx = \frac{1}{d-4} \left( 1 - \frac{1}{bd-4} \right) = \log b + \mathcal{O}(d-4)$$

(3.33)

• dimensionless interaction: $\tilde{\lambda} = \lambda \Lambda^{d-4} S_d/(2\pi)^d$ with $S_d = 2\pi^{d/2}/\Gamma(d/2)$

$$\tilde{\mu}(b) = \tilde{\mu} b^2 \left( 1 - 4\tilde{\lambda} \log b + \mathcal{O}(\tilde{\lambda}(d-4), \tilde{\lambda}^2, \tilde{\lambda} \tilde{\mu}) \right) \approx \tilde{\mu} b^{2-4\tilde{\lambda}+\mathcal{O}(\tilde{\lambda}^2)}$$

(3.34)
3. Wilson’s renormalization group

• now, assume $\hat{\lambda}$ to be small. Consider $b \to \infty$:
  1. for $\tilde{\mu} > 0$, $\tilde{\mu}(b) \to \infty$: superfluid phase (spontaneously broken symmetry)
  2. for $\tilde{\mu} < 0$, $\tilde{\mu}(b) \to -\infty$: normal phase (symmetric)
  3. $\tilde{\mu} = 0$, i.e. $\mu = \mu_c(b)$, $\tilde{\mu}(b) \equiv 0$: fixed point of the transformation
     – fixed point is unstable, $\tilde{\mu}$ grows when perturbed from fixed-point value
     – $\tilde{\mu}$ is a relevant coupling (= tuning parameter of the transition)

• from definition of $y$ ($\mu b^y = \mu_0$, $\mu_0$ fixed) and eq. (3.34)

\[ y = 2(1 - 2\lambda_* + \mathcal{O}(\lambda_*^2)) \]  

(3.35)

where

\[ \lambda_* = \lim_{b \to \infty} \hat{\lambda} \Rightarrow \nu = \frac{1}{2} + \lambda_* + \mathcal{O}(\lambda_*^2) \]  

(3.36)

• Note:
  – in $d > 4$ and $\hat{\lambda} \ll 1$: $\lambda_* = 0$, see eq. (3.29) $\Rightarrow \lambda$ is irrelevant coupling
    → mean-field exponents are exact
    → $\lambda_* = 0$ represents stable fixed point for $d > 4$
  – in $d < 4$: weak interaction grows with $b$ $\Rightarrow$ next-order term must be included
    → $\mu$ and $\lambda$ are relevant couplings at the non-interacting fixed point at $\lambda_* = 0$. 

3.3 Determination of fixed-point coupling below four dimensions

We want to calculate $\lambda_*$ below four dimensions. Therefore, we calculate the $\mathcal{O}(\lambda^2)$-term in eq. (3.29). Explicitly, the $\mathcal{O}(\lambda^2)$ term from eq. (3.19) reads

$$
\frac{\lambda^2}{2} \int_0^\Lambda \frac{d\vec{p}_1...d\vec{p}_4d\vec{q}_1...d\vec{q}_4}{(2\pi)^6} \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \delta(\vec{q}_1 + \vec{q}_2 - \vec{q}_3 - \vec{q}_4)
\times \langle \Phi^*(\vec{p}_4)\Phi^*(\vec{p}_3)\Phi(\vec{p}_2)\Phi^*(\vec{q}_4)\Phi^*(\vec{q}_3)\Phi(\vec{q}_2)\Phi(\vec{q}_1) \rangle_0 >
$$

(3.37)

- To compute the average $\langle \ldots \rangle_0>$: pair up $\Phi^*$ and $\Phi$ with equal wavevectors $\Lambda/b < k < \Lambda$ in all possible ways, similar to procedure in Sec. 3.2.2

- If two unpaired legs remain in the low-energy shell $k < \Lambda/b$ ⇒ contribution to $\mu(b)$

- If four unpaired legs remain in the low-energy shell $k < \Lambda/b$ ⇒ contribution to $\lambda(b)$
  (to calculate the correction to the scaling of $\lambda(b)$ and $\lambda^*$ we need these contributions!)
  ⇒ diagrams contributing to quadratic order in $\lambda$ (joining legs of two vertices):

- diagram (a) can be constructed in 16 different ways.

- diagram (b) can be constructed in 4 different ways.

- only connected diagrams need to be considered

- disconnected diagrams cancel out upon re-exponentiation into $S_\lt$, cf. linked-cluster theorem (later in the lecture)

⇒ upon re-exponentiation the $\mathcal{O}(\lambda^2)$ term becomes:

$$
- \frac{\lambda^2}{2} \int_0^\Lambda \frac{d\vec{k}_1...d\vec{k}_4}{(2\pi)^4} \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \Phi^*(\vec{k}_4)\Phi^*(\vec{k}_3)\Phi(\vec{k}_2)\Phi(\vec{k}_1)
\times \left[ 16 \int_{\Lambda/b}^\Lambda \frac{d\vec{q}}{(2\pi)^d} \frac{1}{(q^2 - \tilde{\mu})(q^2 + \vec{k}_2 - \vec{k}_4)^2 - \tilde{\mu})} + 4 \int_{\Lambda/b}^\Lambda \frac{d\vec{q}}{(2\pi)^d} \frac{1}{(q^2 - \tilde{\mu})(\vec{k}_1 + \vec{k}_2 - \vec{q})^2 - \tilde{\mu})} \right]
$$

(3.38)

- note: we have replaced $\mu$ with $\tilde{\mu}$ as the difference is $\mathcal{O}(\lambda)$ ⇒ affects only next order!

- momentum-shell trafo induced wavevector dependence in the interaction $\sim \Phi^*\Phi^*\Phi\Phi$. 

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3. **Wilson’s renormalization group**

- to define a single interaction coupling constant \(\lambda(b)\): choose value at \(\vec{k}_i = 0\)

\[
\Rightarrow \lambda(b) = b^{4-d} \lambda \left( 1 - 10\lambda \int_{\Lambda/b}^{\Lambda} \frac{d\vec{q}}{(2\pi)^2} \frac{1}{(q^2 - \tilde{\mu})^2} + \mathcal{O}(\lambda^2) \right)
\]  
\[\text{(3.39)}\]

\(\rightarrow\) at critical point \(\tilde{\mu} = 0\) (perform integral, introduce \(\hat{\lambda}\)):

\[
\hat{\lambda}(b) = b^{4-d} \hat{\lambda} \left( 1 - 10\hat{\lambda} \log b + \mathcal{O}(\hat{\lambda}^2) \right) = b^{4-d-10\hat{\lambda}+\mathcal{O}(\hat{\lambda}^2)} \hat{\lambda}
\]  
\[\text{(3.40)}\]

\(\rightarrow\) together with eq. (3.34):

\[
\tilde{\mu}(b) = b^2 \tilde{\mu} \left( 1 - 4\hat{\lambda} \log b + \mathcal{O}(\hat{\lambda}(d-4), \hat{\lambda}^2, \hat{\lambda} \tilde{\mu}) \right) = \tilde{\mu} b^{2-4\hat{\lambda}+\mathcal{O}(\lambda^2)}
\]  
\[\text{(3.41)}\]

this completes the calculation to lowest non-trivial order.
3.4 Epsilon expansion and Wilson-Fisher fixed point

Cast momentum-shell transformations $\mu(b)$ and $\lambda(b)$, eqs. (3.34) and (3.40), into differential form for $|4 - d| \ll 1$, $4 - d = \epsilon$:

$$\frac{d\mu(b)}{d\log b} = \beta_\mu = \bar{\mu}(b) \left(2 - 4\hat{\lambda}(b) + \mathcal{O}(\epsilon\hat{\lambda}, \hat{\lambda}^2, \hat{\lambda}^2\bar{\mu})\right)$$  \hspace{1cm} (3.42)

$$\frac{d\hat{\lambda}(b)}{d\log b} = \beta_\lambda = \epsilon\hat{\lambda}(b) - 10\hat{\lambda}(b)^2 + \mathcal{O}(\epsilon\hat{\lambda}^2, \hat{\lambda}^3, \hat{\lambda}^2\bar{\mu})$$  \hspace{1cm} (3.43)

- this defines the renormalization group $\beta$ functions!

- momentum-shell trafo (=RG trafo) $\Rightarrow$ flow in the space of coupling functions

**Understand the flow in the $\hat{\lambda} - \bar{\mu}$ plane**

- look for **fixed points** ($\beta_\mu = \beta_\lambda = 0$) and analyze their **stability**

- we find two different fixed points:

  1. the (**trivial**) Gaussian fixed point (GFP): $\bar{\mu} = 0$, $\hat{\lambda} = 0$

  2. the (**non-trivial**) Wilson-Fisher fixed point (WFFP): $\bar{\mu} = 0$, $\hat{\lambda} = \frac{\epsilon}{10} + \mathcal{O}(\epsilon^2)$

- for $\epsilon < 0$ the WFFP is unphysical as $\hat{\lambda} < 0$ and further it is unstable in both directions:
  $\rightarrow$ change of $b$ (integrates out more modes) $\Rightarrow$ $\bar{\mu}(b)$ and $\hat{\lambda}(b)$ are driven away from WFFP.

- for $\epsilon > 0$: the WFFP is physical as $\hat{\lambda} > 0$.

**Analysis of stability**

- For analysis of stability we look at $\beta$ functions in $\lambda - \mu$ plane (see Fig. 3.1)

  - the WFFP is stable in the $\hat{\lambda}$ direction and unstable in the $\bar{\mu}$ direction: **critical point**

  - the GFP is unstable in both directions

  $\Rightarrow$ The critical behavior for $\epsilon > 0$ is governed by the nontrivial Wilson-Fisher fixed point

- parameter $\epsilon$ controls the size of the corrections to MFT critical behavior

- consider $\epsilon$ as a continuous parameter

- in $d = 4$: slow flow to zero $\Rightarrow$ logarithmic corrections to scaling
Figure 3.1: The arrows in the plot are $\vec{\beta}(\hat{\lambda}, \tilde{\mu}) = (\beta_{\lambda}, \beta_{\mu})|_{\hat{\lambda}, \tilde{\mu}}$. The black square indicates the GFP and the red square the WFFP. In the $\hat{\lambda}$ direction, the arrows point into the WFFP indicating its stability.

**Flow near fixed points from stability matrix**

$$M_{ij} := \left. \frac{\partial \beta_{x_i}}{\partial x_j} \right|_{x_i = x_i^*} \quad \text{(3.44)}$$

where $x_i = \tilde{\mu}, \hat{\lambda}$.

- eigenvectors of $M$ with positive eigenvalues: *unstable* directions at a given fixed point
  $\rightarrow$ relevant direction
  $\rightarrow$ tuning required to flow to CP (corresponds to tuning to critical temperature)

- eigenvectors of $M$ with negative eigenvalues: *stable* directions at a given fixed point
  $\rightarrow$ irrelevant direction

- **critical point**: only one relevant direction with eigenvalue $y$, cf. eq. (3.36)
  $\Rightarrow$ correlation length exponent $\nu = 1/y$
  - here $M_{ij}$ is already diagonal at WFFP with $y = 2 - 2\epsilon/5$
    $\Rightarrow \nu = \frac{1}{2} + \frac{\epsilon}{10} + O(\epsilon^2)$
  - for $\epsilon = 1(d = 3)$: $\nu \approx 0.6$
  - the experimental value is $\nu_{\text{exp}} \approx 0.670$, i.e. we’re already better than MFT

- a single eigenvalue determines the flow for positive and negative chemical potential
  $\rightarrow$ $\nu$ is equal on both sides of the transition!
Determinatio[n of anomalous dimension

- from rescaling of momenta/fields by momentum-shell trafo:

\[ k^2 \rightarrow Z_k k^2, \quad \mu \rightarrow Z_\mu \mu, \quad \lambda \rightarrow Z_\lambda \lambda \] (3.45)

- define \( \eta = \frac{dZ_\lambda}{d\ln b} \bigg|_{\lambda=\hat{\lambda}} \Rightarrow \eta = \mathcal{O}(\epsilon^2) \)

... calculation ... \( \Rightarrow \) Fisher's scaling law!

Role of newly generated couplings

- for example a sixth order term \( \sim g |\Phi|^6 \) is generated due to the (quartic) coupling \( \lambda \):

\[ \beta_g = \frac{dg}{d\ln b} = 2(3 - d)g + \mathcal{O}(g \hat{\lambda}) \] (3.46)

\( \Rightarrow \) \( g \) is irrelevant at the Wilson-Fisher fixed point near \( d = 4 \)

- in \( d = 3 \): linear term vanishes \( \rightarrow \) coupling becomes marginal
  - i.e. neither relevant nor irrelevant
- higher-order terms in \( \beta_g \) make \( g \) irrelevant at Wilson-Fisher fixed point in \( d = 3 \)

- standard (dimensional) rescaling:

\[ \beta_g = \frac{dg}{d\ln b} = 2(3 - d)g + \mathcal{O}(g \hat{\lambda}) \]

\( \Rightarrow \) \( g \) is irrelevant at the Wilson-Fisher fixed point near \( d = 4 \)

• dimensional analysis: all other couplings are also irrelevant near \( d = 4 \) (next chapter!)
3.5 Summary and comparison to experiment

- near and below $d = 4$:
  - effect of the elimination of high-energy degrees of freedom in GLW theory is only
    the renormalization of already existing couplings
  - all other couplings are ultimatively irrelevant
    $\Rightarrow$ scaling forms of physical quantities (free energy, $\chi$) near transition!
  - corrections to scaling from irrelevant variables (i.e. quartic interaction)

- above $d = 4$:
  - critical behavior is of mean-field type
  - interaction is dangerously irrelevant $\rightarrow$ violation of hyperscaling!

- For a field with $N$ real components and $O(N)$ rotational symmetry one finds:
  \[
  \nu = \frac{1}{2} + \frac{N + 2}{4(N + 8)} \epsilon + \frac{(N + 2)(N^2 + 23N + 60)}{8(N + 8)^3} \epsilon^2 + O(\epsilon^3)
  \] (3.47)

  - series has been computed to 6th order: https://arxiv.org/abs/1705.06483 (May 2017)
  - XY model ($N = 2$ in $d = 3$): critical exponent $\nu$
    \[
    \nu = 0.5 + 0.1\epsilon + 0.055\epsilon^2 - 0.0126091\epsilon^3 + 0.0713876\epsilon^4 - 0.194548\epsilon^5 + 0.707019\epsilon^6 + O(\epsilon^7)
    \]
    - direct substitution of $\epsilon = 1$:
      \[
      \begin{array}{cccccccc}
      \epsilon^0 & \epsilon^1 & \epsilon^2 & \epsilon^3 & \epsilon^4 & \epsilon^5 & \epsilon^6 \\
      0.5 & 0.6 & 0.655 & 0.642 & 0.714 & 0.519 & 1.226 \\
      \end{array}
      \]
      - $\epsilon$ expansion does not converge $\rightarrow$ it is a divergent series: asymptotic series!
      - errors at any given order is bounded by next order term (see tutorial)

- One can extract very accurate results from the $\epsilon$ expansion series using resummation
techniques (e.g. Borel summation): educated guess about the function being expanded
from the finite number of computed terms.

- correlation length exponent $\nu$ for $N \in \{1, 2, 3\}$:
  \[
  \begin{array}{cccc}
  N & \nu^6 \text{ expansion (resummed)} & \nu^6 \text{ experiment} \\
  1 & 0.6292(5) & 0.64 \pm 0.004 \\
  2 & 0.6690(10) & 0.670 \pm 0.007 \\
  3 & 0.7059(20) & 0.72 \pm 0.01 \\
  \end{array}
  \]
  - critical exponents nowadays benchmark for different theoretical methods (RG approaches,
    CFT, Monte Carlo methods)
3.6 Excursion: Relations to QFTs in high-energy physics

Disclaimer: This section cannot replace a full lecture on particle physics QFT, of course. I hope to give you a reasonable account for the bare basics, here and refer to the textbook by Peskin & Schroeder for more details. The focus of the presentation is the application of the RG beta functions from the previous section which allow us to understand two problems in the Standard Model of Particle Physics: the hierarchy problem and the triviality problem.

