TPI: Classical Mechanics

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Preface

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve.
— Eugene Paul Wigner

Chapter 1

Newtonian Mechanics

1.1 Foundations

1.1.1 Galiean space

We start our discussion of classical mechanics by introducing the 'arena' in which it takes place: classical space-time. Mathematically, space time, $\mathbb{E}^4 = \mathbb{E} \times \mathbb{E}^3$ is a four-dimensional **euclidean space** ($\rightarrow$ TP0) . The second factor in the decomposition $\mathbb{E} \times \mathbb{E}^3$ accommodates all points of three-dimensional space, and the first factor all values of time. Points R in space time are often called **events**, or **world points**. We notice three important points about space-time:

- ▷ Being an euclidean space, space-time does not have a canonical origin, no point is special.

- ▷ In the statement above reflects the fact that **space-time is not a vector space**. (In a vector space, the null-vector plays a distinguished role, which point in the universe would be a null-point?). However, we are free to choose any point we like as origin, O. For example, we might say that the center of sun, at this very moment is our space-time origin. The differences $R - O \equiv r \in \mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ are then four-dimensional vectors, sometimes called **four-vectors**. One often uses sloppy notation, which does not distinguish between a point, R, and a vector, r. However, the definition of the latter requires the choice of some origin, which is implicitly assumed. Recall from the definition of euclidean space that the vector representation r of a point R changes upon change of the origin (cf. Fig. 1.1.) Also recall, that the difference $R - Q = (R-O) - (Q-O) = r - q \in \mathbb{R}^4$ between two points is a vector which does *not* depend on the choice of origin. We denote space-time vectors as $r = (t, \mathbf{r})$.[1] Once a basis $\{\mathbf{e}_i\}$ of the vector space $\mathbb{R}^3$ has been chosen, each vector $\mathbf{r} = r^i \mathbf{e}_i$ may be described in terms of its expansion coefficients relative in that basis. We refer to the four components,

[1] For completeness, we note that the assignment of a numerical value t to the time-like component requires a choice of unit. For example, $t = 5\text{sec}$.

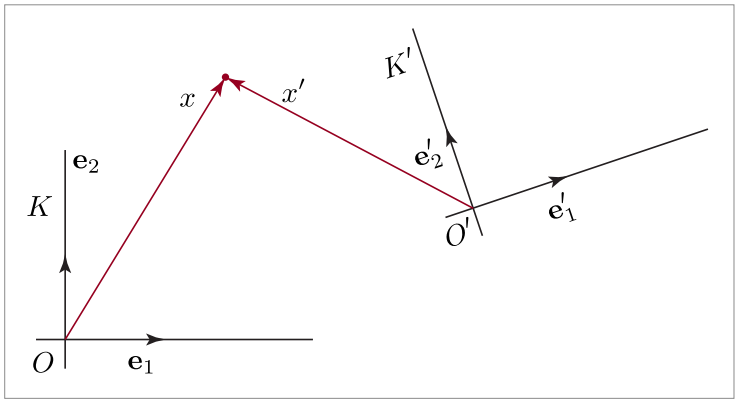


Figure 1.1: Schematic illustrating the role of reference systems in Euclidean space. Discussion, see text.

(t, r^i) identifying r as r^μ, $\mu = 0, 1, 2, 3$ where $r^0 = t$, and $r^i = (\mathbf{r})^i$, $i = 1, 2, 3$ are space like components.[2] We call the choice $(O, \{\mathbf{e}_i\})$ of an origin, O, and of a vector space basis $\{\mathbf{e}_i\}$ a **reference system**, K. See Fig. 1.1 for an illustration of two systems of reference in a two-dimensional cartoon of euclidean space.

Think of some examples of choice-of-origins, points, and vectors to make yourself comfortable with these statements.

▷ In classical mechanics the decomposition $\mathbb{E}^4 = \mathbb{E} \times \mathbb{E}^3$ of space-time into time and space is rigid. There is an absolute time, and an absolute space, and these two exist independently of each other. This view was later abandoned by relativity.

For two events R, Q, we call $(r - q)^0$ the **time** that passed between them. If $(r - q)^0 = 0$, the events are called **simultaneous events**. For two simultaneous events, $R - Q = (0, \mathbf{r} - \mathbf{q})$ can be identified with a three-dimensional space-like vector, and we call $d(R, Q) \equiv |\mathbf{r} - \mathbf{q}| = \sqrt{\langle \mathbf{r} - \mathbf{q}, \mathbf{r} - \mathbf{q} \rangle}$ the **distance** between the events. Here, $\langle \mathbf{v}, \mathbf{w} \rangle = \sum_i v^i w^i$ is the standard scalar product of $\mathbb{R}^3$. Notice that it does not make sense to talk about the distance between non-simultaneous events. Also notice that simultaneousness of events and distance are quantities independent of the choice of origin.

Euclidean space $\mathbb{E}^4$ equipped with the above notions of time and distance is called **Galilean space**. Galilean space is the 'arena' in which classical mechanics takes place. We will usually refer to its points R in terms of four-vectors x, i.e. assuming that a point of reference has been chosen.

[2] Following a standard convention, the space-like components of a space-time point are denoted by Latin indices, $i = 1, 2, 3$. It is also standard to denote space-time vectors, by non-bold face symbols (r vs. $\mathbf{r}$.)

1.1.2 Galilei transformation

In linear algebra, we learn that linear maps are the 'natural maps' compatible with the structure of a vector space. Let us try to identify, purely by mathematical reasoning, a similar set of maps for Galilean space. Later on, we will see that these maps play an important role in the formulation of the laws of classical mechanics.

Much like euclidean space is extends the definition of a vector space (euclidean space = (vector space)+(point of origin)), the 'natural maps' of euclidean space extend the definition of linear maps: we call a map $A : \mathbb{E}^d \to \mathbb{E}^d, R \to AR$ an **affine map**, if there exists a linear map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that

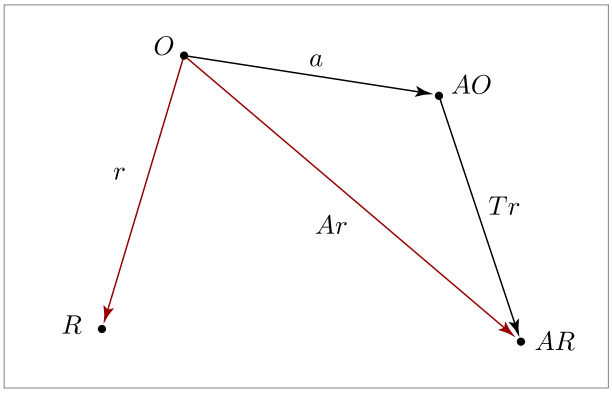
$$\forall R, Q \in \mathbb{E}^4 : \quad AR - AQ = T(R - Q). \tag{1.1}$$

First notice that the definition makes mathematical sense: AR, and AQ are points, but $AR - AQ$ is a vector. So are $R - Q$, and $T(R - Q)$. The definition requires that for arbitrary R, Q the difference vector between the image points, $AR - AQ$ be representable as a linear map T acting on the difference vector between the original points. In other words, an affine map is 'almost' a linear map. To give the definition more concrete meaning, let us consider the particular choice $Q = O$. According to our convention we denote $R - Q = R - O = r$, where the symbol on the r.h.s. now denotes the vector connecting the point r to the origin O. Our definition now requires that

$$\boxed{AR = AO + Tr,} \tag{1.2}$$

i.e. the image point AR is obtained by adding the vector Tr to a generally shifted origin AO. The information describing an affine transform is contained in the shift vector $AO - O$, and in the linear transformation T.

Once an origin, O, has been fixed, and hence a bijection $\mathbb{E}^4 \to \mathbb{R}^4, R \mapsto r = R - O$ been fixed, we may define an affine map $A : \mathbb{R}^4 \to \mathbb{R}^4$, induced by the affine map $A : \mathbb{E}^4 \to \mathbb{E}^4$ (and conveniently denoted by the same symbol). It is defined as $Ar = AR - O = AO - O + Tr \equiv a + Tr$ where we denote the **displacement of the origin as**, by $a \equiv AO - O$. The affine transformation therefore acts on the vector space associated to $\mathbb{E}^4$ as



$$\begin{aligned} &A : \mathbb{R}^4 \to \mathbb{R}^4, \\ &\quad r \mapsto Ar \equiv a + Tr. \end{aligned} \tag{1.3}$$

We will mostly describe affine transformations in this vector space language. Keep in mind, that an affine transformation differs from a linear transformation in that it may contain a finite displacement, a.

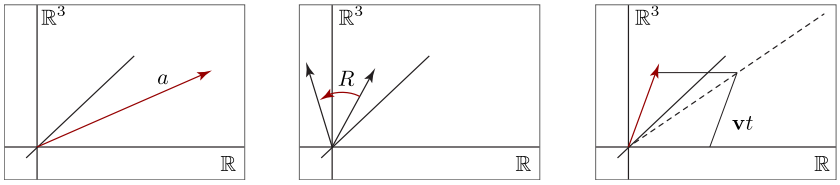
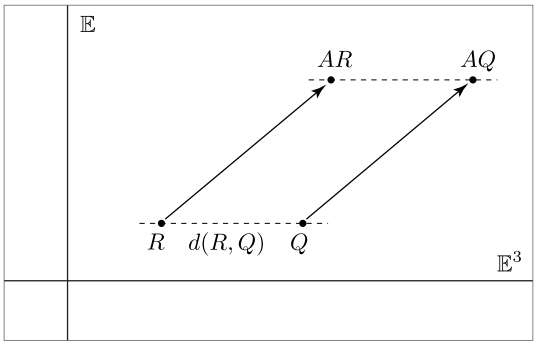


Figure 1.2: The three types of Galilei transformation: translation of origin, rotation of space, homogeneous motion. Discussion, see text.

Of particular interest to classical mechanics are affine maps leaving the Galilei structure of a Galilean space invariant. Affine maps of this type are called **Galilei transformations**. The defining feature of a Galilei transformation is that the time between events does not change, and that the distance between simultaneous events remains the same. In formulas (cf. the figure): $\forall R, Q \in \mathbb{E}^4$:



$$(R-Q)^0 = (AR-AQ)^0,$$
$$(R-Q)^0 = 0 \Rightarrow d(R,Q) = d(AR,AQ).$$

It turns out that there are only three distinct families of Galilei transformations. (cf. Fig. 1.2)

- ▷ **Translations in space and time**: $R \mapsto AR = R + a$, or $r \mapsto Ar = r + a$, where $a = (s, \mathbf{q})$ is a fixed displacement vector which may represent a translation in both, time s and space $\mathbf{q}$. In a vector coordinate representation, $r = (t, \mathbf{r})^T$

$$r = \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix} \mapsto \begin{pmatrix} t+s \\ \mathbf{r}+\mathbf{q} \end{pmatrix}. \tag{1.4}$$

- ▷ **Rotations of space**, $R \mapsto AR = O + Rr$, or $r \mapsto Rr$, such that

$$r = \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix} \mapsto \begin{pmatrix} t \\ O\mathbf{r} \end{pmatrix}, \tag{1.5}$$

  where $O \in \mathrm{O}(3)$ is a rotation matrix ($\rightarrow$ TP0) .

- ▷ **Uniform motion**, $R \mapsto AR = O + Tr$, or $r \mapsto Tr \equiv r + a_{\mathbf{v}}(t)$, such that

$$r = \begin{pmatrix} t \\ \mathbf{r} \end{pmatrix} \mapsto Tr = \begin{pmatrix} t \\ \mathbf{r}+\mathbf{v}t \end{pmatrix}. \tag{1.6}$$

Convince yourself that all three families satisfy the criteria formulated above, and that there are no more Galilei transformations. The composition of two Galilei transformations is again a

Galilei transformation, and each transformation has an inverse, i.e. the set of Galilei transformations forms a group, the **Galilei group**. We finally note that Galilei transformations afford an **active and a passive interpretation**: in the active transformation, you imagine points of space times actually being transformed, for example space being rotated, or translated. In the passive view we leave space untouched. Instead, we imagine a change of coordinate systems, which may include a change of the orthonormal coordinate basis and/or a change of origin O. For example, the third transformation family relates the coordinate systems of two observers in space moving with velocity $\mathbf{v}$ relative to each other, such that at time $t = 0$ their chosen points of origin coincide (think about this statement.) Within the passive framework, coordinate vectors change as indicated in the displayed equations above; points $R \in \mathbb{E}^4$ remain invariant.

The passive interpretation suggests that a Galilei transformation establishes another way of 'looking' at Euclidean space, a rotated perspective, or a translated one. We wouldn't expect the basic laws of physics to depend on the chosen perspectives, e.g. two observers rotated relative to each other by a fixed angle would find that mechanical bodies move according to the same fundamental laws. We will return to this point after we have introduced the laws of classical mechanics in the next section.

INFO It is instructive to count the number of **free parameters of the Galilei group**: a translation in space and time $(\Delta t, \Delta \mathbf{r})$ is described by four parameters (the first family), rotations of space, O, are described by three parameters (second family), and a uniform motion is described by the three parameters of the velocity vector $\mathbf{v}$. A general Galilei transformation is therefore describable in terms of $4 + 3 + 3 = 10$ real parameters.

1.2 Newton axioms

Classical mechanics addresses the problem of describing the motion of 'bodies' in the space time introduced above. Newton made a great contribution to science by exhaustively describing mechanical motion in terms of only three laws. Before translating Newton's laws to formulas, we here quote them in their original prosaic form, as formulated in 1687:

- ▷ **Law I**: Every body persists in its state of being at rest or of moving uniformly straight forward, except insofar as it is compelled to change its state by force impress'd.[3]
- ▷ **Law II**: The alteration of motion is ever proportional to the motive force impress'd; and is made in the direction of the right line in which that force is impress'd.
- ▷ **Law III**: To every action there is always opposed an equal reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

[3] Isaac Newton, *The Principia*, A new translation by I.B. Cohen and A. Whitman, University of California press, Berkeley 1999.

From todays perspective, the revolutionary character of these statements may no longer be apparent. However, Newton's key innovation was a breaking of the tradition to empirically describe nature phenomena, e.g. to track the motion of each planet individually. Rather he made a successful attempt at describing all possible motions in one go, a hitherto unknown degree of powerful abstraction.

In the following we will give the key terms appearing in Newton's laws (body, force, etc.) a precise meaning and formulate the laws in a mathematical language. A **body** is a point–like object, i.e. the position of a body in space and time is specified by an event R of space time. After fixation of an origin, we may specify it by a four-vector $r = (t, \mathbf{r})$. We attribute to each body a scalar quantity m called its **mass**.