To describe non-relativistic interacting bosons in three spatial dimensions, we have employed the action, eq. (2.50)

$$S[\Phi] = \frac{1}{k_B T} \int d\vec{r} \left[ \frac{\hbar^2}{2m} |\nabla \Phi(\vec{r})|^2 - \mu |\Phi(\vec{r})|^2 + \lambda |\Phi(\vec{r})|^4 \right], \tag{3.48}$$

which together with the partition function in functional representation

$$Z = \int \mathcal{D}\Phi^*(\vec{r}) \mathcal{D}\Phi(\vec{r}) e^{-S}, \tag{3.49}$$

allows us to calculate thermodynamic properties from operator averages/correlation functions. This quantum field theoretical formalism that we have developed for the system of many interacting bosons works analogously for other systems with a large/infinite number of coupled degrees of freedom (fields).

- The functional integral approach is also applied in the context of particle physics/high-energy physics!
- in particle physics it is often used to calculate scattering amplitudes (e.g. for large particle collider experiments)
- Particle-physics (PP) is typically formulated as a field theory following the laws of the theory of special relativity

$$\rightarrow (1+3) \text{ dimensional Minkowski spacetime with metric } \eta_M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Particle physics can also be studied in euclidean formulation which is often very convenient

$$\rightarrow \text{ Euclidean space with metric } \eta_E = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Therefore, we need to introduce a Wick rotation \( t = -it_E \) in the time coordinate which then provides an evolution in imaginary time
- Euclidean functional integral for partice physics quantum field theory emphasizes deep connection to statistical mechanics and condensed-matter physics.
- For the calculation of particle physics scattering amplitudes \( \rightarrow \) analytic continuation
- In summary: Euclidean formulation of functional integral is natural for statistical mechanics and convenient for particle physics!
Remarks on relations between particle physics QFT and statistical mechanics

1. Euclidean QFT in $d$-dimensional spacetime $\sim$ classical stat-mech in $d$-dimensional space

2. Euclidean QFT in $d+1$-dim spacetime, $0 \leq \tau \leq \beta \sim$ quantum stat-mech in $d$-dim space

3. Euclidean QFT in $d$-dim spacetime $\sim$ high-$T$ quantum stat-mech in $d$-dim space

Remarks on the Standard Model of Particle Physics

The Standard Model of Particle Physics describes the electromagnetic, weak and strong interactions and classifies all known elementary particles. Does not include gravity (for reasons, which we will learn later in the lecture course). Standard Model includes Higgs boson:

- Higgs boson is a massive scalar particle/field with self-interactions
- it provides a mechanism to give mass to leptons (electron, muon, tauon) and quarks
- “Higgs-like” particle has been measured at LHC in 2012 with mass $m_H \sim 125$ GeV
- generally couples to other fields in Standard Model (e.g. quarks... most importantly the top quark and gauge fields)
- for our purposes it is sufficient to consider the Higgs field individually with a simplified action for a complex scalar field in four-dimensional Euclidean space:

$$S[\Phi] = \int d^4x \left[ |\partial_\mu \Phi(x)|^2 - \mu |\Phi(x)|^2 + \lambda |\Phi(x)|^4 \right], \quad (3.50)$$

- note the striking similarity to the interacting bosons from the previous sections!
- in fact, we have carried out the RG transformations in $4-\epsilon$ euclidean dimensions
- to obtain the RG beta functions for this theory, we just have to set $\epsilon = 0$ in eqs. (3.42)
- this provides us with the leading order beta functions for the Higgs field (without the couplings to other Standard Model fields):

$$\beta_\mu = 2\tilde{\mu}(\tilde{b}) - 4\tilde{\lambda}(\tilde{b})\tilde{\mu}(\tilde{b}) \quad (3.51)$$
$$\beta_\lambda = -10\tilde{\lambda}(\tilde{b})^2 \quad (3.52)$$

- implicit to these equations is the choice of an ultraviolet cutoff $\Lambda$ setting the largest momentum scale of the problem.
- The cutoff $\Lambda$ could be interpreted as the largest energy or momentum scale where the Standard Model of Particle Physics can be well defined.
- Ideally we fantasize that a Standard Model of Elementary Particles can be valid at arbitrarily high energies and momentum and therefore we hope that we can somehow send $\Lambda \to \infty$!
- The next section explores wether this is possible!
3.6. Excursion: Relations to QFTs in high-energy physics

3.6.1 Landau-pole singularity and the triviality problem

We can directly integrate eq. (3.52) by separation of variables. For simplicity of notation, we suppress the tilde on top of the λ and just write λ. Before, we reformulate the derivative with respect to log \( b \):

\[
\frac{d \log b}{db} = \frac{1}{b} \quad \Rightarrow \quad d \log b = \frac{1}{b} db \quad \Rightarrow \quad \frac{d \lambda(b)}{d \log b} = b \frac{d \lambda(b)}{db}.
\] (3.53)

Therefore, we have to solve

\[
b \frac{d \lambda(b)}{db} = -10\lambda(b)^2 \quad \Rightarrow \quad \int_{\lambda_{UV}}^{\lambda_{IR}} \frac{d \lambda}{\lambda^2} = -10 \int_{b_{UV}}^{b_{IR}} \frac{db}{b} \quad \Rightarrow \quad -\frac{1}{\lambda} \bigg|_{\lambda_{UV}}^{\lambda_{IR}} = -10 \log \frac{b_{IR}}{b_{UV}}
\] (3.54)

\[
\Rightarrow \quad \frac{\lambda(\Lambda_{UV})}{1 - 10\lambda(\Lambda_{IR}) \log \left( \frac{\Lambda_{UV}}{\Lambda_{IR}} \right)}
\] (3.56)

- Here, \( \Lambda_{UV} = \Lambda \) is the ultraviolet cutoff, eq. (3.4) which defines the highest momentum-energy scale in the system.
- \( \Lambda_{IR} \) is infrared momentum scale down to which we integrated out the momentum modes.
- Now, suppose, we have a collider experiment at some momentum or energy scale \( \Lambda_{IR} \) where we measure the scattering of particles.
- Note that while the experiment operates at relatively high energies at the collider, these energies ~ \( \Lambda_{IR} \) can be considered low as compared to a ultraviolet scale \( \Lambda_{UV} \) which defines the highest energies/momenta where the model is assumed to be valid.
- The experiment carried out at energies ~ \( \Lambda_{IR} \) shall allow us to fix the interaction between particles \( \lambda_{IR} = \lambda(\Lambda_{IR}) \) defined at that scale.
- this allows us to interpret eq. (3.56): It gives us the value of the interaction \( \lambda(\Lambda_{UV}) \) at scale \( \Lambda_{UV} \) if at scale \( \Lambda_{IR} \) the value \( \lambda_{IR} \) was measured.
- eq. (3.56) has a singularity – the Landau pole singularity – at scale: \( \Lambda_L = \Lambda_{IR} e^{\frac{1}{10\lambda_{IR}}} \).
3. Wilson’s renormalization group

- the presence of the Landau pole indicates that the interaction coupling $\lambda$ grows strong at large momentum scales
  $\Rightarrow$ as we approach the Landau pole our perturbative approach is not reliable anymore.

New physics

- if Landau pole exists beyond perturbation theory $\rightarrow$ theory has limited range of validity
  $\rightarrow$ theory cannot be considered as being a fundamental theory
  $\rightarrow$ we cannot extend it beyond $\Lambda_L$
  $\rightarrow$ new physics is expected to appear, e.g., new degrees of freedom, unknown particles, non-perturbative physics

Triviality problem

- the Landau-pole singularity can also be put in different terms:
  - demand that our theory must be valid on all scales (i.e. also for $\Lambda_{UV} \rightarrow \infty$)
  - this means, we have to send the Landau pole to infinity $\Lambda_L \rightarrow \infty$
  - this requires to have $\lambda_{IR} \rightarrow 0$.
  - once $\lambda_{IR} = 0$ on one scale it remains zero on all scales
  - therefore the theory becomes non-interacting, i.e. trivial on all scales.
  - therefore the Landau pole problem is also called the triviality problem!

Remarks

- A Landau pole also appears in another sector of the Standard Model: Quantum electrodynamics (QED) has a beta function $\frac{d\alpha}{d\log b} \sim -\frac{e^3}{12\pi^2}$ where $e$ is the EM coupling.
- A more elaborate RG analysis predicts: QED-Landau pole is expected to appear at huge ultraviolet scales $\Lambda_{QED} \gg \Lambda_{Planck} \sim 10^{19}$ GeV, where the Planck mass is defined as $m_{Planck} = \sqrt{\hbar c/G}$ with $G$ being the gravitational constant. $m_{Planck}$ defines the scale where quantum gravity effects are expected to occur and a new theory beyond the Standard Model is needed which includes gravity. Therefore, QED-Landau pole is not considered as a problem of practical importance within the Standard Model.
- Further analysis with the Standard Model parameters suggests that also Landau-pole in the Higgs sector only appears beyond $m_{Planck}$. Due to fluctuations of the top quark, however, another problem below $m_{Planck}$ is sometimes discussed: The quartic coupling $\lambda$ in the Higgs sector might drop below zero and render the Higgs potential unbounded from below or induce a second (deeper) minimum with a possibly finite decay rate from the present metastable state. This is often referred to as vacuum instability.
- In case the sign of the beta function is different from the scenarios discussed in this section (e.g. $\phi^3$ theory in $d = 6$ or QCD), the coupling drops to zero for large momentum scales. This behavior is typically referred to as asymptotic freedom (problem set 4). The Landau pole then appears in the infrared and indicates, for example, a change of degrees of freedom, as the system becomes strongly coupled.
3.6.2 Fine-tuning and the hierarchy problem

The second problem which we discuss here and which appears in the Standard Model is the so-called hierarchy problem or naturalness problem. It is related to the beta function for the term which is bilinear in the bosonic fields, eq. (3.51):

$$\beta_\mu = 2\mu(b) - 4\lambda(b)\mu(b).$$  \hspace{1cm} (3.57)

- for simplicity, we assume that we are in the perturbative regime, where \( \lambda \ll 1 \) is small (sufficiently far away from the Landau pole).

\[ \Rightarrow \text{the leading behavior of } \mu(b) \text{ is given by the simplified equation:} \]

$$b \frac{d\mu(b)}{db} = 2\mu(b)$$  \hspace{1cm} (3.58)

- this equation is easily integrated by separation of variables and gives:

$$\mu(\Lambda_{\text{IR}}) = \mu(\Lambda_{\text{UV}}) \left( \frac{\Lambda_{\text{UV}}}{\Lambda_{\text{IR}}} \right)^2$$  \hspace{1cm} (3.59)

- Note that for a precise calculation the \( b \)-dependence of \( \lambda(b) \) should be taken into account.

- Generally, for the Standard Model, there will be further couplings contributing to eq. (3.51), most importantly the coupling to the top quarks (fermions).

- Still, the most important contribution to the beta function of \( \mu \) is the dimensional term \( \propto 2\mu(b) \) which we already determined in Sec. 3.2.1 from zeroth order perturbation theory (i.e. a purely dimensional argument).

- The relation in eq. (3.59) implies that an extreme fine tuning has to be performed on \( \mu(\Lambda_{\text{UV}}) \), when \( \Lambda_{\text{UV}} \gg \Lambda_{\text{IR}} \) and at \( \Lambda_{\text{IR}} \), the parameter \( \mu(\Lambda_{\text{IR}}) \) has to match an experimentally determined (physical) value.

- Let’s make a numerical example:

  - suppose we make a particle collider experiment at energy scale \( \Lambda_{\text{IR}} = 173 \text{GeV} \) (which is at the scale of the mass of the top quark)
  - suppose a measurement allows us to fix the parameter \( \sqrt{\mu(\Lambda_{\text{IR}})} = 125 \text{GeV} \) (which suggests that this parameter can be connected to the Higgs mass!)
  - So far, no direct evidence for physics beyond the Standard Model has been found in particle colliders
    \[ \Rightarrow \text{assume that Standard Model is a valid theory also on much higher scales} \]
    \[ \Rightarrow \text{some ideas suggest a grand unified theory (GUT) at } \Lambda_{\text{UV}} \sim 10^{15} \text{GeV}. \]
  - With these numbers, we conclude that we have to choose a parameter
    $$\mu(\Lambda_{\text{UV}}) = 4.67640625 \cdot 10^{-22} \text{GeV}^2$$  \hspace{1cm} (3.60)

- in the Standard Model this seems unnatural, because there is no “knob” which can be used to adjust this value (hence naturalness problem). Still, for some reason the Standard Model seems to be a system which is tuned close to criticality.

- in the interacting boson system, where we observe critical behavior, this tuning seems to be “natural” as we have to adjust the temperature \( T \) very close to its critical value \( T_c \).
Figure 3.2: Sketch of the hierarchy problem.
4.1 Introductory remarks

In the previous chapter, we have seen that systems in statistical mechanics, condensed-matter physics as well as in particle physics can be described in terms of a functional integral which in short-hand notation can be written as

\[ Z = \int \mathcal{D}\Phi e^{-S[\Phi]}, \]  

with the action \( S[\Phi] \). We have given an example for the action \( S[\Phi] \) in terms of a complex scalar field which finds applications in the superfluid transition of non-relativistic interacting bosons as well as in the Higgs sector of the Standard Model of Particle Physics. More generally, there can be other action functionals \( S[\Phi] \) with different numbers of boson components, fermion fields, gauge fields, other symmetries,... , i.e. the field variable \( \Phi \) collects all relevant degrees of freedom.

Therefore, a profound understanding of quantum and statistical field theories requires a sophisticated toolbox of theoretical methods. Functional methods are suitable for the computation of generating functionals for correlation functions. The generating functionals contain all relevant physical information about a theory. In this chapter and the next one, we will introduce some of these functional methods which are applicable to an extremely broad range of physical systems. Further, these functional methods also provide a deeper conceptual understanding of quantum and statistical field theories. In particular, we will

- introduce generating functionals for more efficient calculations
- analyse divergencies in perturbation theory
- give a classification of perturbatively renormalizable theories
- give an inductive proof of perturbative renormalizability
- introduce an exact renormalization group equation
- learn how to apply functional renormalization
4. Functional methods in quantum field theory

4.2 Generating functional

In quantum and statistical field theory all physical information is stored in correlation functions, e.g., the two-point correlation function from eq. (2.26) or the susceptibility eq. (2.27). In a particle collider experiment with two incident beams and \((N - 2)\) scattering products, this process can be described by the \(N\)-point correlation function. In the following, we will refer to quantum or statistical field theory just shortly as QFT.

- in Euclidean QFT the correlation functions are defined as
  \[
  \langle \Phi(x_1)\ldots\Phi(x_n) \rangle := N \int \mathcal{D}\Phi \Phi(x_1)\ldots\Phi(x_n)e^{-S[\Phi]},
  \]
  where \(x_i\) are suitable coordinates in position or momentum space.
- we fix the normalization \(N\) such that \(\langle 1 \rangle = 1\).
- functional integral measure includes implicit regularization with ultraviolet cutoff \(\Lambda\)
  \[
  \int \mathcal{D}\Phi \rightarrow \int_{\Lambda} \mathcal{D}\Phi.
  \]
- the regularized measure shall preserve the symmetries of the theory.
- in case we have to deal with Minkowski-valued correlators, we assume that these can be defined from Euclidean ones by analytic continuation.
- with minor modifications, the following discussion also works for fermions.

All \(n\)-point correlation functions are summarized by the generating functional:

\[
Z[J] = \int \mathcal{D}\Phi e^{-S[\Phi]+\Phi\cdot J},
\]
with the source term \(\Phi \cdot J = \int d^d x \Phi(x)J(x)\).

- Recall that at finite temperature, we had \(Z[0] = \exp[-\beta F]\), where \(F\) is the free energy.