INFO At first sight, the idealization of **point-like bodies** may look strange. However, it is less of an abstraction than one might think: we will learn later that to each (rigid) extended body we may associate a special point, called its 'center of mass'. In the presence of external forces, the body at large moves as if all its mass was concentrated in the center of mass. For example, when viewed from earth the motion of the quite extended planet Mars approximates the motion of a point particle of Mars' mass.

The **motion of a body** is described by a curve ($\to$ TP0) in three-dimensional (!) affine space:

$$\begin{aligned} \gamma : I &\to \mathbb{E}^3, \\ t &\mapsto \mathbf{R}(t). \end{aligned} \tag{1.7}$$

If a point of origin has been chosen, we may equally describe the motion by a curve in $\mathbb{R}^3$ as $\gamma : I \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$. The **velocity** of the body is given by

$$\mathbf{v}(t) = d_t \mathbf{r}(t), \tag{1.8}$$

and the product of mass and velocity, $m\mathbf{v} \equiv \mathbf{p}$ defines its **momentum**. We further define the body's instantaneous **acceleration** as

$$\mathbf{a}(t) = d_t \mathbf{v}(t) = d_t^2 \mathbf{r}(t). \tag{1.9}$$

A non-accelerated body, $\mathbf{a} = 0$, moves at constant velocity $\mathbf{v} = \text{const.}$, i.e. it performs a 'constant' motion. Newton's first and second law state that changes in the motion of a body are due to the action of a force, $\mathbf{F}$. A **Force** is a vector field, $\mathbf{F} : \mathbb{E}^4 \to \mathbb{R}^3, R \to \mathbf{F}(R)$ assigning to space time points a vector $\mathbf{F}(R)$ quantifying direction and magnitude of what we call 'force'. Force is an instance that changes motion in the following way: during an infinitesimally short time window Δt, the velocity of a body changes as $\mathbf{v}(t + \Delta t) - \mathbf{v}(t) = \Delta t m^{-1} \mathbf{F}$, i.e. the change in velocity is proportional to the acting force, inversely proportional to the body's mass m, and proportional to the duration of action Δt. Dividing by Δt and taking the limit $\lim_{\Delta t \to 0} \Delta t^{-1}(\mathbf{v}(t + \Delta t) - \mathbf{v}(t)) = \mathbf{a}(t)$, we express the statement as

$$\boxed{\mathbf{F} = m\mathbf{a},} \tag{1.10}$$

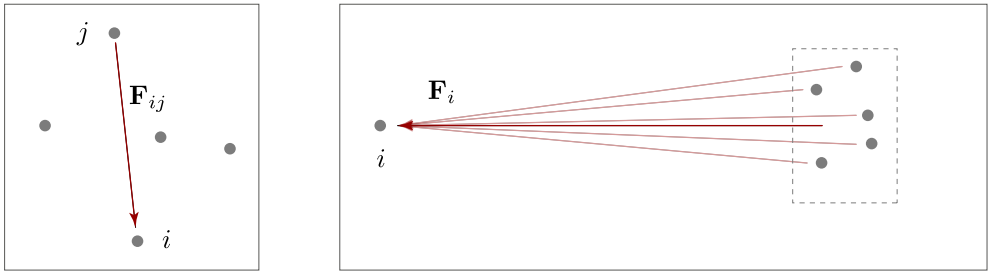


Figure 1.3: Left: a system of particles interacting via pair forces. Right: an 'external' forces generated by a remote system of particles. Discussion, see text.

which is the **quantitative formulation of Newton's first and second law**. Within Newtonian mechanics, neither force nor mass can be reasoned out of deeper principles. Rather, they are defined by the second law as instances causing $(\mathbf{F})$ or inhibiting (m) the acceleration of bodies.

Eq. (1.10) implements Newtons first and second law, but not the third one. The equation assumes the presence of an 'extraneous' force, $\mathbf{F}$, but does not tell us where this force came from. Ultimately, however, the force must be due to the presence of other bodies, interacting with our given test body. A more self contained representation of the theory is obtained if we consider an assembly of N bodies of mass m_i at space time vectors r_i. We assume that the jth body exerts a force $\mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j)$ on the ith body, which (i) depends on the vector $\mathbf{r}_i - \mathbf{r}_j$ between the two bodies,[4] (ii) acts along the line connecting them, $\mathbf{F}_{ij} \propto \mathbf{r}_i - \mathbf{r}_j$, at a strength $|\mathbf{F}_{ij}|$ depending on the distance $|\mathbf{r}_i - \mathbf{r}_j|$, and (iii) is opposite to the force exerted by i on j, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. These properties, too, are axiomatic and cannot be proven from more fundamental principles. Point (iii) implements **Newton's third law**.

The total force acting on body no. i is given by the sum $\mathbf{F}_i \equiv \sum_{j\neq i} \mathbf{F}_{ij}$. Substituting this into the Newton equation (1.10), $\mathbf{F}_i = m_i \mathbf{a}_i$, we obtain the set of N coupled equations

$$\boxed{m_i \ddot{\mathbf{r}}_i = \sum_{j\neq i} \mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j).} \tag{1.11}$$

The Newton equation (1.10) with its presumed 'external' force can be interpreted as a special case of Eqs. (1.11). To see this, assume that the bodies $j \neq i$ are spatially separated from i acting on it via a net force $\mathbf{F}_i$ (Fig. 1.3, right panel) which we interpret as an external force.

To **summarize**, Newton's equations (1.11) provide the mathematical formulation of all three Newton's laws formulated in prosaic form above: if the total force $\mathbf{F}_i$ acting on particle no. i vanishes, its velocity remains constant, $d_t \mathbf{v}_i = 0$ (first law). In the presence of force, the velocity changes $d_t \mathbf{v}_i \propto \mathbf{F}_i$ in a direction set by the force (second law), and the force exerted by body i on j is opposite to the force of j on i, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ (third law).

[4] This is not the most general conceivably force: $\mathbf{F}_{ij} = \mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j, \dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)$ might depend on coordinates and velocities. Instead of two-body forces F_{ij}, we might consider three body forces $F_{i,jk}(\mathbf{r}_i, \mathbf{r}_j, \mathbf{r}_k)$ describing the correlated action of j and k on i, or even more complicated types of forces. However, for the purposes of our introductory discussion, the ansatz above is general enough.

Finally notice that the Eqs. (1.11) imply methods to **measure forces and masses**. Consider, for example a two-body system $N = 2$. From $\mathbf{F}_{12} = -\mathbf{F}_{21}$, we obtain the equation $m_1\ddot{\mathbf{r}}_1 = -m_2\ddot{\mathbf{r}}_2$, or

$$\frac{m_1}{m_2} = \frac{|\ddot{\mathbf{r}}_1|}{|\ddot{\mathbf{r}}_2|}. \tag{1.12}$$

The ratio of the two participating masses can be determined from the ratio of their acceleration. You may introduce a 'test body' to define the unit of mass (much like the kilogram stored in Paris defines the reference mass of our system of units.) The mass of any other body may then be fixed in relation to the test body by a measurement of acceleration. Similarly, you may subject the test body to different forces. Measuring its response acceleration, $\ddot{\mathbf{r}}$ then defines the magnitude of the acting force $|\mathbf{F}| = m|\ddot{\mathbf{r}}|$ as a function of measurable quantities.

The Newton equations (1.11) describe how massive bodies move in response to the presence of forces. However, as with any physics law the scope of Newtonian mechanics is limited: it describes motion at velocities $|\dot{\mathbf{r}}| \ll c$ much smaller than the speed of light, c. And it is limited to 'macroscopic bodies' large enough that the effects of quantum mechanics remain negligibly small. Within the confines imposed by these conditions, Newtonian mechanics remains an integral part of physics to this date.

INFO Newton's equations refer to quantities carrying a physical dimension, time, length, mass, etc. We will write

$$[X] = L, T, M, \ldots \tag{1.13}$$

to indicate that a physical quantity carries the dimension of length, time, mass,.... For example, $[\dot{\mathbf{r}}] = L/T$. To quantify dimensionful variables, we need to decide on a **system of units**. In our discussion of classical mechanics, we will employ the SI system, in which time is measured in **seconds** (s), mass in **kilogram** (kg), and length in **meters**, (m). These three units are fundamental, and all others derive from them. For example, the unit of velocity is $\mathrm{m/s}$, etc. The canonical unit of force is the **Newton**,

$$1\mathrm{N} = 1\mathrm{kg} \times \frac{1\mathrm{m}}{1\mathrm{s}^2}.$$

Occasionally, we will discuss the mechanics of electrically charged bodies, and this will require a fourth fundamental unit of the SI system, the **Ampere** (A), which is a unit for the strength of electric current. $(1\mathrm{A}$ is the strength of current flowing through two idealized infinitely long parallel straight wires separated by a distance of $1\mathrm{m}$ if they exert a force of $2 \times 10^{-7}\mathrm{N}$ on each other.) An important derived unit is that of electric charge, the **Coulomb** $1\mathrm{C} = 1\mathrm{A} \times 1\mathrm{s}$.

1.3 Fundamental aspects of Newtonian mechanics

In this section we discuss various fundamental structures inherent to Newtonian mechanics. This will prepare our later discussion of more concrete problems and solution strategies.

1.3.1 Newton equations as differential equations

Technically, Newton's equations (1.24) are second order ordinary differential equations for the $f = 3N$ functions $x^i_j(t)$, $j = 1, \ldots, N$. To obtain a definite solution of these equations, we need to provide initial conditions. Being second order, each equation has to be supplied with two initial conditions, i.e. we need $6N$ conditions in total. A common choice is to fix all initial coordinates, $r^i(0) = r^i_0$, and all initial velocities, $d_t r^i(0) = v^i_0$. This conforms with our intuition that, say, the motion of a flying ball can be predicted once you know it's point of origin *and* the intial velocity. Another frequent choice is to fix the initial and the finital coordinates, $r^i(0) = r^i_{\rm i}$, $r^i(t) = r^i_{\rm f}$. However, with this choice, the solution is not, in general unique:

EXAMPLE Consider the Newton equation of a single particle in a constant force in 3-direction: $d_t^2\mathbf{r} = c\mathbf{e}_3$, where c is a constant. The general solution of this equation reads $\mathbf{r}(t) = \mathbf{u}+t\mathbf{v}+c\frac{t^2}{2}\mathbf{e}_3$, with constant vectors $\mathbf{u}, \mathbf{v}$. One may now choose an initial coordinate and velocity, $\mathbf{r}_0, \mathbf{v}_0$ and compare: $\mathbf{r}_0 = \mathbf{r}(0) = \mathbf{u}$ and $\mathbf{v}_0 = d_t\mathbf{r}(0) = \mathbf{v}$ to fix the constants. Alternatively, we may fix $\mathbf{r}(0) = \mathbf{r}_{\rm i}$ and $\mathbf{r}(t_{\rm f}) = \mathbf{r}_{\rm f}$. To obtain $\mathbf{r}_{\rm i} = \mathbf{u}$, $\mathbf{r}_{\rm f} = \mathbf{u} + t_{\rm f}\mathbf{v} + c\frac{t_{\rm f}}{2}\mathbf{e}_3$.
To understand why fixation of $\mathbf{r}_{{\rm i},f}$ does not in general fix a unique solution, consider a particle confined to a move on a ring, in the absence of external forces. Now fix $\mathbf{r}_{\rm i} = \mathbf{r}_{\rm f} = \mathbf{r}_0$, where $\mathbf{r}_0$ is arbitrary on the ring. The problem is solved by the trivial trajectory $\mathbf{r}(t) = \mathbf{r}_0$, but also by trajectories spinning around the ring an integer number of times, before returning to the point of origin.

NOTATION Our so far discussion assumed the usage of cartesian coordinates x^j. When expressed in generalized coordinates – think of spherical coordinates for concreteness – the mathematical form of the equations may change. (We will see examples below.) However, several of the concepts introduced below retain validity. We therefore follow the widespread convention to denote the position coordinates describing a mechanical system by q^j, and in the follwing discuss a number of these invariant concepts of the theory.

Newton's equations for an N-particle system,

$$m_i\ddot{q}^j_i = F^j_i(\{q, \dot{q}\}) \tag{1.14}$$

are $3N$ ordinary differential equations of second order for the coordinate functions q^j_i, $i = 1, \ldots, N$, $j = 1, 2, 3$. The number of variables required to specify the problem is commonly called the number of **degrees of freedom**, f. For a Newton equation describing the unconstrained [5] motion of particles that number equals $f = 3N$. The notation $F^j_i(\{q, \dot{q}\})$ indicates that the jth component of the force $\mathbf{F}_i$ acting on the ith particle may depend on the coordinates of all other particles, and on their velocities. It is often advantageous to apply a

[5] Constraints are present if some or all coordinates of the problem are confined to a submanifold ($\rightarrow$ TP0) of $\mathbb{R}^3$. For example, to describe the motion of a ball rolling on a plane only two coordinates are required, the motion of a roller coaster cart can be described in terms of one coordinate, etc. Below, we will introduce means to efficiently describe problems with constraints.

standard procedure in the theory of differential equations to convert the system into a system of $2 \times 3N$ equations of *first* order.

To this end, we use the definition of momentum $m_i \dot{\mathbf{q}}_i \equiv \mathbf{p}_i$ to write,

$$\begin{aligned} \dot{q}_i^j &= \frac{1}{m_i} p_i^j, \\ \dot{p}_i^j &= F_i^j(\{q,p\}), \end{aligned} \tag{1.15}$$

where $F(\{q,p\})$ is a shorthand notation for the forces depending on coordinates and momenta. [6] It is customary to define a $6N$ dimensional vector

$$\mathbf{x} \equiv \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix}, \tag{1.16}$$

bundling all variables relevant to the problem into a single vector. For a system containing $f \equiv 3N$ coordinate variables, the $2f$-dimensional space spanned by the vector $\mathbf{x}$ is called the **phase space** of the problem. Phase space is a concept of crucial importance to the modern formulation of mechanics, to which we will turn below. For the moment, we just note that the phase space formulation of Newton's equations reads as

$$\boxed{\dot{\mathbf{x}}(t) = \mathbf{X}(\mathbf{x}(t)),} \tag{1.17}$$

where

$$\mathbf{X}_i^j = \begin{pmatrix} \frac{1}{m_i} p_i^j \\ F_i^j(\{q,p\}) \end{pmatrix}. \tag{1.18}$$

Technically, this is a system of $6N$ ordinary differential equations of first order in time. According to a fundamental result of the theory of differential equations, such systems possess a unique solution provided an 'inital condtion' has been specified, and the functions $\mathbf{X}$ defining the right hand side are sufficiently smooth. [7] The precise statement goes as follows: for each **initial condition** $\mathbf{x}(0)$, the system of differential equations (1.17) possesses a unique **solution trajectory** $\mathbf{x}(t)$. The trajectory $\mathbf{x}(t)$ varies smoothly (i.e. differentiably) on the initial condition $\mathbf{x}(0)$. The physical interpretation of these statements is that if an initial set of coordinates $\mathbf{q}(0)$ and momenta $\mathbf{p}(0)$ is specified (think of the initial coordinate $\mathbf{q}(0)$ and momentum $\mathbf{p}(0) = m\mathbf{v}(0)$ of a thrown ball) then the Newton equations specify the future development $\mathbf{q}(t)$ and $\mathbf{p}(t)$ of these variables. The motion responds smoothly to changes in the initial conditions (e.g., if you slightly change the initial velocity of the ball, then its trajectory will change smoothly in response [8])

[6] Since $\dot{q}_i^j = p_i^j/m_i$, $F(\{q,p\})$ is trivially obtained from $F(\{q,\dot{q}\})$.