4.2.1 Correlation functions

- the functional derivative acts as:
  \[
  \delta \Phi(x) \over \delta \Phi(y) = \delta^d(x - y), \quad \text{or} \quad \delta J(x) \over \delta J(y) = \delta^d(x - y).
  \]

- therefore, we obtain
  \[
  \frac{\delta}{\delta J(x)} e^{\int d^d y J(y)\Phi(y)} = \Phi(x)e^{\int d^d y J(y)\Phi(y)}
  \]

  \[
  \Rightarrow \langle \Phi(x_1)\ldots\Phi(x_n) \rangle = \left( \frac{1}{Z[J]} \frac{\delta}{\delta J(x_1)} \ldots \frac{\delta}{\delta J(x_n)} Z[J] \right) \bigg|_{J=0}
  \]

  \[
  \Rightarrow Z[J] \text{ contains full information about the theory.}
  \]

- Definition: \(n\)-point correlator in the presence of a source \(J\):
  \[
  \langle \Phi(x_1)\ldots\Phi(x_n) \rangle_J = \frac{1}{Z[J]} \int \mathcal{D}\Phi \Phi(x_1)\ldots\Phi(x_n)e^{-S[\Phi]+\Phi\cdot J}.
  \]
4.2. Generating functional

What does $Z[J]$ generate?

- $Z[J]$ is the generating functional and generates the correlation functions $\langle \Phi(x_1)\ldots\Phi(x_n) \rangle$

- For example, the two-field Green’s function can be pictorially represented as

$$\langle \Phi(x_1)\Phi(x_2) \rangle = x_2 x_1$$ (4.9)

- Observation: Perturbative calculation of the four-field Green’s function

$$\langle \Phi(x_1)\Phi(x_2)\Phi(x_3)\Phi(x_4) \rangle$$ (4.10)

(e.g. using the Wick theorem) contains disconnected contributions of the form:

$$\langle \Phi(x_1)\Phi(x_2) \rangle \langle \Phi(x_3)\Phi(x_4) \rangle = x_4 x_3 x_2 x_1$$ (4.11)

- these contributions can be expressed in terms of the exact two-field Green’s function

$$\langle \Phi(x_1)\Phi(x_2) \rangle = x_2 x_1$$ (4.12)

⇒ they do not exclusively contain information about correlations involving four fields. ⇒ we carry along redundant information!

- for more details: see App. A

Dyson-Schwinger equations

- the quantum equations of motion follow from the translational invariance of $D\Phi$:

$$\int D\Phi \frac{\delta}{\delta \Phi(x)} \left( e^{-S[\Phi]+\Phi \cdot J} \right) = 0$$ (4.13)

⇒ $\int D\Phi \left( J(x) - \frac{\delta S}{\delta \Phi(x)} \right) e^{-S[\Phi]+\Phi \cdot J} = 0$ (4.14)

⇒ $\left[ J(x) - \frac{\delta S}{\delta \Phi(x)} \right]_J = 0$ (4.15)

- These are the Dyson-Schwinger equations (DSE). They can also be written as

$$\left( J(x) - \frac{\delta S}{\delta \Phi(x)} \left[ \frac{\delta}{\delta J} \right] \right) Z[J] = 0$$ (4.16)

- The DSEs provide an infinite hierarchy of equations relating the $\Phi$-field correlation functions with each other

  - for example, 2-point correlation function can be written in terms of 4-point correlation function (problem set 5)
4. Functional methods in quantum field theory

4.2.2 Schwinger functional

A more efficient way to store the information about the correlation functions is given by the Schwinger functional:

\[ W[J] = \log Z[J] = \log \int \mathcal{D}\Phi e^{-S[\Phi] + J \cdot \Phi} \]  

- Schwinger functional is generating functional for the connected Green's functions.

- Observation: \( W[J] \) contains diagrams of the form:

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{diagram.png}
\end{array}
\]

\( \Rightarrow \) \( W[J] \) still carries redundant information in particular cases.

- we would like to have a generating functional for the one-particle-irreducible (1PI) Green's functions.

4.3 Effective action

We define the classical field:

\[
\varphi(x) := \langle \Phi \rangle_J = \frac{1}{Z[J]} \int \mathcal{D}\Phi \Phi(x) e^{-S[\Phi] + J \cdot \Phi} = \frac{\delta W}{\delta J(x)} \]

\[ \Rightarrow \varphi = \varphi[J] \]  

- invert \( \varphi[J] \) and compute \( J[\varphi] \).

We define the effective action \( \Gamma \) as the Legendre transform of \( W[J] \):

\[
\Gamma[\varphi] = \sup_J (J \cdot \varphi - W[J]) \]

- for any given \( \varphi \) a special \( J = J_{\text{sup}} \) is singled out for which \( J \cdot \varphi - W[J] \) approaches its supremum (\( J = J[\varphi] \)).

- this definition of \( \Gamma \) guarantees that \( \Gamma \) is convex.

- at \( J = J_{\text{sup}} \):

\[
\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = J(x) + \int_y \frac{\delta J[y]}{\delta \varphi(x)} \varphi(y) - \int_y \frac{\delta W[J]}{\delta J[y]} \frac{\delta J[y]}{\delta \varphi(x)} = J(x) \] 

use eq. (4.20)

\[ \Rightarrow \frac{\delta \Gamma[\varphi]}{\delta \varphi} = J \]  

- eq. (4.21) is the quantum equation of motion (all quantum/statistical effects included).

- for comparison, the classical field equation is \( \frac{\delta S}{\delta \Phi(x)} = 0 \).
4.3. Effective action

**Functional integro-differential equation for effective action**

\[ e^{-\Gamma[\varphi]} + J \cdot \varphi = e^{W[J]} = \int \mathcal{D} \Phi e^{-S[\Phi]} + J \cdot \Phi \]  
(4.23)

\[ e^{-\Gamma[\varphi]} = \int \mathcal{D} \Phi e^{-S[\Phi]} \frac{\delta \Gamma[\varphi]}{\delta \varphi} + J \cdot \varphi \]  
(4.24)

\[ e^{-\Gamma[\varphi]} \xrightarrow{\Phi \to \Phi + \varphi} \int \mathcal{D} \Phi e^{-S[\Phi + \varphi]} + \int \frac{\delta \Gamma[\varphi]}{\delta \varphi} \Phi \]  
(4.25)

- functional integro-differential equation
- exact determination of \( \Gamma[\varphi] \) available only for rare special cases.
- a solution of eq. (4.25) can be attempted by a *vertex expansion*:

\[ \Gamma[\varphi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{x_1, \ldots, x_n} \Gamma^{(n)}(x_1, \ldots, x_n) \varphi(x_1) \ldots \varphi(x_n) \]

\[ = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{p_1, \ldots, p_n} \delta(p_1 + p_2 + \ldots + p_n) \Gamma^{(n)}(p_1, \ldots, p_n) \varphi(p_1) \ldots \varphi(p_n) \]  
(4.26)

- the expansion coefficients \( \Gamma^{(n)} \) are the *one-particle irreducible (1PI) proper vertices*.
- inserting eq. (4.26) into eq. (4.25) and comparing coefficients of the field monomials in an infinite tower of coupled integro-differential equations for the \( \Gamma^{(n)} \):

*Dyson-Schwinger equations (DSE)*

- functional method of constructing approximate solutions by truncated (= finite ansatz for the series in eq. (4.26)) DSEs has highly developed applications in gauge theories.
- \( n \)-point correlation functions can be reconstructed from the \( \Gamma^{(n)} \).
- in particular the analysis of divergencies is simpler using the \( \Gamma^{(n)} \).

**Inverse propagator**

We consider the two times functional derivative of the effective action with respect to the classical fields:

\[ \Gamma^{(2)}(x, y) = \frac{\delta^2 \Gamma}{\delta \varphi(x) \delta \varphi(y)} \]  
(4.27)

we find \( \Gamma^{(2)} = G^{-1} \) with \( G(x, y) = \frac{\delta^2 W}{\delta J(x) \delta J(y)} = W^{(2)}(x, y) \)

\[ \Rightarrow \Gamma^{(2)} W^{(2)} = 1 \]  
(4.28)

or more explicitly

\[ \int \frac{d^dq''}{(2\pi)^d} \Gamma^{(2)}(q, q'') W^{(2)}(q'', q') = (2\pi)^d \delta^{(d)}(q - q'). \]  
(4.29)
4. Functional methods in quantum field theory

Perturbation theory for $\Gamma[\varphi]$

From eq. (4.25) we obtain:

$$\Gamma[\varphi] = -\log \int_{\Lambda} \mathcal{D}\Phi e^{-S[\Phi + \varphi] + \int_x \frac{\delta \Gamma[\varphi]}{\delta \Phi(x)} \Phi(x)}$$  \hspace{1cm} (4.30)

with

- classical/background field $\varphi$
- fluctuation field $\Phi$

We can rewrite this as an implicit equation

$$\Gamma[\varphi] = S[\varphi] + \Gamma_l[\varphi]$$  \hspace{1cm} (4.31)

with

$$\Gamma_l[\varphi] = -\log \int_{\Lambda} \mathcal{D}\Phi e^{-\left(S[\Phi + \varphi] - S[\varphi]\right) + \int_x \frac{\delta \Gamma[\varphi]}{\delta \Phi(x)} \Phi(x)}$$  \hspace{1cm} (4.32)

Now, we make a saddle-point approximation (expand around $\Phi = 0$)

$$S[\varphi + \Phi] = S[\varphi] + \int_x \frac{\delta S}{\delta \varphi(x)} \Phi(x) + \frac{1}{2} \int_{x,y} \frac{\delta^2 S}{\delta \varphi(x) \delta \varphi(y)} \Phi(x)\Phi(y) + ...$$  \hspace{1cm} (4.33)

$$\Rightarrow \Gamma_l[\varphi] = -\log \int_{\Lambda} \mathcal{D}\Phi \exp \left\{ -\frac{1}{2} \int_{x,y} S^{(2)}(x,y) \Phi(x)\Phi(y) + \int_x \frac{\delta \Gamma_l}{\delta \varphi(x)} \Phi(x) + O(S^{(3)}) \right\}$$  \hspace{1cm} (4.34)

neglecting part marked in the above equation gives the 1-loop contribution to effective action:

$$\Gamma_l[\varphi] = -\log \left( \det_{\Lambda} S^{(2)} \right)^{-\frac{1}{2}}$$  \hspace{1cm} (4.35)

$$\Rightarrow \Gamma_l[\varphi] = \frac{1}{2} \text{Tr} \log S^{(2)}[\varphi]$$  \hspace{1cm} (4.36)

now use $\log \det A = \text{Tr} \log A$

$$\Rightarrow \Gamma_l[\varphi] = \frac{1}{2} \text{Tr} \log S^{(2)}[\varphi]$$  \hspace{1cm} (4.37)

This provides us with important expression for perturbation theory for $\Gamma[\varphi]$ at one-loop order:

$$\Gamma[\varphi] = S[\varphi] + \frac{1}{2} \text{Tr} \log S^{(2)}[\varphi]$$  \hspace{1cm} (4.38)

In momentum space, the one-loop integral explicitly reads:

$$\Gamma_l = \frac{1}{2} \int \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{d^d \mathbf{q}'}{(2\pi)^d} (2\pi)^d \delta^{(d)}(\mathbf{q} - \mathbf{q}') \left( \log S^{(2)}[\varphi] \right)(\mathbf{q}, \mathbf{q}')$$  \hspace{1cm} (4.39)
4.4 Perturbative renormalization

Here, we want to understand more systematically how renormalization in perturbation theory works. This will lead us to a classification of perturbatively renormalizable theories.

4.4.1 Divergencies in perturbation theory

Example: $\phi^3$ theory on the 1-loop level. The corresponding action can be written as:

\begin{equation}
S[\varphi] = \int d^d x \left\{ \frac{1}{2} (\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 + \frac{1}{3!} g \varphi^3 \right\}
\tag{4.40}
\end{equation}

- Remark: this theory is unphysical as the potential is not bounded from below:

\begin{equation}
\varphi
\end{equation}

- However, there is a well-defined perturbative expansion and further the vertex structure appears in many physically relevant theories.

\begin{equation}
S^{(2)}[\varphi] = -\partial^2 + m^2 + g \varphi(x)
\tag{4.41}
\end{equation}

\begin{equation}
\Rightarrow \Gamma_{1l}[\varphi] = \frac{1}{2} \text{Tr} \log (-\partial^2 + m^2 + g \varphi(x))
\tag{4.42}
\end{equation}

\begin{equation}
= \frac{1}{2} \text{Tr} \left[ \log \left( 1 + \frac{1}{-\partial^2 + m^2 g \varphi(x)} \right) \right] + \text{const.}
\tag{4.43}
\end{equation}

\begin{equation}
= -\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left[ \left( \frac{-g}{-\partial^2 + m^2 \varphi(x)} \right)^n \right] + \text{const.}
\tag{4.44}
\end{equation}

- after Fourier transformation to momentum space, we find the proper vertices:

\begin{equation}
\Gamma^{(n)}_{1\text{-loop}} = -\frac{(n-1)!}{2} (-g)^n \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} \frac{1}{(q + p_1)^2 + m^2} \cdots \frac{1}{(q + p_1 + \cdots + p_{n-1})^2 + m^2}
\tag{4.45}
\end{equation}

- Feynman diagram for eq. (4.45) (1-loop 1PI diagram):
4. Functional methods in quantum field theory

- ultraviolet (= large momentum) behavior of the loop-integration for $q^2 \gg p_1^2, ..., p_n^2, m^2$:

$$\Gamma^{(n)}_{\text{1-loop}} \sim \int d^d q q^{-2n}$$

therefore, it diverges for $d \geq 2n$.

- more explicitly:
  - $d = 1$: all correlation functions are finite (quantum mechanics)
  - $d = 2$: 1-point function diverges (tadpole):

$$\Gamma^{(1)}_{\text{1-loop}} = \frac{g}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2}$$

- ...
  - $d = 6$: 1-, 2-, and 3-point functions diverge

- Regularization:
  - cut the momentum integral by integrating in a sphere $|q| < \Lambda$ with $\Lambda^2 \gg p_1^2, ..., p_n^2, m^2$ → cutoff regularization, see also Eq. (3.4)

Now, focus on the case $d = 6$: Determination of divergencies by expansion in $p_1^2, ..., p_n^2, m^2$:

$$\Gamma^{(1)}_{\text{1-loop}} = \frac{g}{2} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + m^2} = \frac{g}{2} \int \frac{d^d q}{(2\pi)^d} \left( \frac{1}{q^2} - \frac{m^2}{q^4} + \frac{m^4}{q^6} + O(\frac{1}{q^8}) \right)$$

where we employed a Taylor series expansion of the integrand in the external momenta and the mass $p_1^2, ..., p_n^2, m^2$. Further, we use the spherical integral:

$$\int_{\Lambda} d^6 q = V_{S^5} \int_{\Lambda} dq q^5$$

where $V_{S^5} = \pi^3$. Therefore, we get:

$$\Gamma^{(1)}_{\text{1-loop}} = \frac{g}{2 m^3} \left[ \frac{\Lambda^4}{4} - \frac{m^2 \Lambda^2}{2} + m^4 \log \frac{\Lambda}{m} + O(1) \right]$$

- The log-term contains an infrared divergence

- therefore, we have introduced an infrared cutoff at $m$

- Here, we will focus on the analysis of the ultraviolet divergencies

- similarly, we obtain:

$$\Gamma^{(2)}_{\text{1-loop}} = \frac{g^2}{2 \pi^3} \left[ \frac{\Lambda^2}{2} + (2m^2 + p^2) \log \frac{\Lambda}{m} + O(1) \right]$$

$$\Gamma^{(3)}_{\text{1-loop}} = \frac{g^3}{2 \pi^3} \log \frac{\Lambda}{m} + O(1)$$
why $d = 6$?

- $d = 6$ is special as divergencies appear only in vertices that maximally are part of the original action:

$$S[\varphi] = \int \Gamma_{0-\text{loop}}^{(1)} \varphi + \frac{1}{2} \int \Gamma_{0-\text{loop}}^{(2)} \varphi^2 + \frac{1}{3!} \int \Gamma_{0-\text{loop}}^{(3)} \varphi^3 \quad (4.53)$$

- here we have introduced the tree level or bare vertices:

$$\Gamma_{0-\text{loop}}^{(1)} = 0 \quad (4.54)$$

$$\Gamma_{0-\text{loop}}^{(2)} = p^2 + m^2 \quad (4.55)$$

$$\Gamma_{0-\text{loop}}^{(3)}(p_1, p_2, -(p_1 + p_2)) = g \quad (4.56)$$

- note that $\Gamma_{0-\text{loop}}^{(1)}$ can be generated by a shift $\varphi \to \varphi + \text{const.}$

- furthermore, the dependence of divergencies on the external momenta equals the momentum dependence of the bare vertices $\Gamma_{0-\text{loop}}^{(n)}$.

- on the contrary in $d = 8$: $\Gamma_{0-\text{loop}}^{(4)} = 0$ and $\Gamma_{1-\text{loop}}^{(4)}$ diverges
4.4.2 Removal of divergencies

Divergent part of 1-loop 1PI functional in $d = 6$:

$$
\Gamma_{1-\text{loop,div}} = \int d^6x \left[ ga_1(\Lambda)\varphi(x) + \frac{1}{2}g^2a_2(\Lambda)(\partial_\mu\varphi)^2(x) + \frac{1}{2}g^2a_3(\Lambda)\varphi^2(x) + \frac{1}{3!}g^3a_4(\Lambda)\varphi^3(x) \right]
$$

(4.57)

where from eqs. (4.50), (4.51) and (4.52) we have:

$$
a_1(\Lambda) = \frac{1}{2^7\pi^3} \left[ \frac{\Lambda^4}{4} - \frac{m^2\Lambda^2}{2} + m^4 \log \frac{\Lambda}{m} \right]
$$

(4.58)

$$
a_2(\Lambda) = \frac{1}{3 \cdot 2^7\pi^3} \log \frac{\Lambda}{m}
$$

(4.59)

$$
a_3(\Lambda) = \frac{1}{2^7\pi^3} \left[ -\frac{\Lambda^2}{2} + 2m^2 \log \frac{\Lambda}{m} \right]
$$

(4.60)

$$
a_4(\Lambda) = \frac{1}{2^6\pi^3} \log \frac{\Lambda}{m}
$$

(4.61)

*Note*: the origin of these divergencies is that fluctuations on all scales contribute to the vertex functions.

- **Physical problem**: How do the low-energy observables (susceptibility, magnetization, coupling, scattering amplitude, particle masses), that are given by the full vertex functions, relate to the parameters in the bare action $g, m^2, \varphi$ that describe the high-energy parameters of a microscopic theory at scale $\Lambda$?

- **Idea**: Fix the coupling, mass, chemical potential, etc. at a renormalization point/scale $\mu$ (scale of measurement) to the (physical) values $m_R, g_R, \varphi_R$ by renormalization conditions for the full vertices:

The renormalized physical parameters are fixed by means of the effective action:

$$
\Gamma^{(2)}(p^2 \rightarrow 0) = m_R^2 \quad \text{(fixes physical mass)}
$$

(4.62)

$$
\frac{\partial}{\partial p^2} \Gamma^{(2)}(p^2 \rightarrow 0) = 1 \quad \text{(fixes normalization of field)}
$$

(4.63)

$$
\Gamma^{(3)}(p_1^2 = p_2^2 = p_3^2 = 0) = g_R \quad \text{(fixes physical coupling)}
$$

(4.64)

- This makes sure that the renormalized parameters coincide with the measurements from a (low-energy) experiment.

- **More general**: fix parameters at $p^2 \rightarrow \mu^2$, where $\mu$ is the renormalization point/scale

$$
\Gamma^{(2)}(p^2 = \mu^2) = m^2 \neq m_R^2(p = 0)
$$

(4.65)

- **Note**:
  - $m_R$ and $g_R$ are real physical input
  - eq. (4.63) is a normalization prescription. Finite rescaling is still permitted.
Renormalization in perturbation theory:

Expansion in the renormalized coupling $g_R$:

$$S[\varphi] \equiv S_\Lambda[\varphi] = \int \left[ \frac{1}{2} \varphi (-\partial^2 + m^2) \varphi + \frac{1}{3!} g \varphi^3 \right]$$

$$= S_R[\varphi_R] + \delta S_R[\varphi_R]$$

$$= \int \left[ \frac{1}{2} \varphi_R (-\partial^2 + m_R^2) \varphi_R + \frac{1}{3!} g_R \varphi_R^3 \right.$$  

$$+ \frac{1}{2} (Z_\varphi - 1)(\partial_{\mu} \varphi_R)^2 + \frac{1}{2} \delta m^2 \varphi_R^2 + \frac{1}{3!} g_R (Z_g - 1) \varphi_R^3 \left.] \right\} \text{counterterms}$$

with

$$\varphi = Z_{\varphi}^{1/2} \varphi_R$$

$$g = g_R \frac{Z_g}{Z_{\varphi}^{1/2}}$$

$$m^2 = (m_R^2 + \delta m^2) \frac{1}{Z_{\varphi}}$$

and the parametrization of the relation between bare and renormalized quantities is given by

- $Z_{\varphi}$: wave function renormalization
- $Z_g$: coupling renormalization
- $m_R, g_R, \varphi_R$: renormalized quantities

Now, assume that $Z_{\varphi}, Z_g, \delta m^2$ have an expansion in $g_R$ such that

$$(Z_{\varphi} - 1), \delta m^2, (Z_g - 1) = \mathcal{O}(g_R^2)$$

$\Rightarrow$ second line in eq. (4.69) is of the same order as 1-loop contribution, cf. Eq. (4.57)

$\Rightarrow$ 1-loop calculation has to be performed with $S_R[\varphi_R]$

$\Rightarrow$ we then have:

$$\Gamma[\varphi_R] = S_\Lambda[\varphi] + \frac{1}{2} \text{Tr} \log S_\Lambda^{(2)}[\varphi] + \mathcal{O}(2\text{-loop})$$

$$= S_R[\varphi_R] + \delta S_R[\varphi_R] + \frac{1}{2} \text{Tr} \log S_R^{(2)}[\varphi_R] + \mathcal{O}(2\text{-loop})$$

now the renormalization conditions, eqs. (4.62), fix the relations $Z_{\varphi}, Z_g, \delta m^2$:

In particular, for the divergent parts, we obtain:

$$\Gamma[\varphi_R] = \mathcal{O}(\Lambda^0)$$

$\Rightarrow \delta S_R[\varphi_R] + \frac{1}{2} \text{Tr} \log S_R^{(2)}[\varphi_R] = \mathcal{O}(\Lambda^0)$
with eq. (4.57) for perturbative calculation with $g_R$:

$$\frac{1}{2} \text{Tr} \log S_R^{(2)}[\varphi_R] = \int d^6x \left[ g_R a_1(\Lambda) \varphi_R(x) + \frac{1}{2} g_R^2 a_2(\Lambda) (\partial_\mu \varphi_R)^2(x) + \frac{1}{2} g_R^2 a_3(\Lambda) \varphi_R^2(x) + \frac{1}{3!} g_R^3 a_4(\Lambda) \varphi_R^3(x) \right]$$ (4.78)

and $a_i$ from eq. (4.58), we obtain:

$$Z_\varphi = 1 - g_R^2 a_2(\Lambda)$$ (4.79)
$$Z_g = 1 - g_R^2 a_4(\Lambda)$$ (4.80)
$$\delta m^2 = g_R^2 a_3(\Lambda)$$ (4.81)

$\Rightarrow$ the assumption eq. (4.73) is selfconsistent

$\Rightarrow \Gamma[\varphi_R]$ is finite (on a 1-loop level) and only depends on physical parameters!

**Interpretation:**

The *original* parameters in $S[\varphi]$ ($m^2, g, ...$) are *not* the ones measured in a (low-energy) experiment but they are connected to the latter ones by transformations.

This means the original (= *bare*) parameters *are functions of the cutoff*:

$$m = m(\Lambda), \quad g = g(\Lambda)$$ (4.82)

- This tuning of the bare parameters does *not* increase the number of parameters in the $\varphi^3$ theory for $d \leq 6$.
- For example: in $d = 8$, the vertex $\Gamma^{(4)}$ is also divergent, so $\delta S$ has to include a $\varphi^4$ term (+ 1 parameter!). This term generates further divergencies in higher orders, such that the number of parameters increases without limit.
4.4.3 Divergencies in perturbation theory: general analysis

- Is it possible to choose the counter terms \( \delta m^2, \delta Z_\phi = Z_\phi - 1 \), etc. for any theory such that the low-energy physics is finite?
- under which conditions the theory remains predictive?

**Canonical dimension of a field:**

- consider the free propagator of a field:

\[
\Delta(p) \equiv \left( \Gamma_{0}^{(2)}(p) \right)^{-1} \sim \frac{1}{|p|^\sigma}
\]

(4.83)

where we have used the assumptions that the limit \( p \to \infty \) is \( O(d) \) symmetric

- then the **canonical dimension** of a field is defined as

\[
[\phi] := \frac{d - \sigma}{2}
\]

(4.84)

- **Examples:**
  - scalar field:

\[
\Delta(p) = \frac{1}{p^2 + m^2} \sim \frac{1}{p^2} \Rightarrow [\phi] = \frac{d - 2}{2}
\]

(4.85)

- fermion field (Dirac):

\[
\Delta(p) = \frac{1}{p\mu + m} \sim \frac{1}{p} \Rightarrow [\phi] = \frac{d - 1}{2}
\]

(4.86)

- Note: In these examples, the canonical dimension can also be inferred from the action (dimensionless). This is not possible for higher spin fields.

- in theories with polynomial interactions, the interaction can be written as a sum of vertices:

\[
V(\phi) \sim \int d^d x \left( \frac{\partial}{\partial x} \right)^k \phi_1(x)^{n_1} \phi_2(x)^{n_2} ... \phi_s(x)^{n_s}
\]

(4.87)

where the \( \phi_i \) are different types of fields.

- pictorially, such a vertex can be represented as:
4. Functional methods in quantum field theory

- the *dimension of a vertex* is:

\[
\delta[V] = -d + k + \sum_{i=1}^{s} n_i |\phi_i| \tag{4.88}
\]

- vertex in momentum space

\[
V(\varphi) \sim \int d^d p_1 ... d^d p_{n_1 + n_2 + ... + n_s} \delta^{(d)}(p_1 + ... + p_{n_1 + ... + n_s}) p^k \varphi_1(p_1) ... \varphi_s(p_{n_1} + ... + p_{n_s}) \tag{4.89}
\]

⇒ vertex contains momentum-conserving \( \delta \) function

- divergencies in the \( n \)-point proper vertices \( \Gamma^{(n)} \) appear in the integrations over the fluctuation momenta. Perturbation theory sorts the contributions to \( \Gamma^{(n)} \) according to the number of independent momentum integrations (\( = \) number of loops).

**Superficial degree of divergence of a diagram \( \gamma \):**

If all integration momenta in a diagram \( \gamma \) are scaled by a factor \( \lambda \gg 1 \)

⇒ diagram is scaled by a factor \( \lambda^{\delta(\gamma)} \) with:

\[
\delta(\gamma) = dL - \sum_i I_i \sigma_i + \sum_{\alpha} v_{\alpha} k_{\alpha} \tag{4.90}
\]

where

- \( L \): number of loops
- \( I_i \): number of internal lines
- \( v_{\alpha} \): number of vertices of type \( \alpha \)
- \( k_{\alpha} \): number of momenta/derivatives

Then we can distinguish three different cases:

1. if \( \delta(\gamma) > 0 \) a regularized diagram diverges at least like \( \Lambda^{\delta(\gamma)} \).
2. if \( \delta(\gamma) = 0 \) a diagram diverges at least like \( \log \Lambda \).
3. if \( \delta(\gamma) < 0 \) a diagram is *superficially* convergent (divergencies can only come from subdiagrams).
• Example ($\varphi^3$ theory):
  
  – here, we have $\sigma = 2$, $k = 0$ \implies $\delta(\gamma) = dL - 2I$
  
  – in $d = 6$ and $L = 1$ (1-loop level): $\delta(\gamma) = 6 - 2I$
  
  \implies for $I = 1, 2, 3 \longrightarrow \delta(\gamma) = 4, 2, 0$

  ![Diagram](image)

• Different expression of $\delta(\gamma)$ by topological relations:
  
  – be $E_i$ the number of external lines of a field $\varphi_i$
  
  – we find

  $$
  \delta(\gamma) = d - \sum_i [\varphi_i] E_i + \sum_\alpha v_\alpha \delta(V_\alpha)
  $$

  (4.91)

  – with this equation, we can give a

  *classification of (perturbatively) renormalizable theories...*
4.4.4 Classification of perturbatively renormalizable theories

Suppose an analysis of the superficial degree of divergence is sufficient to get the information about all possible divergencies. This is the statement of the BPHZ theorem (Bogoliubov, Parasiuk, Hepp, Zimmermann):

*All divergencies of a perturbatively renormalizable theory can be removed by counter terms that correspond to superficially divergent amplitudes. (very technical proof!)*

In the following: \([\phi_i] \geq 0\) then a theory is:

- **Non-renormalizable**, if at least one vertex has a positive dimension \(\delta(V_\alpha)\).
  
  By considering diagrams with increasing number \(v\) of vertices (higher loops) of this type, we can make the degree of divergence arbitrarily large and this for any 1PI correlation function \(\Gamma^{(n)}\) (any number of external legs \(E_i\))
  
  \[ \Rightarrow \text{the removal of divergencies then requires infinitely many counter terms i.e. } \infty\text{-many physical parameters} \]
  
  \[ \Rightarrow \text{Loss of predictivity!} \]

- **Super-renormalizable**, if all \(\delta(V_\alpha) < 0\)
  
  \[ \Rightarrow \text{only a finite number of diagrams are superficially divergent.} \]

  Example (\(\phi^4\) theory in \(d = 3\)):

  \[ \delta(V) = -3 + 0 + 4 \cdot \frac{1}{2} = -1 \quad (4.92) \]

  \[ \Rightarrow \delta(\gamma) = 3 - \frac{1}{2}E - v \quad (4.93) \]

  which gives the divergent diagrams:

  - \(E = 2\):

    \[
    \begin{array}{c}
    \text{v = 1} \\
    \text{v = 2} \\
    \text{v = 2}
    \end{array}
    \]

  - \(E = 4\) has at least \(v = 2\) \[ \Rightarrow \delta(\gamma)|_{E \geq 2} < 0. \]

- **Renormalizable**, if at least one \(\delta(V_\alpha) = 0\) (and there is no \(\delta(V_\alpha) > 0\)).
  
  \[ \Rightarrow \infty\text{-many diagrams have }\delta(\gamma) > 0, \text{ however, at fixed } E_i: \delta(\gamma) \text{ independent from } v_\alpha \]

  \[ \Rightarrow \text{only a finite number of } \Gamma^{(n)} \text{ is divergent} \]

  \[ \Rightarrow \text{only a finite number of counter terms is required} \]

  \[ \Rightarrow \text{finite number of physical parameters} \]

  \[ \Rightarrow \text{Theory is predictive!} \]
4.4. Perturbative renormalization

Examples:

\[ \delta(V) = -d + k + \sum_{i=1}^{s} n_i |\varphi_i| \] (4.94)

1. scalar fields without derivative interactions \((k = 0)\):

\[ \varphi = \frac{d - 2}{2} \Rightarrow \delta(V) = -d + n \frac{d - 2}{2} = d \left( \frac{n}{2} - 1 \right) - n \] (4.95)

- \(\varphi^3\) is renormalizable in \(d = 6\)
- \(\varphi^4\) is renormalizable in \(d = 4\)
- \(\varphi^6\) is renormalizable in \(d = 3\)

2. spin-1/2, \([\Psi] = \frac{d - 1}{2}\) (Dirac field)

- \((\bar{\Psi}\Psi)^2\) in \(d = 2\) (Gross-Neveu model)
- \(\varphi\bar{\Psi}\Psi\) in \(d = 4\) (Yukawa model)
- \(\bar{\Psi}\Psi^2\) in \(d = 3\)

3. spin-1 (no gauge theories): \([V_\mu] = \frac{d}{2}\)

- renormalizable only in \(d = 2\), \((S_{\text{kin}} \sim \int \partial V \partial V)\), e.g., with \(\bar{\Psi}\gamma_\mu V_\mu \Psi\) interaction

4. spin \(\geq 3/2\): no perturbatively renormalizable theories.

5. gauge theories, spin 1: \([A_\mu] = \frac{d - 2}{2}\)

- \(F_{\mu\nu}^a F_{\mu\nu}^a\) in \(d = 4\) with interactions \(\bar{\Psi}\gamma_\mu A_\mu \Psi\), \(\partial_\mu \varphi A_\mu \varphi\), \(\varphi^2 A^2\) renormalizable

6. No physically acceptable (from the point of view of particle physics) and renormalizable theory exists for \(d > 4\) (perturbatively)!