[7] More precisely, ($\rightarrow$ TP0) the functions $\mathbf{X}$ need to specify a so-called Lipschitz continuity condition.

[8] Notice however, that 'smooth' changes need not remain small at large times. It is a defining property of chaotic mechanical systems that even small adjustments of the initial conditions lead to massive changes in the evolution trajectories at large times!

1.3.2 The invariance group of Newtonian mechanics

Newton's equations make reference to a set of coordinates $(\mathbf{r}, t)$ parameterizing points X of Galilean space, $\mathbb{E}^4$. To define these coordinates, we need a reference system K, which in turn implies the definition of a point of origin, $\mathbb{E}^4 = O + \mathbb{R}^4$, and the choice of a basis spanning $\mathbb{R}^4$. If we choose a different reference system K' the space time coordinates of X will change to $(\mathbf{r}', t')$. We call Newton's equations **invariant** under the transformation, if their representation in the new system reads

$$m_i \frac{d^2\mathbf{r}'_i}{dt'^2} = \sum_{j\neq i} \mathbf{F}'_{ij}(\mathbf{r}'_i - \mathbf{r}'_j), \tag{1.19}$$

where $\mathbf{F}'_{ij}$ are the force vectors $\mathbf{F}_{ij}$ represented in the new coordinate system. 'Invariance' means that the Equations are structurally equivalent to the equations in K. To appreciate the significance of this criterion, consider the transformation to a system K' whose point of origin $O' = O - \frac{t^2}{2}\mathbf{a}_0$ performs accelerated motion relative to that of K in space. The coordinates of the two systems are related to each other as

$$\begin{aligned} t' &= t, \\ \mathbf{r}' &= \mathbf{r} + \frac{t^2}{2}\mathbf{a}_0. \end{aligned} \tag{1.20}$$

The transformation of a K-Newton equation $m d_t^2 \mathbf{r} = \mathbf{F}$ to the new system reads $m d_{t'}^2 \mathbf{r}' = m d_t^2(\mathbf{r} + \frac{t^2}{2}\mathbf{a}_0) = \mathbf{F} + m\mathbf{a}_0 = \mathbf{F}' + m\mathbf{a}_0$, where we used that $\mathbf{F} = \mathbf{F}(\{\mathbf{r}_i - \mathbf{r}_j\}) = \mathbf{F}(\{\mathbf{r}'_i - \mathbf{r}'_j\}) = \mathbf{F}'$ does not change under the transformation. The appearance of an additional term $m\mathbf{a}_0$ signifies non-invariance of the Newton equation. Specifically, an equation that is force-free in K, $\mathbf{F} = 0$, appears to be contain a force $m\mathbf{a}_0$ in K'. To understand the origin of this force, imagine a body at rest in K. In K' it will appear to be at position $\mathbf{r}' = t^2\mathbf{a}_0/2$, i.e. it moves at accelerated velocity $\mathbf{v}' = t\mathbf{a}_0$. From a K' perspective, a force $m\mathbf{a}_0$ causing this acceleration must act. We call forces that can be removed by a transformation of coordinates **fictitious forces**. Forces of this this type appear and disappear under changes of coordinate systems. However, before furthering on this point, let us identify the set of coordinate transformations that leave the Newton equations invariant.

All points in Galilean space are equal, and so are all 'directions'. We therefore expect that transformations changing the point of origin, or the geometric orientation of coordinate axes leave Newton's equations invariant. Indeed, it is straightforward to verify that the following families of transformations do satisfy the invariance criterion:

- ▷ **Translation in both space and time**:

$$\begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix} = \begin{pmatrix} t + s \\ \mathbf{r} + \mathbf{q} \end{pmatrix}, \tag{1.21}$$

  with constant $s, \mathbf{q}$.

▷ **Rotation of space**:

$$\begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix} = \begin{pmatrix} t' \\ R\mathbf{r} \end{pmatrix}, \tag{1.22}$$

with a rotation matrix $R \in \mathrm{SO}(3)$.

▷ **Uniform motion** of the reference systems:

$$\begin{pmatrix} t' \\ \mathbf{r}' \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{r} + \mathbf{v}t \end{pmatrix}, \tag{1.23}$$

with fixed $\mathbf{v}$.

It is less straightforward to prove that there are no other transformations satisfying the invariance criterion. [9]

EXERCISE Convince yourself of the invariance of Newton's equations under the transformations above. Observe that the rotation of space, requires that all vectors, including the forces entering Newton's equation, $\mathbf{F} \mapsto R\mathbf{F}$ be transformed. Try to convince yourself that no other invariance transformations can be found.

Compositions of the elementary transformations above, e.g., rotation followed by translation, are again invariance transformations, and each transformation has an inverse. The set of transformations therefore forms a group, the **invariance group of classical mechanics**. You will have noticed that this group equals the Galilei group, i.e. we have arrived at the conclusion that

The Galilei group is the invariance group of classical mechanics.

Two coordinate systems connected to each other by a Galilei transformation are called **relative intertial** to each other. We have seen above how starting from a force free system K we may transform by a *non*-Galilei transformation into a system K' in which a force acts. However, this force is fictitious in that it can be eliminated by a coordinate transformation, viz. the reverse transformation to K. This observation motivates the definition of **inertial systems** as the class of those reference systems K in which no fictitious forces are present. The complementary class of forces, i.e. forces that cannot be eliminated by coordinate transformation are called **intrinsic forces**. The Newton equations in inertial systems contain only those, and in this sense are distinguished for a maximal degree of simplicity. Keep in mind, however, that even fictitious forces can be quite real in the systems where they act. For example, an observer

[9] In fact, there *are* more general coordinate transformations preserving the form of Newton's equations, cf. O. Jahn, V. V. Sreedhar, *The Maximal Invariance Group of Newtons's Equations for a Free Point Particle*, Am.J.Phys. **69**, 1039 (2001). However, generalization beyond the Galilei group do not appear to play a significant role in the practice of mechanics (note the publication date of the given reference!) and will not be discussed here.

locked into a windowless cabin dragged at accelerated speed $\mathbf{v}_0 = \text{const.} \times t$ through an otherwise force-free universe will feel a *real* (though fictitious) force pressing her to the bottom of the cabin. However, an observer at rest (relative to the cabin) wouldn't feel any force.