It is not known whether this property is logically connected with the fact that spacetime has just four dimensions or is a mere coincidence. (from the book by J. Zinn-Justin)
Functional renormalization group

The functional renormalization group provides a nonperturbative and practical theoretical framework for quantum field theoretical calculations. More specifically, it represents the implementation of Wilson’s idea for the renormalization group with the functional methods which we have learned in the previous chapter. The central object of the functional RG is a flow equation – the Wetterich equation, which describes the evolution of correlation functions or the generating functional under successive application of momentum-shell integrations in terms of a functional differential structure.

5.1 Effective average action

We will define the effective average action $\Gamma_k$ as an action functional which depends on a momentum-shell parameter $k$ (also called "infrared cutoff scale" not to be confused with the ultraviolet cutoff $\Lambda$). $\Gamma_k$ will be defined such that it interpolates between two important limits:

1. For $k \to \Lambda$: $\Gamma_k$ corresponds to the bare action to be quantized, i.e. $\Gamma_{k \to \Lambda} \simeq S_{\text{bare}}$.

2. For $k \to 0$: $\Gamma_k$ corresponds to the full effective action, i.e. $\Gamma_{k \to 0} = \Gamma$.

To achieve this, we modify the generating functional $Z[J]$, cf. eq. (4.4) by adding a term $\Delta S_k[\Phi]$ to the action:

$$e^{W_k[J]} \equiv Z_k[J] = \int_{\Lambda} D\Phi e^{-S[\Phi]-\Delta S_k[\Phi]+J \cdot \Phi}$$

where the additional term reads

$$\Delta S_k[\Phi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \Phi(-q)R_k(q)\Phi(q).$$

It is called a regulator term which is quadratic in the field $\Phi$ (like the derivative term or the “mass” term $m^2$). Therefore, it can be interpreted as a momentum-dependent mass term.
5. **Functional renormalization group**

5.1.1 **The regulator term**

The regulator function $R_k(q)$ has to satisfy three different conditions to qualify for the desired regularization properties for the effective average action $\Gamma_k$:

1. It should implement *infrared regularization* which is realized by the condition:

$$\lim_{k^2 \to 0} R_k(q) > 0$$  \hspace{1cm} (5.3)

2. We want to recover the standard effective action $\Gamma$ when $k \to 0$, i.e. when we remove the infrared cutoff. The regulator should therefore vanish in this limit:

$$\lim_{k^2 \to 0} R_k(q) = 0$$ \hspace{1cm} (5.4)

3. The functional integral should be dominated by the stationary point of the action in the limit of large $k \to \Lambda \to \infty$. This can be achieved by the condition:

$$\lim_{k \to \Lambda \to \infty} R_k(q) \to \infty$$ \hspace{1cm} (5.5)

This filters out the classical field configuration of the action $\Gamma_{k \to \Lambda} \to S + \text{const.}$

---

![Figure 5.1: Sketch of a representative regulator function $R_k(q^2)$ (black solid line). For later purpose, we also sketch its derivative $\partial_q R_k(q^2)$ (orange dashed). For this plot, we chose $R_k(q^2) = 2k^2/(e^{(q^2/k^2)^4} + 1)^{-1}$. You can check that another suitable choice would be $R_k(q^2) = (k^2 - q^2)\Theta(k^2 - q^2)$.](image-url)
5.2 Wetterich equation

We now want to study the trajectory of the interpolating functional $\Gamma_k$ between the limits $k \to 0$ and $k \to \infty$ which we have constructed to correctly give $\Gamma$ and $S_{bare}$, respectively.

Therefore it is convenient to use the short-hand notation:

$$t = \log \frac{k}{\Lambda} \quad \partial_t = k \frac{\partial}{\partial k}.$$  \hspace{1cm} (5.6)

We keep the source term $J$ independent from $k$ and start with the derivative of the generating functional

$$\partial_t e^{W_k} = \partial_t \left( \int_{\Lambda} D\Phi e^{-S[\Phi] - \Delta S_k[\Phi] + J \cdot \Phi} \right)$$  \hspace{1cm} (5.7)

$$\Rightarrow \frac{e^{W_k}}{Z_k} \partial_t W_k = \int_{\Lambda} D\Phi (-\partial_t \Delta S_k[\Phi]) e^{-S[\Phi] - \Delta S_k[\Phi] + J \cdot \Phi}$$  \hspace{1cm} (5.8)

$$\Rightarrow \partial_t W_k = -\frac{1}{2} \int_q \left( \partial_t R_k(q) \left[ \frac{1}{Z_k} \int_{\Lambda} D\Phi \Phi(-q) \Phi(q) e^{-S[\Phi] - \Delta S_k[\Phi] + J \cdot \Phi} \right] \right)$$  \hspace{1cm} (5.9)

We now recall that the (k-modified) connected propagator is

$$G_k(q) = \frac{\delta W_k}{\delta J \delta J} (q) = \langle \Phi(-q) \Phi(q) \rangle - \varphi(-q) \varphi(q)$$  \hspace{1cm} (5.10)

where according to eq. (4.18), we have

$$\varphi(q) = \langle \Phi(q) \rangle = \frac{\delta W_k[J]}{\delta J(q)}$$  \hspace{1cm} (5.11)

and for convenience we have suppressed the index $J$ for the expection values $\langle ... \rangle_J$.

So, with eq. (5.9) we obtain

$$\partial_t W_k = -\frac{1}{2} \int_q \partial_t R_k(q) \left( G_k + \varphi(-q) \varphi(q) \right)$$  \hspace{1cm} (5.12)

$$= -\frac{1}{2} \int_q \partial_t R_k(q) G_k - \partial_t \Delta S_k[\varphi]$$  \hspace{1cm} (5.13)

Note the modified argument in $\Delta S_k$ in the above equation!

Before, we go on with the calculation, we can now define the average effective action as a modified Legendre transform reading

$$\Gamma_k[\varphi] = \sup_J (J \cdot \varphi - W_k[J]) - \Delta S_k[\varphi]$$  \hspace{1cm} (5.14)

cf. eq. (4.20).
Accordingly, the corresponding quantum equation of motion, cf. eq. (4.21), is also modified:

\[ J(x) = \frac{\delta \Gamma_k[\varphi]}{\delta \varphi(x)} + (R_k \varphi)(x). \] (5.15)

This can be used to obtain:

\[ \frac{\delta J(x)}{\delta \varphi(y)} = \frac{\delta^2 \Gamma_k[\varphi]}{\delta \varphi(x) \delta \varphi(y)} + R_k(x, y). \] (5.16)

Furthermore, we also have

\[ \frac{\delta \varphi(y)}{\delta J(x')} = \frac{\delta^2 W_k[J]}{\delta J(x') \delta J(y)} \equiv G_k(y - x'). \] (5.17)

From the previous two equations, we can now deduce the important equation

\[ \delta(x - x') = \frac{\delta J(x)}{\delta J(x')} = \int_y \frac{\delta J(x)}{\delta \varphi(y)} \frac{\delta \varphi(y)}{\delta J(x')} \]

\[ = \int_y \left( \frac{\delta^2 \Gamma_k[\varphi]}{\delta \varphi(x) \delta \varphi(y)} + R_k(x, y) \right) G_k(y - x') \] (5.19)

\[ = \int_y \left( \Gamma^{(2)}_k[\varphi] + R_k \right)(x, y) G_k(y - x'). \] (5.20)

In compact notation this reads:

\[ 1 = (\Gamma^{(2)}_k + R_k)G_k \] (5.21)

With all these preparations, we can now derive an expression for the scale derivative of the average effective action at fixed \( \varphi \) and at \( J = J_{\text{sup}} \):

\[ \partial_t \Gamma_k[\varphi] = \partial_t W_k[J] + (\partial_t J) \cdot \varphi - \partial_t \Delta S_k[\varphi] = -\partial_t W_k[J] - \partial_t \Delta S_k[\varphi] \] (5.22)

\[ = \frac{1}{2} \int_q \partial_t R_k(q) G_k \] (5.23)

Together with eq. (5.21), this gives the Wetterich equation

\[ \partial_t \Gamma_k[\varphi] = \frac{1}{2} \text{Tr} \left[ (\Gamma^{(2)}_k[\varphi] + R_k)^{-1} \partial_t R_k \right]. \] (5.24)

This equation will be the starting point for further investigations. Often this equation is also referred to as Wetterich’s flow equation, functional RG flow equation or just the flow equation.
### 5.2. Wetterich equation

#### 5.2.1 Some properties of the Wetterich equation

1. Eq. (5.24) is an exact equation and the propagator appearing in the trace operation on the r.h.s. of the equation is the exact propagator. At this stage no approximations have been made.

2. Despite being exact, eq. (5.24) displays a one-loop structure as signaled by the trace operation on the r.h.s.

3. Eq. (5.24) is a functional differential equation for the functional $\Gamma_k$. No functional integral has to be performed in contrast to eq. (4.4).

4. Eq. (5.24) was derived from a generating functional involving a functional integral. As an alternative perspective, we may also define QFT based on eq. (5.24) without using the starting point of the functional integral.

5. The regulator term guarantees IR regularization by construction. Further, the derivative of the regulator $\partial_t R_k$ appears on the r.h.s. of the Wetterich equation. This also ensures ultraviolet regularization, cf. fig. 5.1. Moreover, the peaked structure of fig. 5.1 also implements the *Wilsonian idea* of momentum-shell integration.

6. The solution of the Wetterich equation corresponds to a renormalization group trajectory in so-called *theory space* which is spanned by all possible symmetry-compatible field operators in the space of action functionals. The RG trajectory interpolates between $S_{\text{bare}}$ and $\Gamma = \Gamma_{k \to 0}$.

7. The regulator term has to fulfill the basic conditions given in Sec. 5.1.1. Apart from these conditions, it can be chosen arbitrarily. The precise RG trajectory then depends on the choice of $R_k$ and reflects the RG scheme dependence. The final point of the RG trajectory, however, is independent of the regulator choice.

8. The Wetterich equation can be used to derive perturbation theory: The loop-expansion in perturbation theory expands the action in powers of $\hbar$: $\Gamma_k = S + \hbar \Gamma_k^{1\text{-loop}} + \mathcal{O}(\hbar^2)$. Therefore, to one-loop order, we can use $\Gamma_k^{(2)} = S^{(2)}$ on the r.h.s. of the Wetterich equation. This gives:

$$
\partial_t \Gamma_k^{1\text{-loop}} = \frac{1}{2} \text{Tr} \log (S^{(2)} + R_k)
$$

$$
\Rightarrow \quad \Gamma_k^{1\text{-loop}} = S + \frac{1}{2} \text{Tr} \log S^{(2)} + \text{const.} \quad (5.25)
$$
5. Functional renormalization group

5.2.2 Method of truncations

The functional RG provides an exact formulation of the renormalization group. For practical calculations, however, approximations for the action functionals have to be made, i.e. the action functional that are plugged into eq. (5.24) have to be truncated.

Vertex expansion

One such truncation or approximation scheme is the vertex expansion as given before:

$$\Gamma_k[\varphi] = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{x_1,\ldots,x_n} \Gamma_k^{(n)}(x_1,\ldots,x_n) \varphi(x_1)\ldots\varphi(x_n)$$  \hspace{1cm} (5.27)

Inserting this expansion into the Wetterich equation provides RG flow equations for the vertex functions $\Gamma_k^{(n)}$.

Derivative expansion

Another important example is the derivative expansion which is often used for the description of scalar field theories:

$$\Gamma_k[\varphi] = \int_x \left( \frac{1}{2} Z_k(\varphi)(\partial_\mu \varphi)^2 + O(\partial^4) + U_k(\varphi) \right)$$  \hspace{1cm} (5.28)

where $U_k(\varphi)$ is the effective potential which again can be expanded in field monomials, e.g., $U_k = \frac{1}{2} m_k^2 \varphi^2 + \frac{1}{4!} \lambda_k \varphi^4 + \ldots$. 

5.2.3 Example: Local potential approximation and for a scalar field

In this section, we study the local potential approximation (LPA) for scalar field theories. The LPA is a reasonable ansatz/truncation when the momentum dependence of the vertex functions is not important and the anomalous dimension is small. This is the case for the critical behavior of $O(N)$ symmetric scalar models.

The LPA for the Wetterich equation, eq. (5.24), is given by the following ansatz for the flowing effective action

$$\Gamma[\varphi] = \int \frac{1}{2} [\nabla \varphi]^2 + U_k(\varphi). \tag{5.29}$$

- The only cutoff dependence is in the effective potential $U_k(\varphi)$.
- $U_k(\varphi)$ is an arbitrary function of the real field $\varphi = \vec{\varphi} = (\varphi_1, \varphi_2, ..., \varphi_N)$.
- For a discussion of the ground state and, e.g., its preserved or spontaneously broken symmetries the effective potential $U_k(\varphi)$ is the most important quantity. It corresponds to the discussion of the free energy in Sec. 2.6.2.
- We consider a $(d-1)$-dimensional system at $T = 0$ (or equivalently a $d$-dimensional classical system) $\Rightarrow \varphi$ depends on the $d$ dimensional vector $\vec{x}$.
- We assume that the theory shall be an $O(N)$ invariant scalar theory, i.e. the action is invariant under the symmetry $\varphi_a \rightarrow R_{ab} \varphi_b$ with any $N \times N$ orthogonal matrix $R$ $\Rightarrow$ effective potential only depends on inner product $\rho = \frac{1}{2} \sum_{a=1}^{N} \varphi_a \varphi_a = \frac{1}{2} \vec{\varphi} \cdot \vec{\varphi}$.

We will use the following cutoff insertion

$$\Delta S_k[\varphi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \varphi_a(-q) R_k(q) \varphi_a(q), \tag{5.30}$$

where summation over the index $a$ is implied and with regulator function

$$R_k(p) = (k^2 - p^2) \Theta(k^2 - p^2). \tag{5.31}$$

$\Theta(x)$ is the step function.

We can now insert the ansatz eq. (5.29) into the Wetterich equation, eq. (5.24). In a first step, we obtain

$$\partial_t U_k[\varphi] = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \text{Tr} \left[ q^2 \delta_{ab} + \frac{\delta^2 U_k(\varphi)}{\delta \varphi_a \delta \varphi_b} + R_k(q) \delta_{ab} \right]^{-1} \partial_t R_k(q) \tag{5.32}$$

where the Tr operation goes over internal indices of the field $a, b$.

We now use that the effective potential only depends on the field invariant $\rho = \frac{1}{2} \vec{\varphi} \cdot \vec{\varphi}$, i.e. $U_k(\varphi) = U_k(\rho)$. To that end, we rewrite the derivatives of $U_k$ with respect to the fields $\varphi$ as derivatives w.r.t. $\rho$

$$\frac{\delta U_k(\rho)}{\delta \varphi_a} = U_k'(\rho) \varphi_a \Rightarrow \frac{\delta^2 U_k(\rho)}{\delta \varphi_a \delta \varphi_b} = U_k''(\rho) \varphi_a \varphi_b + U_k'(\rho) \delta_{ab} \tag{5.33}$$
To study the ground state properties of the theory, we now use that a constant field configuration yields a minimum for the effective potential, i.e. \( \varphi_a(x) = \text{const.} \), cf. Sec. 2.6.2.

Just as in section 2.6.2, the value of an \( O(N) \) symmetric potential \( U_k(\varphi_a) = -\mu \rho + \lambda \rho^2 + \ldots \) will only depend on the length of the vector \( \vec{\varphi}_0 \), the direction is arbitrary.

\[ \vec{\varphi}_0 = (v, 0, \ldots, 0) \quad \text{with} \quad v = \sqrt{2\rho} \quad (5.34) \]

Inserting all of this into eq. (5.32), we obtain

\[ \partial_t U_k(\rho) = \frac{1}{2} \int \frac{d^d q}{(2\pi)^d} \partial_t R_k(q) \text{Tr} \left[ \left( q^2 \delta_{ab} + R_k(q) \delta_{ab} + U'_k(\rho) \delta_{ab} + 2 \rho U''_k(\rho) \delta_{a1} \right)^{-1} \right] \quad (5.35) \]

Inserting now the specific form of the regulator function, we can simplify this expression

\[ \partial_t U_k(\rho) = k^{d+2} S_d \left[ \frac{1}{k^2 + U'_k(\rho)} + \frac{N - 1}{k^2 + U'_k(\rho)} \right], \quad (5.37) \]

with \( S_d = \pi^{d/2}/[(2\pi)^d \Gamma(d/2 + 1)] \) (tutorial).

The flow equation eq. (5.37) exhibits that the flow of the effective potential is driven by two essential contributions, i.e. (1) the contribution from a radial mode \( \sim k^2 + U'_k(\rho) + 2 \rho U''_k(\rho) \) and (2) the contribution from \( N - 1 \) Goldstone modes \( \sim k^2 + U'_k(\rho) \) as expected in a scenario with a spontaneously broken continuous symmetry.

For vanishing external sources, the Goldstone modes are massless.

A simple extension of this approximation scheme consists in multiplying a uniform wave function renormalization \( Z_k \) to the kinetic term in the action \( \propto q^2 \varphi(-q) \varphi(q) \). Then the anomalous dimension can be given in terms of the flow equation \( \eta = -\partial_t \log Z_k \).

The set of flow equations for \( U_k \) and \( Z_k \) is already well-suited to describe many properties of scalar field theories in \( 2 \leq d \leq 4 \) (and not only close to \( d = 4 \), e.g.:

- critical properties of scalar field theories in \( d = 3 \), see sec. 5.3.1!
- (multi-)critical properties of scalar field theories in \( d < 3 \)
- aspects of the Kosterlitz-Thouless transition for \( d = 2, N = 2 \)
- non-linear sigma model
- Mermin-Wagner-Hohenberg theorem in \( d = 2 \)
- ... (for further reading, see e.g., https://arxiv.org/pdf/hep-ph/0005122.pdf)
5.3 Functional RG fixed points and the stability matrix

Quite generally, we can expand the action functional for a quantum or statistical field theory as before in terms of

- **running couplings** $\tilde{g}_{i,k}$ and field operators $O_i$:

  $$
  \Gamma_k[\varphi] = \sum_i \tilde{g}_{i,k} O_i[\varphi], \text{ e.g. } O_i[\varphi] = \{\varphi^2, \varphi^4, (\partial \varphi)^2, \ldots \}.
  $$ (5.38)

In the following, we will be interested in fixed-point solutions, as these allow us to evaluate the scaling behavior of physical quantities, for example, close to a second-order phase transition. To that end we study the RG flow of the

- **dimensionless couplings**:

  $$
  g_{i,k} = \tilde{g}_{i,k} k^{-d_{g_i}},
  $$ (5.39)

where $d_{g_i}$ is the canonical dimensionality of the coupling.

Note that the canonical dimension of the coupling is related to the **canonical dimension of the vertex** by the simple relation:

$$
    d_{g_i} = -\delta(V_{g_i}).
$$ (5.40)

The reason for choosing dimensionless couplings is that a fixed point, which is a scale-free point, is hard to identify when dimensionful couplings are present, since each dimensionful coupling corresponds to a scale.

The RG flow of the couplings is given in terms of

- **$\beta$ functions**:

  $$
  \beta_{g_i}(\{g_n\}) = \partial_t g_{i,k} = -d_{g_i} g_{i,k} + f_i(\{g_n\}),
  $$ (5.41)

where $f_i(\{g_n\})$ are functions of the dimensionless couplings.

The first term reflects the scale-dependence due to the canonical dimensionality, whereas the second term carries the quantum/statistical corrections to the scaling.

**Fixed point**

A **fixed point** is defined by a set of values for the dimensionless couplings $\{g_n^*\}$ where the all the beta functions vanish simultaneously:

$$
    \beta_{g_i}(\{g_n^*\}) = 0 \quad \forall \ i.
$$ (5.42)
5. Functional renormalization group

Linearized flow and stability matrix

To determine the critical exponents, which enter the scaling of observable quantities in the vicinity of a second-order phase transition, we linearize the flow around the fixed point \( \{ g_{n*} \} \).

We find for the linearized flow

\[
\partial_t g_{i,k} = \beta_{g_i} = B_{ij} (g_{j,k} - g_{j,k}^*) + \mathcal{O}((g - g^*)^2),
\]

where we have introduced the stability matrix

\[
B_{ij} = \left. \frac{\partial \beta_i}{\partial g_j} \right|_{g^*}.
\]

We diagonalize the stability matrix \( B_{ij} \), introducing the negative eigenvalues \( \theta_I \)

\[
B_{ij} V_I^j = -\theta_I V_I^i.
\]

Here, the \( V_I^i \) are the eigenvectors enumerated by the index \( I \) which labels the order of the \( \theta_I \) according to their real part, starting with the largest one, \( \theta_1 \).

Solution of linearized flow

The RG eigenvalues allow for a classification of physical parameters by analyzing the solution of the coupling flow in the fixed-point. The solution to this linearized equation is given by

\[
g_{i,k} = g_{i,k}^* + \sum_I C_I V_I^i \left( \frac{k}{k_0} \right)^{-\theta_I},
\]

Herein, \( C_I \) is a constant of integration and \( k_0 \) is a reference scale.

- Eigendirections with \( \text{Re}\{\theta_I\} > 0 \) are called relevant directions and drive the system away from the fixed point as we evolve the flow towards the infrared. Those infrared unstable directions determine the macroscopic physics.

- Eigendirections with \( \text{Re}\{\theta_I\} < 0 \) die out and flow into the fixed point towards the infrared. They are thus called the irrelevant (infrared stable) directions.

- For the marginal directions \( \text{Re}\{\theta_I\} = 0 \), it depends on the higher-order terms in the expansion about the fixed point.

For physics which takes place in the vicinity of the fixed point, we then have:

*The number of relevant and marginally-relevant directions determines the number of physical parameters to be fixed.*

For the flow away from the fixed point, the linearized fixed-point flow (5.43) generally is insufficient and the full nonlinear \( \beta \) functions have to be taken into account.
5.3. Functional RG fixed points and the stability matrix

5.3.1 Example: critical exponents for scalar field theories

Flow of dimensionless potential

To study the fixed point properties of $O(N)$ models, we can use the flow equation for the effective potential, which we derived previously, eq. (5.37). For the fixed-point analysis, we introduce dimensionless quantities, i.e. we rescale the potential and the field with appropriate powers of the cutoff $k$, i.e. $\tilde{\rho} = k^{d-2} \rho$ and $u_k(\tilde{\rho}) = k^{-d} U_k(\rho)$.

Then, we obtain from eq. (5.37),

$$\partial_t u_k(\tilde{\rho}) = -du_k(\tilde{\rho})' \tilde{\rho} + S_d \left( \frac{1}{1 + u_k(\tilde{\rho})'} + \frac{N - 1}{1 + u_k(\tilde{\rho})'} \right).$$

Quartic potential and beta functions in $d = 3$

To evaluate this equation further, we expand the dimensionless effective potential $u_k$ in terms of a running non-vanishing minimum $\kappa_k$ and a four-boson interaction $\lambda_k$:

$$u_k = \frac{\lambda_k}{2} (\tilde{\rho} - \kappa_k)^2.$$  (5.48)

Projection prescription for beta functions of the couplings

Instead of the flow of the function $u_k$, we would like to have beta functions for the individual couplings, i.e. in this case $\kappa_k, \lambda_k$. Therefore, we can use projection prescriptions, for example, the coupling $\lambda_k$ can be obtained from the potential $u_k$ by the following projection:

$$\lambda_k = \frac{\partial^2 u_k(\tilde{\rho})}{\partial \tilde{\rho}^2} \bigg|_{\tilde{\rho} = \kappa_k}.$$  (5.49)

Accordingly, we obtain the flow equation for the quartic coupling from the prescription:

$$\partial_t \lambda_k = \frac{\partial^2 (\partial_t u_k(\tilde{\rho}))}{\partial \tilde{\rho}^2} \bigg|_{\tilde{\rho} = \kappa_k}.$$  (5.50)

For the flow of $\kappa_k$, we use the fact that the first derivative of $u_k$ by definition of the potential vanishes at the minimum, i.e. $u'_k(\kappa_k) = 0$. This implies:

$$0 = \partial_t u'_k(\kappa_k) = \partial_t u'_k(\tilde{\rho})\big|_{\tilde{\rho} = \kappa_k} + (\partial_t \kappa_k) u''(\kappa_k)$$

$$\Rightarrow \partial_t \kappa_k = -\frac{1}{u''(\kappa_k)} \partial_t u'_k(\tilde{\rho})\big|_{\tilde{\rho} = \kappa_k}.$$  (5.51)

This procedure yields beta functions/flow equations for $\kappa_k$ and $\lambda_k$ in $d = 3$:

$$\beta_\kappa = \partial_t \kappa_k = -\kappa_k + \frac{1}{6\pi^2} \left( (N - 1) + \frac{3}{(1 + 2\kappa_k \lambda_k)^2} \right),$$

$$\beta_\lambda = \partial_t \lambda_k = -\lambda_k + \frac{\lambda_k^2}{3\pi^2} \left( (N - 1) + \frac{9}{(1 + 2\kappa_k \lambda_k)^2} \right).$$  (5.54)
5. Functional renormalization group

Non-Gaussian fixed points for $N = 2$

A numerical search for fixed points of the above beta functions $\beta_\kappa = \beta_\lambda = 0$, e.g., for the choice $N = 2$ yields a solution:

$$\kappa^* = 0.0337737... \quad \text{and} \quad \lambda^* = 10.8375778.... \quad (5.55)$$

Stability matrix and critical exponents

The matrix of first derivatives reads

$$B = \begin{pmatrix}
\frac{-2\lambda}{\pi^2(2\kappa\lambda+1)^3} - 1 & \frac{-2\kappa}{\pi^2(2\kappa\lambda+1)^3} \\
\frac{18\lambda^3}{\pi^2(2\kappa\lambda+1)^3} & \frac{2\lambda(\frac{g-3\kappa\lambda}{2\kappa\lambda+1} + 1)}{3\pi^2} - 1
\end{pmatrix}. \quad (5.56)$$

Evaluation at the fixed point values, provides the stability matrix $B$ which has negative eigenvalues:

$$\theta_1 = 1.611, \quad \theta_2 = -0.3842. \quad (5.57)$$

So, we have one relevant ($\theta_1 > 0$) and one irrelevant direction ($\theta_2 < 0$) at this non-Gaussian fixed point. This corresponds the Wilson-Fisher fixed point which we have studied with Wilson’s RG approach.

The eigenvalue $\theta_1$ is related to the scaling of the field bilinear at the Wilson-Fisher fixed point: $\theta_1 = y$, cf. (3.35). Therefore, the functional RG prediction for the correlation length exponent at that level of truncation (LPA4) is:

$$\nu_{\text{LPA4}} = \frac{1}{\theta_1} \approx 0.62. \quad (5.58)$$

Higher orders in the potential and anomalous dimension

This estimate can easily be improved with the functional RG approach, by allowing for a more general expansion of the effective potential, i.e.:

$$u_k = \sum_{i=2}^{n_{\text{max}}} \frac{\lambda_{i,k}}{i!}(\bar{\rho} - \kappa_k)^i. \quad (5.59)$$

and by amending the equation for the anomalous dimension $\eta = -\frac{\partial}{\partial \ln Z_k}$. For example with $n_{\text{max}} = 6$ and including $\eta \neq 0$, we find the following results:

<table>
<thead>
<tr>
<th>Application in $d = 3$</th>
<th>$\nu_{\text{FRG}}$</th>
<th>$\nu_{\text{ZJ}}$</th>
<th>$\eta_{\text{FRG}}$</th>
<th>$\eta_{\text{ZJ}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N = 0$</td>
<td>0.59</td>
<td>0.5882(11)</td>
<td>0.040</td>
<td>0.0284(25)</td>
</tr>
<tr>
<td>$N = 1$</td>
<td>0.64</td>
<td>0.6304(13)</td>
<td>0.044</td>
<td>0.0335(25)</td>
</tr>
<tr>
<td>$N = 2$</td>
<td>0.68</td>
<td>0.6703(15)</td>
<td>0.044</td>
<td>0.0354(25)</td>
</tr>
<tr>
<td>$N = 3$</td>
<td>0.73</td>
<td>0.7073(35)</td>
<td>0.041</td>
<td>0.0355(25)</td>
</tr>
<tr>
<td>$N = 4$</td>
<td>0.77</td>
<td>0.7411(6)</td>
<td>0.037</td>
<td>0.035(4)</td>
</tr>
<tr>
<td>$N = 10$</td>
<td>0.89</td>
<td>0.859</td>
<td>0.021</td>
<td>0.024</td>
</tr>
</tbody>
</table>

Table 5.1: Critical exponents for $O(N)$-models in three dimensions.

Here, we compare our results to high accuracy computations by J. Zinn-Justin et al. using resummed perturbation expansions at five-loop order. Given the simplicity of the FRG approximation these results show a very reasonable agreement.
5.4 Ultraviolet completion of quantum field theories

5.4.1 Preliminaries

Wilson’s RG construction as well as the functional RG generally work in the context of (euclidean) quantum and statistical field theories, as they are just a strategy to evaluate the functional integral.

\[ S = \int d^d x \left[ \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \]  

(5.60)

\[ Z = \int_\Lambda D\phi \exp(-S[\phi]) \]  

(5.61)

with UV cutoff \( \Lambda \).

- Momentum-shell transformation/integration \( \equiv \) transformation in space of couplings
- RG transformation \( \Rightarrow \) flow in the space of couplings!

**RG transformations close to the Gaußian fixed point:**

\[ S^*_G = \int d^d x \frac{1}{2} (\partial_\mu \phi)^2, \]  

(5.62)

\[ m^2|_G = \lambda^*|_G = \lambda^0|_G = 0 \]  

(5.63)

- In the Gaußian fixed point regime: neglect higher-order corrections to rescaling of couplings (à la Wilson):

\[ m(b)^2 = m^2 b^2, \quad \lambda(b) = \lambda b^{4-d}, \quad \lambda_6(b) = \lambda_6 b^{6-2d}, ... \]  

(5.64)

- Or à la FRG, cf. eq. (5.46):

\[ m_k^2 \propto k^{-2}, \quad \lambda_k \propto k^{d-4}, \quad \lambda_{6,k} \propto k^{2d-6}, ... \]  

(5.65)

- Generally:

\[ g(b) = b^\theta g, \quad b > 1, \quad \text{or (FRG)} : \quad g_k \propto k^{-\theta}. \]  

(5.66)

with the classification:

- *relevant* if \( \theta > 0 \) (driven away from Gaußian FP under RG trafos \( k \to 0 \))
- *irrelevant* if \( \theta < 0 \) (vanish under RG trafos = IR stable)
- *marginal* if \( \theta = 0 \) (higher orders determine stability)
• Generally, the coefficient \( g_{l,n_i} \) of an operator with \( l \) derivatives and \( n_i \) powers of the field \( \phi_i \) (with canonical dimension \([\phi_i]\)) has the following behavior under RG transformations close to the Gaussian FP:

\[
g_{l,n_i}(b) = b^{-(l+\sum_i n_i[\phi_i] - d)} g_{l,n_i} = b^{-\delta(V)} g_{l,n_i}
\]

or

\[
g_{k,j,n_i} \propto k^{l+\sum_i n_i[\phi_i] - d} = k^{\delta(V)}
\]

where \( \delta(V) \) is the canonical dimension of the vertex, cf. eq. (4.88).

⇒ The classification \{relevant, irrelevant, marginal\} corresponds to the classification \{super-renormalizable, non-renormalizable, renormalizable\}.

**Consequence in \( d = 4 \):**

It is found in particle physics that all theories that play a role are perturbatively renormalizable theories:

→ **Standard Model of Particle Physics!**

Until now, it seemed to be a mere coincidence that all the (fundamental) particle physics theories that appear in nature are (perturbatively) renormalizable.

**Wilson’s interpretation:**

• Suppose there is a more fundamental theory that has the symmetries of the Standard Model of Particle Physics \((SU(3) \times SU(2) \times U(1))\) at a ultraviolet cutoff scale \( \Lambda \) and here it can be described by a quantum field theory.