INFO Observe that in classical mechanics time and space stand on similar footing – both can be translated, etc. – while there are also peculiar 'asymmetries'. For example, space can be rotated within itself, but rotations 'mixing' time and space are excluded. On the one hand, this seems natural (what would a 'rotation' between space and time be like anyway?). On the other hand, the **mathematical asymmetry between space and time** does not feel quite right and hints at the existence of a generalized theory where it is lifted. The separation between space and time is indeed abandoned in relativistic mechanics, a generalization of Newtonian mechanics capable of handling fast motion at velocities comparable to the speed of light. From the perspective of relativistic mechanics, the separation between space and time emerges as an artefact in the limit of slow motion dynamics.

On a related note, the definition of **inertial systems** as given above, also has a sense of vagueness to it. How to we really discriminate between real and fictitious forces. For example, an observer locked into a windowless cabin which is dragged at uniformly accelerated motion will have no constructive way to discriminate the acting force (a 'fictitious force' according to our discussion) from a gravitational force. Conversely, imagine a universe in which a constant 'real' force field acts. Assume that all bodies in the universe respond to the force by accelerated motion in its direction. A reference frame fixed performing this accelerated motion will then feel inertial (why?). But according to our discussion above it is not. These seemingly paradoxical examples show that the problem lies with unambiguous global definitions of inertial frames, whatever 'global' means in our universe. In the nineteenth century there has been a lot of discussion of this point. However, the proper interpretation of of the term 'inertial' became clear only after the advent of reality.

1.3.3 Center of mass and relative coordinates

Technically, Newton's equations are second order ordinary differential equations for the $3N$ functions $x_i^j(t)$, where N is the number of particles. Only in few cases can such equations be solved explicitly. However, there exist a number of strategies by which the problem can be simplified by using appropriate systems of coordinates and/or exploiting so called conservation laws. In this section, we introduce a system of variables, suitably to the description of mechanic's principal conservation laws below.

Consider the general system of Newton's equations

$$m_i \ddot{\mathbf{r}}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_{\mathrm{e},i}, \tag{1.24}$$

for a system of N particles, subject to both 'internal forces' $\mathbf{F}_{ij}$ exerted by particle no.j on i, and 'external' forces $\mathbf{F}_{\mathrm{e},i}$ exerted by the 'rest of the universe' on particle no.i. For example, the $\mathbf{r}_i$ might be the coordinates of charged (and hence interacting) particles moving in an

accelerator, and $\mathbf{F}_{\mathrm{e},i}$ would be the forces caused by an external magnetic field keeping the particles in focus, etc.

The sum,

$$\mathbf{F}_{\mathrm{e}} \equiv \sum_i \mathbf{F}_{\mathrm{e},i}, \tag{1.25}$$

is called the total **external force** acting on the system. And the sum of all particle momenta

$$\mathbf{P} \equiv \sum_i \mathbf{p}_i \tag{1.26}$$

is called the **total momentum** of the system. The total momentum changes in response to the external force acting on the system,

$$\boxed{\dot{\mathbf{P}} = \mathbf{F}_{\mathrm{e}}.} \tag{1.27}$$

To prove, equation (1.27), we compute

$$d_t\mathbf{P} = \sum_i m_i \ddot{\mathbf{r}}_i \overset{(1.24)}{=} \sum_{i\neq j} \mathbf{F}_{ij} + \sum_i \mathbf{F}_{\mathrm{e},i} = \sum_i \mathbf{F}_{\mathrm{e},i} = \mathbf{F}_{\mathrm{e}}, \tag{1.28}$$

where in the crucial third equality we used that in $\sum_{i\neq j} \mathbf{F}_{ij}$ we sum over $\mathbf{F}_{ij}$ and $\mathbf{F}_{ji} = -\mathbf{F}_{ji}$, i.e. the sum vanishes.[10] Notice how equation (1.27) resembles the equation of motion of a 'superparticle' of momentum $\mathbf{P}$ and subject to an external force $\mathbf{F}_{\mathrm{e}}$. We further this interpretation by defining the **total mass of the system**,

$$M \equiv \sum_i m_i, \tag{1.29}$$

and its **center of mass**,

$$\mathbf{R} \equiv \frac{1}{M} \sum_i m_i \mathbf{r}_i, \tag{1.30}$$

The center of mass is that point in which the mass of the system appears to be concentrated if we look at it from far away. For example, the center of mass of the system (earth/Mars) is situated somewhere on the axis connecting the center of these planets. The total momentum can now be expressed as $\mathbf{P} = M d_t\mathbf{R}$, in analogy to the formula of a point particle.

It is often convenient to split the coordinates of the particles constituting the system as

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i, \tag{1.31}$$

[10] To see this more explicitly, write $\sum_{i\neq j} \mathbf{F}_{ij} = \sum_{i<j} \mathbf{F}_{ij} + \sum_{i>j} \mathbf{F}_{ij} = \sum_{i<j} \mathbf{F}_{ij} + \sum_{i<j} \mathbf{F}_{ji} = \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) = 0$, where in the second last step we relabeled summation indices $i \leftrightarrow j$

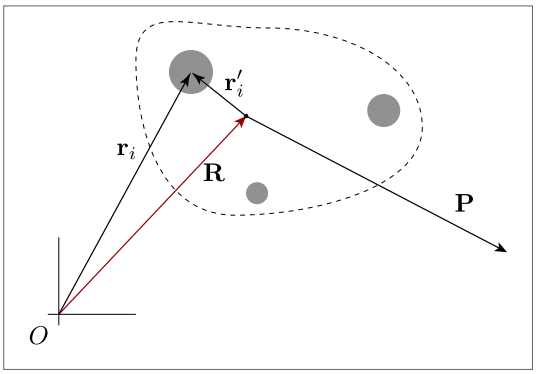
where the vector describing the separation of particle i from the center of mass, $\mathbf{r}'_i$, is called its **relative coordinate**. Similarly, we define $\mathbf{v}'_i = d_t\mathbf{r}'_i$ as the **relative velocity**, and $\mathbf{p}'_i = m_i\mathbf{v}'_i$ as the **relative momentum**. The relative coordinate, velocity and momentum describe the motion of a particle in a reference system whose origin is the center of mass $\mathbf{R}+O$. In many cases, it advantageous to address the dynamics of an N particle system within the system of its relative coordinates. The relative coordinates aren't all independent, because

$$\sum_i m_i\mathbf{r}'_i = \sum_i m_i\mathbf{r}_i - \sum_i m_i\mathbf{R} = M\mathbf{R} - M\mathbf{R} = 0. \tag{1.32}$$

So, if you know $N-1$ of the relative vectors $\mathbf{r}_i$, the Nth one follows as $\mathbf{r}'_N = m_N^{-1}\sum_{i=1}^{N-1} m_i\mathbf{r}'_i$. A typical solution strategy will first aim to understand the motion of the center of mass by solution of the relatively simple Eq. (1.27). One may then turn to the system of relative coordinates, and aim to solve the Newton equation for these. For $N \gg 1$ there are many more relative coordinates than center of mass coordinates, i.e. the relative system is generally harder to solve.

1.3.4 Conservation laws

Newton's equations simplify considerably if **conserved quantities** are present. Technically, a conserved quantity is a function $f(\mathbf{r}_1,\ldots,\mathbf{r}_n,\dot{\mathbf{r}}_1,\ldots,\dot{\mathbf{r}}_n)$ of the coordinates and perhaps the velocities of the problem, such that $d_t f = 0$, i.e. f does not change in time. The algebraic equation $f = \text{const.}$ enables us to express one of the unknown variables in terms of the others, i.e. the number of unknowns has effectively reduced by one. What here sounds like an abstract statement is familiar to all of us from daily experience. For example, we all know that in the absence of forces acting in 1-direction, a mass will retain its velocity in 1-direction, i.e. we have a conserved quantity $\dot{x}_1 = \text{const.}$. Intuitive knowledge of this fact is used when we aim to catch a flying ball and subconsciously 'calculate' its point of impact using the approximate conservation of it's velocity component parallel to the ground.



Momentum

The equations $f = \text{const.}$ stating the conservation of a quantity are called **conservation laws**. For the one-body problem Eq. (1.10), the conservation law above is formulated as $\mathbf{F} = 0 \Rightarrow d_t m\dot{\mathbf{r}} = d_t\mathbf{p} = 0$, i.e. $\mathbf{p} = \text{const.}$:

The absence of forces implies the conservation of momentum.

This is no more than a reformulation of Newton's first law. Also notice that $\mathbf{p} = \text{const.}$ gives us *three* scalar conservation laws, $d_t r^i = \text{const.}$ The most general motion compatible with the conservation law reads $r^i = a^i t + b^i$, where the constants have to be adjusted so as to conform with the intial conditions of the problem. In this simple case, the conservation law enables us to trivially solve the problem.

Applied to an N-particle system, the correspondence above reads

> In the absence of external forces, $\mathbf{F}_\text{e} = 0$, the total momentum of an N-particle system is conserved, $\mathbf{P} = \text{const.}$.

This is our first example of a conservation law. Try to make yourself familiar with its intuitive meaning. For example, a galaxy moving through space will be subject to all kinds of internal forces. However, if there aren't any substantial forces acting on the system as a whole, the galaxy will appear to move as one big 'metaparticle' through empty space. The precise formulation of this statement is that it's total momentum does not change.

Angular momentum

Another conservation law familar from daily life relates to the motion of spinning bodies. A wheel in motion opposes changes in its rotation axis, a frisbee disc tends to maintain its rotation axis, etc.

The physical quantity behind these phenomena is called **angular momentum**. The angular momentum of a point particle at $\mathbf{r}$ and carrying momentum $\mathbf{p}$ relative to a point $\mathbf{r}_0$ is defined as

$$\boxed{\mathbf{l} = (\mathbf{r} - \mathbf{r}_0) \times \mathbf{p}.} \tag{1.33}$$

Notice that angular momentum is always defined relative to a reference point $\mathbf{r}_0$. Changes in angular momentum are caused by the action of a **torque**, $\mathbf{n} \equiv (\mathbf{r} - \mathbf{r}_0) \times \mathbf{F}$, where $\mathbf{F}$ is the force acting on the body:

$$\boxed{d_t \mathbf{l} = \mathbf{n}.} \tag{1.34}$$

Using $(d_t \mathbf{r}) \times \mathbf{p} = m\mathbf{v} \times \mathbf{v} = 0$, the equation is seen to be a consequence of the product rule and Newton's equation. As with angular momentum, torque is defined relative to a reference point. The equation implies another important conservation law,

> In the absence of torque, the angular momentum of a particle is conserved.

For example, the angular momentum of a satellite at $\mathbf{r}$ relative to earth's center $\mathbf{r}_0$ is conserved as long as forces other than the gravitational force F_g may be ignored. The reason is that $F_g \propto (\mathbf{r} - \mathbf{r}_0)$ acts towards the gravitational center, and this means $\mathbf{n} = 0$. Angular momentum

is conserved, e.g., for motion on a circular orbit at constant velocity, $|\mathbf{v}| = \mathrm{const}$. However, as we will see, it may also be conserved along more complex trajectories.

The angular momentum $\mathbf{L}$ of a system of particles relative to $\mathbf{r}_0$ is given by the sum of partial angular momenta

$$\boxed{\mathbf{L} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{p}_i.} \tag{1.35}$$

Changes in this quantity are caused by the total torque

$$\mathbf{N}_\mathrm{e} = \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_{\mathrm{e},i} \tag{1.36}$$

due to the *external* forces $\mathbf{F}_{\mathrm{e},i}$ acting on the particles:

$$\boxed{d_t \mathbf{L} = \mathbf{N}_\mathrm{e},} \tag{1.37}$$

i.e. the internal forces of the system do not alter its angular momentum. The equation is proved by straightforward calculation:

$$\begin{aligned} d_t \mathbf{L} &= \sum_i [\underbrace{\dot{\mathbf{r}}_i \times \mathbf{p}_i}_{0} + (\mathbf{r}_i - \mathbf{r}_0) \times \dot{\mathbf{p}}_i] = \\ &= \sum_{i \neq j} (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_{ij} + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_{i,\mathrm{e}} = \\ &= \frac{1}{2} \sum_{i \neq j} \underbrace{[(\mathbf{r}_i - \mathbf{r}_0) - (\mathbf{r}_j - \mathbf{r}_0)] \times \mathbf{F}_{ij}}_{0} + \mathbf{N}_\mathrm{e} = \mathbf{N}_\mathrm{e}, \end{aligned} \tag{1.38}$$

where in the third line, we symmetrized the sum, $\sum_{i \neq j} X_{ij} = \frac{1}{2}(\sum_{i,j} X_{ij} + \sum_{j,i} X_{ji}) = \frac{1}{2} \sum_{i,j} (X_{ij} + X_{ji})$, and used that $(\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = 0 \propto (\mathbf{r}_i - \mathbf{r}_j) \times (\mathbf{r}_i - \mathbf{r}_j) = 0$ due to the directional orientation of the pair forces. The qualitative reason for the vanishing effect of the internal forces on the angular momentum is that the torque exerted by the pair force $\mathbf{F}_{ij}$ on the system is compensated by the opposite effect of the force $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$ (think about this point.) We conclude that

> In the absence of external torque $\mathbf{N}_\mathrm{e}$ the angular momentum $\mathbf{L}$ of an N-particle system is conserved.

As an example, consider the dynamics of a rotating wheel. The angular momentum of the wheel relative to a point on the wheel axis points in the direction of that axis. It is not affected by the complicated system of forces giving the wheel its static stability. Only if an external force with a component perpendicular to the rotation axis is applied will the angular momentum change.

We finally note that the angular momentum of an N-particle system affords a decomposition

$$\mathbf{L} = (\mathbf{R} - \mathbf{r}_0) \times \mathbf{P} + \sum_i (\mathbf{r}'_i \times \mathbf{p}'_i), \tag{1.39}$$

into a contribution resembling the angular momentum of a point particle of momentum $\mathbf{R}$ at the center of mass coordinate $\mathbf{R}$, plus an 'internal' contribution summing over the angular momenta of the compound particles relative to the center of mass.

EXERCISE Prove the formula, and think of examples illustrating its contents.