• Generically, this QFT will have an action with all the allowed \( \infty \)-many couplings

• If the coupling coefficients are not too large, i.e. they are close to the Gaussian FP

⇒ infrared physics will be dominated solely by the relevant and the marginal couplings

⇒ this corresponds to a renormalizable QFT!

**Example:**

Potential of the \( \phi^4 \) sector of the standard model:

\[
V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 + \frac{\lambda_6}{6!} \phi^6 + ... \tag{5.69}
\]

with \( \lambda_i = \mathcal{O}(1) \).

⇒ RG:

\[
\lambda_{k,i \geq 6} \propto k^{(d/2-1)-d} \xrightarrow{d=4} k^{1-4} \rightarrow 0 \quad \text{for} \quad k \rightarrow 0.
\] 

Accordingly, \( \lambda_k \) decreases slowly (logarithmically) and \( m_k^2 \) generally grows large quickly (fine-tuning/hierarchy problem) for \( k \rightarrow 0 \).
5.4. Ultraviolet completion of quantum field theories

5.4.2 Asymptotic safety scenario (Weinberg 1976, Gell-Mann-Low 1954)

- Perturbatively renormalizable theories automatically arise in the infrared (IR), if the QFT system is dominated by the Gaußian fixed point.

- What happens if the system is dominated by a non-Gaußian (i.e. interacting) fixed point?

- Can theories be renormalizable if perturbation theory fails?

Why do we need renormalizability?

- IR physics should be separated from UV (cutoff Λ independent).

- number of physical parameters < ∞, i.e. QFT should be predictive.

Assumption:

Suppose there is a (non-Gaußian/interacting) ultraviolet fixed point Γ* in the theory space of action functionals.

→ we can express fixed-point action in terms of suitable expansion of field operators $O_i$:

$$\Gamma^*[\varphi] = \sum_i g_i^* O_i[\varphi], \text{ e.g. } O_1[\varphi] = \{\varphi^2, \varphi^4, (\partial \varphi)^2, \ldots\} \quad \text{(5.71)}$$

- the corresponding beta functions of the dimensionless couplings $g_i$ all vanish:

$$\beta_{g_i}(\{g_n^*\}) = 0 \quad \text{(5.72)}$$

and at least one coupling $g_i^* \neq 0$ (to make it non-Gaußian).

- Now, if we can find RG trajectory connecting Γ* with a meaningful physical theory $\Gamma_{IR}$ at some infrared scale.

⇒ We have found a QFT, which can be extended to arbitrarily high scales, since for $\Lambda \to \infty$ we just run into the fixed point.

Such a theory is called asymptotically safe, as it is free from pathological divergencies.
5. **Functional renormalization group**

**Ultraviolet fixed-point behavior**

- We already discussed the critical behaviour in the vicinity of a fixed point, however, with a perspective where we evolve the RG scale \(k\) towards the infrared.
- For the concept of asymptotic safety, where the fixed-point is supposed to govern the theory in the ultraviolet we invert the discussion.
- For convenience we display again eq. (5.46) for linearized flow in fixed-point regime

\[
g_i = g_i^* + \sum_l C^l V^l_i \left( \frac{k_0}{k} \right)^{\theta^l}.
\]

- This gives us modified *classification scheme* when following flow towards higher scales:
  - \(\theta_I < 0\): *ultraviolet repulsive* (RG irrelevant)
  - \(\theta_I > 0\): *ultraviolet attractive* (RG relevant)
  - \(\theta_I = 0\): ... higher orders ... (RG marginal)

**Construction of asymptotic safety**

- The directions with \(\text{Re}\{\theta_I}\} < 0\) run away from the fixed point as we increase \(k\) (green arrows in plot below).
- The directions with \(\text{Re}\{\theta_I\} > 0\) run into from the fixed point as we increase \(k\) (blue arrows in plot below).
- We define the *critical hypersurface* \(S\):
  \(S\) is the *set of all points* in theory space that run towards the fixed point as \(k \to \infty\), i.e. the points lying on RG relevant trajectories.

![Diagram of Theory Space and Critical Hypersurface](image-url)
5.4. Ultraviolet completion of quantum field theories

- Consequently, as we decrease $k$, irrelevant directions rapidly approach critical surface:

$\Rightarrow$ Observables in the IR are all dominated by the properties of the fixed point, independently of whether the flow has started exactly on or near the critical surface.

- The tangential space of $S$ is spanned by the relevant RG eigenvectors $V_i^I$.

- The number of linearly independent relevant directions at the fixed point corresponds to the dimension of $S$:

$$\dim S \equiv \Delta = \text{number of relevant directions with } \theta_I > 0.$$ (5.73)

- **Renormalization**: Set all $C_I = 0$ for all irrelevant directions ($\theta_I < 0$)

$\Rightarrow$ Limit $k \to \Lambda \to \infty$ exists and the system runs into the FP

$\Rightarrow$ Everything is finite!

$\Rightarrow$ Cutoff independence

- **How many physical parameters are there?**

  - (number of physical parameters) = (number of non-trivial initial conditions) = $\Delta$

    = $\dim S < \infty$ if $\Delta < \infty$

  - This establishes the predictivity within the asymptotic safety scenario!
5. Functional renormalization group

5.4.3 Candidate theories for asymptotic safety

Gravity

Einstein’s theory of gravity is perturbatively non-renormalizable.

→ suggested from the dimensional/canonical analysis of the Einstein Hilbert action:

\[
S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-\det(g_{\mu\nu})} (R - 2\lambda) \tag{5.74}
\]

where

- \( g_{\mu\nu} \) is the metric tensor
- we work in 4-dimensional spacetime
- \( R \) is the Ricci scalar
- \( \lambda \) is the cosmological constant
- \( G_N \) is the gravitational constant
- we have set the speed of light \( c = 1 \)

Variation of Einstein-Hilbert action w.r.t. metric field yields Einstein’s field equations.

Perturbative non-renormalizability of gravity

A simple way to see that the theory of gravity is perturbatively non-renormalizable without going into the details of the general theory of relativity is given as follows:

- We first recall Coulomb’s law, reading

\[
V(r) = \alpha \frac{1}{r}, \tag{5.75}
\]

where \( \alpha = \frac{e^2}{4\pi} \) is the fine-structure constant which is a dimensionless quantity \( \alpha \approx \frac{1}{137} \).

(Also \( e \) is dimensionless coupling, stating that QED is perturbatively renormalizable.)

- Now, look at gravity: In the non-relativistic limit and for low gravitational fields Einstein’s gravity is consistent with Newton’s gravity.

- In Newton’s gravity, we have Newton’s law, reading:

\[
V(r) = G_N \frac{M_1 M_2}{r}. \tag{5.76}
\]

- Comparison with to Coulomb’s law, this implies that Newton’s gravitational constant \( G_N \) has canonical dimension \(-2\) (in units of mass or momentum).\(^1\)

- Therefore, the theory is perturbatively non-renormalizable!

\(^1\)Recall that if vertex has dimension \( \delta[V] = \Delta \), then the corresponding coupling \( g \) has dimension \( \delta[g] = -\Delta \).
Asymptotically safe gravity

The above analysis is valid at the Gaussian fixed point, i.e. when an expansion about $G_N^* = 0$ is justified. The question is now, whether the Einstein-Hilbert action admits a non-Gaussian fixed point with a finite number of relevant parameters in the sense of the asymptotic safety scenario. Evidence for this scenario has been collected with the FRG methods with increasing efforts in the last 15 years, see for example


![Figure 5.2: Figure taken from https://arxiv.org/pdf/hep-th/0110054.pdf](https://arxiv.org/pdf/hep-th/0110054.pdf)

Gross-Neveu models in $2 < d < 4$

Gross-Neveu models are models of Dirac fermions with four-fermion interactions, e.g.:

$$S_{GN} = \int d^d x \left( \bar{\psi} \phi \psi + \lambda_\psi (\bar{\psi} \psi)^2 \right).$$

These models are perturbatively non-renormalizable as the coupling carries a negative mass dimension. This conclusion, however, is only an artifact of the perturbative quantization procedure. In fact they are non-perturbatively renormalizable in $2 < d < 4$ at a non-Gaussian fixed point and hence can be extended to arbitrarily high scales, see, e.g.:

Remark on generating functional

A.1 Connected and disconnected Green’s functions


- Example: $\Phi^4$ theory in 4 dimensions

$$S[\Phi] = S_0[\Phi] + S_I[\Phi] = \int d^4x \left( \frac{1}{2} \Phi(-\partial^2 + m^2)\Phi + \frac{1}{4} \lambda \Phi^4 \right) \quad (A.1)$$

- where $\Phi$ is a real scalar field
- and $\partial^2 = \partial_\mu \partial^\mu$ and instead of $-\mu$ we now use $m^2$ as parameter for the field bilinear
- and $S_0[\Phi] = \int d^4x \left( \frac{1}{2} \Phi(-\partial^2 + m^2)\Phi \right)$ and $S_I[\Phi] = \int d^4x \left( \frac{1}{4} \lambda \Phi^4 \right)$

- with eq. (4.6), we then obtain:

$$Z[J] = \int D\Phi e^{-S[\Phi]+\Phi\cdot J} \quad (A.2)$$

$$= e^{-\frac{\lambda}{4} \int d^4x \left( \frac{4}{3J(x)} \right)^4} Z_0[J] \quad (A.3)$$

with

$$Z_0[J] := \int D\Phi e^{-S_0[\Phi]+\Phi\cdot J} \quad (A.4)$$

- we rewrite $Z_0[J]$

$$Z_0[J] = \int D\Phi e^{-\frac{1}{2} \int d^dxd^dy \Phi(x)K(x,y)\Phi(y)+\int d^d x J(x)\Phi(x)} \quad (A.5)$$

- where $K(x,y) = (-\partial^2 + m^2)\delta^{(d)}(x - y)$
- and the inverse of $K$:

$$\int d^d z G(x,z)K(z,y) = \delta^{(d)}(x - y) \quad (A.6)$$
A. Remark on generating functional

• now, parametrize

\[ \Phi(x) = \tilde{\Phi}(x) + \int d^dz G(x,z)J(z) \]  \hspace{1cm} (A.7)

\[ \Rightarrow \quad Z_0[J] = \int D\tilde{\Phi} e^{-\frac{1}{2} \int d^dxd^dy \tilde{\Phi}(x)K(x,y)\tilde{\Phi}(y)} e^{\frac{1}{2} \int d^dxd^dy J(x)G(x,y)J(y)} \]  \hspace{1cm} (A.8)

\[ = \int D\tilde{\Phi} e^{-S_0[\tilde{\Phi}]} e^{\frac{1}{2} \int d^dxd^dy J(x)G(x,y)J(y)} \]  \hspace{1cm} (A.9)

• here, we have used \( D\Phi(x) = D\tilde{\Phi}(x) \) \iff translational invariance of \( D\Phi \) (measure)

• so, we obtain for the generating functional

\[ Z[J] = e^{-S_I[\frac{\lambda}{\pi \hbar}]} \int D\tilde{\Phi} e^{S_0[\tilde{\Phi}]} e^{\frac{1}{2} \int d^dxd^dy J(x)G(x,y)J(y)} \]  \hspace{1cm} (A.10)

\[ Z[J] = \frac{1}{\tilde{N}} e^{-S_I[\frac{\lambda}{\pi \hbar}]} e^{\frac{1}{2} \int d^dxd^dy J(x)G(x,y)J(y)} \]  \hspace{1cm} (A.11)

• where \( \tilde{N} = \det^{1/2}(-\partial^2 + m^2)/\sqrt{2\pi} \) from Gaussian integration (cf. problem set 3).

• schematically \( Z[J] \) can be represented by the following picture:

\[ Z[J] = \frac{1}{\tilde{N}} (1 - \frac{3}{4} \sum \delta x - \frac{3}{2} \sum \delta y + \frac{\lambda}{4} (\sum \delta x)^2 ) e^{\frac{1}{2} \int d^dxd^dy \cdot J \cdot \delta} \]

• derivatives of \( Z[J] \) with respect to the source fields \( J \)

\[ \rightarrow \delta^n Z[J] \text{ generate the connected + disconnected Green’s functions} \]
A.2 Example: two-point function

• for example the 2-point function is generated by

\[
\frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} = e^{-S_I[\frac{1}{2}J]} \frac{\delta^2}{\delta J(x) \delta J(y)} e^{\frac{1}{2}J \cdot G \cdot J} \tag{A.12}
\]

• with

\[
J \cdot G \cdot J = \int d^d x d^d y \ J(x)G(x,y)J(y) = J \quad \quad (A.13)
\]

• where the second equation provides a pictorial representation of the integral over the two sources \( J \) with \emph{propagator} \( G \).

• this yields

\[
\frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} = e^{-S_I[\frac{1}{2}J]} \{ G(x,y) + (G \cdot J)(x)(G \cdot J)(y) \} e^{\frac{1}{2}J \cdot G \cdot J} \tag{A.14}
\]

with

\[
(G \cdot J)(x) = \int d^d y G(x,y)J(y) = (J \cdot G)(x) \tag{A.15}
\]

as \( G(x,y) = G(y,x) \). \text{CHECK THE FOLLOWING EQS.}

\[
\Rightarrow \frac{\delta^2 Z[J]}{\delta J(x) \delta J(y)} = \left[ G(x,y) - \frac{\lambda}{2} \int d^d z G(x,z)G(y,z) \left( \frac{\delta}{\delta J(z)} \right)^2 \right. \\
- \frac{\lambda}{2} \int d^d z (G \cdot J)(x)G(y,z) \left( \frac{\delta}{\delta J(z)} \right)^3 \\
+ \left. \frac{7}{4} \lambda^2 \int d^d z d^d z' G(x,z)G(y,z') \left( \frac{\delta}{\delta J(z)} \right)^3 \left( \frac{\delta}{\delta J(z')} \right)^3 \right] Z[J] \tag{A.16}
\]

⇒ redundant information!
Appendix B

Divergencies beyond 1-loop

In this chapter, we analyse the appearing divergencies beyond 1-loop, discuss the Callan-Symanzik equations and give an idea of the inductive proof of perturbative renormalizability.
B. Divergencies beyond 1-loop

B.1 Recap: Renormalized action and 1-loop divergencies

Scalar field theory: $\Phi^4$-theory in $d=4$

1. Renormalized action and 1-loop divergencies

action: $S_{\Phi^4} = \int d^4x \left( \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{1}{4!} \lambda \Phi^4 \right) \Rightarrow$ vertex dimension $\delta(\lambda \Phi^4) = 0$ (III.1)

- Power counting: Superficial degree of divergence of $n$-point proper vertex $\Gamma^{(n)}$:
  \[ \delta(\gamma_{\Gamma^{(n)}}) = 4 - \text{dim}(\Phi^4) - E = 4 - n \text{ with } E \neq n \] (III.2)
  \[ \Rightarrow \text{All } \Gamma^{(n)} \text{ with more than 4 external lines are superficially convergent.} \]

- Also consider operator insertions of $\Phi^k(x)$:
  \[ \Gamma^{(n)}(\Phi, \ldots, \Phi; x_1, \ldots, x_n) = 2^{-E} \left< \Phi^k(x_1) \cdots \Phi^k(x_n) \Phi^k(x_n) \cdots \Phi^k(x_n) \right>_{\Gamma^{(n)}} \] (III.3)
  \[ \Rightarrow \text{superficial degree of divergence: } \delta(\gamma_{\Gamma^{(n)}}) = 4 - n - 2E \] (III.4)

Examples of superficially divergent diagrams:

- $\Gamma^{(2)}$: $\ell=0$, $n=2$, $\delta(\gamma) = 2$:
  \[ \begin{array}{c}
  \includegraphics[width=0.2\textwidth]{example1}
  \end{array} \]

- $\Gamma^{(4)}$: $\ell=0$, $n=4$, $\delta(\gamma) = 0$:
  \[ \begin{array}{c}
  \includegraphics[width=0.2\textwidth]{example2}
  \end{array} \]

- $\Gamma^{(4)}$: $\ell=4$, $n=2$, $\delta(\gamma) = 0$:
  \[ \begin{array}{c}
  \includegraphics[width=0.2\textwidth]{example3}
  \end{array} \]

\[ \Rightarrow \text{Remark: Diagrams with } n=0 \text{ never arise as subdiagrams.} \]

One-loop analysis:

action in renormalized quantities:

\[ S_{\Lambda} = S_{\Phi^4} + \delta S_R [S_R] = \int d^4x \left[ \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} m^2 \Phi^2 + \frac{1}{4!} \lambda \Phi^4 \right] + \frac{1}{2} \Phi^k \left( - \delta Z_\Phi \partial^2 + \delta m^2 \right) \Phi^k + \frac{1}{4!} \lambda \Phi^4 \delta Z_\lambda \Phi^4 \] (III.5)

with $\Phi^k = Z_\Phi^{\frac{1}{2}} \Phi^k$, $\lambda = \lambda \delta Z_\lambda$, $m^2 = (m^2 + \delta m^2) \frac{1}{2} \Phi^4$

(III.6)

\[ \delta Z_\Phi = Z_\Phi - 1, \quad \delta Z_\lambda = Z_\lambda - 1 \]

4. bare parameters are functions of $m_\Phi, \lambda_R, \Lambda$ such that all $\Phi^k$-correlation functions have a finite limit.

4. Counterterms are formal series in $\lambda_R$: $\delta m^2 = a_0(\lambda) \lambda_R + a_1(\lambda) \lambda_R^2 + \ldots$

4. Correlation functions shall have finite limit order by order in $\lambda_R$

4. Effective action at 4-loop: $\Gamma^{(4)}_{\Phi^4} = \Gamma^{(4)}_{\Phi^4} + \delta S_R [S_R]$ (III.7)

where $\Gamma^{(4)}_{\Phi^4} = \frac{1}{2} \ln (A + \lambda R m^2 \frac{1}{2} \Phi^4) \sim \ldots \begin{array}{c}
  \includegraphics[width=0.2\textwidth]{example4}
  \end{array}$ (III.8)
B.1. Recap: Renormalized action and 1-loop divergencies

\[ \Gamma^{(1)}(p) = \frac{1}{2} \int d^4 x \, \frac{1}{m^2} \ln \frac{m_R^2}{m^2} + O(1) \]  
\[ \Gamma^{(2)}(p^2) = -\frac{1}{6} \lambda_R \int d^4 x \, \frac{1}{q^2 + m^2} \cdot \frac{1}{(q^2 + p^2)^2 + m^2} = -\frac{2\lambda_R}{6\pi^2} \ln \frac{m_R^2}{m^2} + O(1) \]

\( \Gamma^{(2)}(p^2) \) is independent from \( p \) \( \Rightarrow \) No \( (\partial_{\mu} \phi)^2 \)-term is generated at 1-loop level

\( \Rightarrow \) no term \( \sim \lambda_R \Rightarrow Z = 1 + O(\lambda_R^2) \)

This is a peculiarity of \( \phi^4 \) at 1-loop order.

\[ S_{\text{ren}} = \frac{1}{2} \int d^4 x \, \lambda_R \left( \frac{\partial^2}{\partial x^2} \right) \left( \frac{1}{2} - m^2 \right) \ln \frac{m_R^2}{m^2} + O(1) \]

\[ Z_R = 1 + \frac{2}{6\pi^2} \lambda_R \ln \frac{m_R^2}{m^2} + O(1) \]

\[ \frac{\partial}{\partial p^2} \Gamma_R^{(2)}(p^2) \rightarrow 0 \]

\( \Gamma^{(1)} \) and \( \Gamma^{(2)} \) into \( \Gamma_R^{(2)} \).

Divergences are removed such that \( R \Gamma_R^{(2)} \) is finite at one-loop order

and the renormalization conditions,

\[ \Gamma_R^{(2)}(p \rightarrow 0) = m^2 \]

\[ \frac{\partial}{\partial p^2} \Gamma_R^{(2)}(p \rightarrow 0) = 1 \]

\[ \Gamma_R^{(2)}(p^2 \rightarrow 0) = \lambda_R \]

are fulfilled.

Note: \( R \Gamma_R^{(2)} = \sum_{n} \frac{1}{n!} \int d^n x \, d^n y \, \Gamma_R^{(2)}(x_n, \ldots, x_1) \phi_R(x_n) \ldots \phi_R(x_1) \).
B. Divergencies beyond 1-loop

B.2 Divergencies beyond 1-loop

2.1. Superficially convergent diagrams \((\not p^4, n \geq 4)\)

At higher orders divergent subdiagrams appear

\[ n = 6 \]

\[ \sim \Gamma^{(2)} \frac{A^2}{\epsilon} \ln \frac{\Lambda}{m} \]

Here we can identify the divergent subdiagrams with the one-loop divergencies of the 2-point/4-point function. For those \(\not p^4\) counterterms have already been provided.

At this order the following diagrams contribute to the perturbation theory:

- 1-loop counter-term for 2-point function is inserted on propagator
- 1-loop counter-term of 4-point function

Two-loop contributions from 1-loop counterterms!

\(\not p^4\) counterterms which render divergent functions finite at higher orders also cancel the divergent subdiagrams of superficially convergent functions.

**Skeleton diagrams:** A skeleton diagram is a really convergent diagram, i.e. it is superficially convergent and has no divergent subdiagrams.

- e.g. 1-loop diagrams of 2n-point functions, \(n \geq 2\)

**Skeleton expansion:** Replace all propagators and vertices in skeleton diagram by full propagators and vertices: \(\Gamma^{(2)}\) and \(\Gamma^{(4)}\) and expand in \(\not p^4\)

**Weinberg's theorem:** In dressed skeletons the power-counting is modified only logarithmically.

\[ \delta (p^4) < 0 \] (superficial degree of divergence) then the diagram is really convergent (on a \((L+4)\)-loop level) provided \(\Gamma^{(2)}\) and \(\Gamma^{(4)}\) have been renormalized on \(L\)-loop level.

Problem of renormalizability reduces to renormalization of divergent proper vertices \(\Gamma^{(2)}\) and \(\Gamma^{(4)}\) (also \(\Gamma^{(2,1)}\), later)!

2.2. Superficially divergent diagrams with subdivergencies

Skeleton argument not applicable!

2.3. Overlapping divergencies

Concept of insertion of diagrams of lower order is no longer well-defined.

III.2.2 and III.2.3 make proofs of perturbative renormalizability difficult!
B.3 Callan-Symanzik equations

**Observation:**

\[
\left. \frac{\partial}{\partial m^2} \right|_{\lambda_A} \int \mathcal{D} \bar{\phi} \phi \cdots e^{-\int \cdots \frac{1}{2} m^2 \phi^2 \cdots} = - \int \mathcal{D} \phi \int \mathcal{D} \bar{\phi} \phi \cdots e^{-\int \cdots}
\]

\[
\Rightarrow \frac{\partial}{\partial m^2} \left|_{\lambda_A} \right. \Gamma^{(n)}(q_1, \ldots, q_t) = \Gamma^{(n+1)}(0, q_1, \ldots, q_t, p_1, \ldots, p_n) \quad (\text{III.14})
\]

\[
\Rightarrow \text{By differentiating w.r.t. bare mass, we can relate superficially divergent correlation functions to functions which have a skeleton expansion.}
\]

Supposed the regularized and renormalized theory exists, then the full quantum effective action can be expressed in both, the bare as well as the renormalized variables:

\[
\Gamma'[\Phi'] = \Gamma'[\Phi]
\]

_for the proper vertices we then obtain (\Phi = \Phi_0 (\Phi_0)):

\[
\Gamma^{(n)} = \Gamma_0^{(n)} \Gamma^{(n)} \quad \text{and} \quad \Gamma^{(m_e)} = \Gamma_0^{(m_e)} \Gamma^{(m_e)} \quad (\text{III.15})
\]

\[
\text{New renormalization constant } \Gamma_0^{(m_e)} \text{ for superficially divergent vertex } \Gamma^{(m_e)}
\]

\[
\text{with renormalization condition } \Gamma_0^{(m_e)}(q=0, p_1=\ldots=p_n=0) = 1.
\]

Now, express (Eq. III.14) in terms of renormalized quantities: \((\lambda_A, m_0, q_0) \Gamma'_{\lambda_A}(\Phi_0) \rightarrow (\lambda_{\text{ren}}, m_{\text{ren}}, q_{\text{ren}}) \Gamma'_{\lambda_{\text{ren}}}(\Phi_{\text{ren}})\)

\[
\text{Start with: } m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} \Gamma^{(n)} = m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} \left( m^2 \right) \frac{\partial}{\partial m^2} \Gamma^{(m_e)} = (m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} m^2 ) \Gamma^{(m_e)}(q_{\text{ren}}) \quad (\text{III.16})
\]

\[
\text{Now use Eq. (III.15) to replace bare by renormalized functions in Eq. (III.16) and apply chain rule to transform differentiation w.r.t. } m_R \text{ at fixed } (\lambda_A) \text{ into differentiation at fixed } (\lambda_{\text{ren}}).
\]

**Definitions:**

\[
m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} m^2 = \beta
\]

\[
m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} \ln Z_T = \left( m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} + \beta \frac{\partial}{\partial \lambda_A} \right) \ln Z_T = \eta
\]

\[
m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} \ln Z_T^2 = \left( m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} + \beta \frac{\partial}{\partial \lambda_A} \right) \ln Z_T^2 = \eta
\]

\[
\frac{\partial}{\partial \lambda_A} \left( m_R \left| \frac{\partial}{\partial m^2} \right|_{\lambda_A} m^2 \right) = m_R \sigma
\]

\[
\lambda, \eta, \sigma, \beta, \sigma, \gamma \text{ are functions of } \lambda_A \text{ and } m_R, \text{ and they are dimensionless.}
\]

If the limit \( \lambda \to \infty \) exists, they are only functions of \( \lambda_A \).
B. Divergencies beyond 1-loop

Insert Eq. (III.15) into Eq. (III.16):
\[
\Rightarrow Z_T^{\ell} Z_P^{3-\ell} \left. \frac{\partial}{\partial m_R} m_R^{1/2} \Gamma^R_m(e) \right|_{m^2} = \frac{Z_T^{\ell}}{Z_P^{3-\ell}} \left. \frac{\partial}{\partial m_R} m_R^{1/2} \Gamma^R_m(e) \right|_{m^2}
\]

\[
\Rightarrow \left[ m_R \frac{\partial}{\partial m_R} + \beta \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \lambda_R - \lambda_R^2 \right] \Gamma^R_m(e) = m_R^{2/2} \sigma \Gamma^R_m(e)
\]

(Callan-Symanzik equation (CSE))

1. The CSE describes the change of the \( n \)-point vertices as a response to a change of the renormalized mass \( \sigma \) for different physical masses.

2. For given renormalization functions \( \beta, \gamma, \delta, \sigma \) and given boundary conditions (e.g. \( m_R \to \infty \), no dynamics) all \( n \)-point vertices can be computed.

Include renormalization conditions:

1. \( n=2, \ell=0: \) \( \Gamma^R_m(\alpha^2 \to 0) = m_R^{2} \) and \( \Gamma^R_m(\alpha^2=0, \beta_1=0, \beta_2=0) = 0 \)

\[
\Rightarrow \text{CSE: } (m_R \frac{\partial}{\partial m_R} - \gamma) m_R^{2} = m_R^{2/2} \sigma \Rightarrow \sigma = 2 - \gamma
\]

Further:
\[
\Rightarrow \Gamma^R_m(\alpha^2 \to 0) = m_R \frac{\partial}{\partial m_R} \Gamma^R_m(\alpha^2 \to 0) \Rightarrow \Gamma^R_m(\alpha^2 \to 0) = \sigma \lambda_R^2
\]

2. \( n=4, \ell=0: \) \( \Gamma^R_m(0,0,0,0) = \lambda_R^2 \)

\[
\Rightarrow \text{CSE: } (\beta \frac{\partial}{\partial \lambda_R} - 2\gamma) \lambda_R^2 = \beta - 2\gamma \lambda_R = m_R^{2}(2-\gamma) \Gamma^R_m(0,0,0,0)
\]

First diagram contributing to \( \Gamma^R_m(4) \) is of order \( \lambda_R^2 \):

\[
\Rightarrow \beta = \sigma \lambda_R^2
\]

1. First diagram contributing to \( \Gamma^R_m(4) \) is of order \( \lambda_R^2 \):

\[
\Rightarrow \beta = \sigma \lambda_R^2
\]

Relate all renormalization functions \( \beta, \gamma, \delta, \sigma \) in Eq. (III.18) to values of proper vertices at zero momentum.

From these relations, if renormalized proper vertices have limit when \( \Lambda \to 0 \)

\[
\Rightarrow \beta, \gamma, \delta, \sigma \text{ have a limit } \Rightarrow \text{CSE exists.}
\]
B.4 Inductive proof of perturbative renormalizability (idea)

► From considerations about the skeleton expansion:

All correlation functions are finite at a given order, if the superficially divergent vertex functions have been renormalized in the corresponding order.

► In, in $\phi^4$ theory $\Gamma_2^{(2)}$, $\Gamma_2^{(4)}$ and $\Gamma_4^{(4)}$ have a $\lambda \to \infty$-limit on a $L$-loop level.

$\Rightarrow$

All vertices that have a skeleton expansion are finite on a $(L+1)$-loop level.

► A proof of renormalizability is now given, if one can conclude inductively, that now also $\Gamma_2^{(2)}$, $\Gamma_2^{(4)}$ and $\Gamma_4^{(4)}$ have a finite $\lambda \to \infty$-limit on a $(L+1)$-loop level.

► Start of induction: Renormalization at $(L+1)$-loop level, cf. Sec. III.4.

► Proof is possible with the CSE in the form:

$$m_R \frac{\partial}{\partial m_R} \Gamma_2^{(n)} = \left( -\beta \frac{\partial}{\partial \lambda_R} + \frac{\beta}{2} + \xi \right) \Gamma_2^{(n)}(\lambda_R) + m_R^2 (2-\xi) \frac{\partial}{\partial m_R} \Gamma_2^{(n+1)}$$

► Eq. (III.22) relates $L$-loop renormalized quantities (RHS) with $(L+1)$-loop order (LHS).

► From the CSE (III.22) the finite $\lambda \to \infty$-limit on $(L+1)$-loop level follows by finiteness on $(L+1)$-loop level.

► Example: Four-point vertex $\Gamma_2^{(4)}$, $(L=0, n=4)$:

CSE (III.22): $m_R \frac{\partial}{\partial m_R} \Gamma_2^{(4)} = \left( -\beta \frac{\partial}{\partial \lambda_R} + \frac{\beta}{2} + \xi \right) \Gamma_2^{(4)}(\lambda_R) + m_R^2 (2-\xi) \frac{\partial}{\partial m_R} \Gamma_2^{(4)}$

► Suppose $\Gamma_2^{(4)}(\lambda_R)$ is finite up to $L$ loops $\Rightarrow$ finite up to $\lambda_R^{L+1}$.

$\Rightarrow$ $\Gamma_2^{(4)}$ (superficially convergent, has skeleton expansion) is finite up to $(L+1)$-loop.

$\Rightarrow$ finite up to $\lambda_R^{L+2}$.

$\Rightarrow$ (since $\beta$ finite up to $\lambda_R^{L+2}$ $\Rightarrow$ $\beta$ at least finite up to $\lambda_R^{L+1}$)

$\Rightarrow$ with separation of $\Gamma_2^{(4)}$ into leading and subleading terms $\frac{\partial}{\partial \lambda_R} \Gamma_2^{(4)}(\lambda_R) + \frac{\partial}{\partial m_R} \Gamma_2^{(4)}(\lambda_R) = -\beta + 2\lambda_R \xi$ is finite $\Rightarrow$ $\lambda_R^{L+2}$

$\Rightarrow$ finite up to $\lambda_R^{L+2}$.

Integrate Eq. (III.23) to show that $\Gamma_2^{(4)}(\lambda_R)$ itself is finite at $(L+1)$-loop:

$\Gamma_2^{(4)} = \int \frac{d m_R}{m_R^2} \frac{\partial}{\partial m_R} \Gamma_2^{(4)}$ (RHS of III.23) $\Rightarrow$ No divergencies by integration (in particular at upper bound), as large $\beta$ masses suppress contributions. WORKS ANALOGOUSLY FOR $\Gamma_4^{(4)}(\lambda_R)$. 

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