Energy

Let $\mathbf{r}(t)$ be the solution of Newton's equations over the time interval $I = [t_1, t_2]$. The assignment $t \mapsto \mathbf{r}(t)$ defines a curve $\gamma : I \to \mathbb{R}^3, t \mapsto \mathbf{r}(t)$ in three-dimensional space.[11] Now consider the **work**

$$W \equiv \int_\gamma d\mathbf{s} \cdot \mathbf{F} \tag{1.40}$$

done along this curve against a *time independent* force $\mathbf{F}(\mathbf{r})$ acting on the particle. We compute this quantity as

$$\begin{aligned} W &= \int_{t_1}^{t_2} dt \, \frac{d\mathbf{r}}{dt} \cdot \mathbf{F} = \int_{t_1}^{t_2} dt \, \frac{d\mathbf{r}}{dt} m \frac{d^2\mathbf{r}}{dt^2} = \\ &= \frac{m}{2} \int_{t_1}^{t_2} dt \, \frac{d}{dt} \left(\frac{d\mathbf{r}}{dt} \right)^2 = \frac{m}{2} \left(\mathbf{v}^2(t_2) - \mathbf{v}^2(t_1) \right). \end{aligned} \tag{1.41}$$

We denote the quantity

$$\boxed{T \equiv \frac{m}{2} \mathbf{v}^2} \tag{1.42}$$

as the **kinetic energy** of the particle. The formula above states that the work done against the force acting on the particle equals the change in kinetic energy. An even stronger connection between work against the force and kinetic energy emerges if the **force vector field**

$$\begin{aligned} \mathbf{F} : \mathbb{R}^3 &\to \mathbb{R}^3, \\ \mathbf{r} &\mapsto \mathbf{F}(\mathbf{r}) \end{aligned} \tag{1.43}$$

is a **conservative** vector field. Recall ($\to$ TP0) that a vector field is conservative if there exists a function $U(\mathbf{r})$ such that $\mathbf{F}(\mathbf{r}) = -\nabla U(\mathbf{r})$[12] The function U is called the **potential of**

[11] More precisely, $\mathbf{r}(t)$ defines a specific parameterization of the geometric curve γ, think about this difference.

[12] The minus sign appearing in this formula is a matter of convention.

the force. Also recall that the line integral against a conservative force field is obtained as

$$W = \int_\gamma d\mathbf{s}\cdot\mathbf{F} = -\int_{t_1}^{t_2} dt\,\frac{d\mathbf{r}}{dt}\cdot\nabla U(\mathbf{r}) = -\int_{t_1}^{t_2} dt\,\frac{dU(\mathbf{r}(t))}{dt} = -(U(\mathbf{r}(t_2)) - U(\mathbf{r}(t_1))). \tag{1.44}$$

Comparing our two formulas for W, we obtain

$$\frac{m}{2}\left(\mathbf{v}^2(t_2) - \mathbf{v}^2(t_1)\right) = -(U(\mathbf{r}(t_2)) - U(\mathbf{r}(t_1))), \tag{1.45}$$

or

$$T(\dot{\mathbf{r}}(t_2)) + U(\mathbf{r}(t_2)) = T(\dot{\mathbf{r}}(t_1)) + U(\mathbf{r}(t_1)). \tag{1.46}$$

Since the time arguments t_2, t_1 can be chosen arbitrarily, we conclude that the sum $T+U$ does not change along the trajectory $\mathbf{r}(t)$. This observation motivates the definition of the **energy of the particle** as

$$\boxed{E = T + U}. \tag{1.47}$$

We may think of the energy as a function of both, the particles instantaneous position $\mathbf{r}$, and it's velocity $\mathbf{v}$, $E = T(\mathbf{v}) + U(\mathbf{r})$. Equivalently, we may think of it as a characteristic of the particle's trajectory: given a trajectory $\mathbf{r}(t)$ (i.e. a solution of the Newton equation), we may compute the velocity vector $\mathbf{v}(t) = d_t\mathbf{r}(t)$ at any instance of time to obtain the energy as $E = T(d_t\mathbf{r}) + U(\mathbf{r})$. The **conservation of energy** along the trajectory may be confirmed as

$$d_t E = d_t(T(\dot{\mathbf{r}}) + U(\mathbf{r})) = \partial_{\dot{\mathbf{r}}} T\cdot\ddot{\mathbf{r}} + \partial_{\mathbf{r}} U\cdot\dot{\mathbf{r}} = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} - \mathbf{F}(\mathbf{r})\cdot\dot{\mathbf{r}} = (\underbrace{m\ddot{\mathbf{r}} - \mathbf{F}}_{0})\cdot\dot{\mathbf{r}}. \tag{1.48}$$

The physical interpretation of energy conservation is illustrated in Fig. 1.4. The particle moves in a 'potential landscape' described by the function $U(\mathbf{r})$. A force $\mathbf{F} = -\nabla U$ pointing in the direction of the fastest *decrease*[13] of U. If the particle moves in the direction of that force, U decreases. At the same time, the kinetic energy T (and hence the magnitude of it's velocity) increases. The decrease of U and the increase of T are balanced such that $E = T+U$ remains constant.

The conservation law corresponding to our observations above states that

> The energy of a particle subject only to conservative forces is conserved, $d_t E = 0$.

EXAMPLE As an example of a conversion of potential to kinetic energy consider a ball of mass m being dropped from a hight h in the presence of a downward gravitational force $\mathbf{F} = -mg\mathbf{e}_z$[14] This force is conservative and it's potential energy is given by $U(\mathbf{r}) = mgz$. When the ball is

[13] Recall that for a function f, ∇f is a vector pointing in the direction of fastest increase fo f.

[14] Here, $g \simeq 9.81\mathrm{m/s}^2$ is the so called **gravity of earth**.

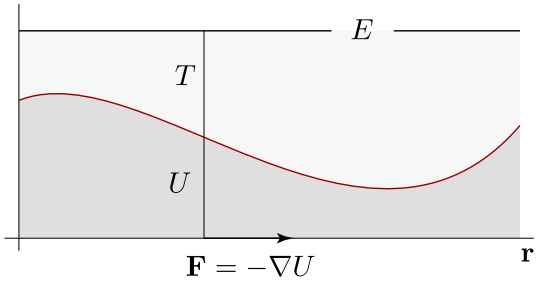


Figure 1.4: On the conservation of energy along the trajectory of a particle. Discussion, see text.

released it has energy $E = T + U = 0 + mgh$. The solution of Newton's equations subject to the initial conditions $\mathbf{r}(0) = h\mathbf{e}_z$ and $\dot{\mathbf{r}}(0) = 0$ reads $\mathbf{r}(t) = (h - \frac{gt^2}{2})\mathbf{e}_z$. The ball hits the surface $z = 0$ at time $t_0 = \sqrt{2h/g}$ at a velocity $\mathbf{v}(t_0) = -gt_0\mathbf{e}_z = -\sqrt{2hg}\mathbf{e}_z$. At the point of impact, its energy is given by $E = T + U = \frac{m}{2}\mathbf{v}(t_0)^2 + 0 = mgh$, i.e. the potential energy has been fully converted into kinetic energy.
As an example for non-conservative forces, consider the winds of a circulating weather system, e.g. a hurrican. The integral of the force field around the storm's center is non-vanishing, and this signifies the non-gradientness of the acting forces. A particle subject to such forces will tend to spin around the center, at growing speed. It's kinetic energy increases, but there is no compensating potential energy.

The total **energy of an N-particle system** can be defined in similar terms. Assume that the pair forces are conservative in the sense that they derive from a potential as [15]

$$\mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j) = -\partial_{\mathbf{r}_i} U(|\mathbf{r}_i - \mathbf{r}_j|), \tag{1.49}$$

i.e. that there exists a **pair potential** function

$$\begin{aligned} U : \mathbb{R}^3 &\to \mathbb{R}, \\ \mathbf{r} &\mapsto U(|\mathbf{r}|), \end{aligned} \tag{1.50}$$

such that $U(|\mathbf{r}_i - \mathbf{r}_j|)$ measures the 'interaction potential' of two particles separated by a distance $|\mathbf{r}_i - \mathbf{r}_j|$. It is customary to omit the $|\cdot|$ in the notation and to write $U(\mathbf{r}) = U(|\mathbf{r}|)$ for simplicity. Further assume that the external forces are conservative, too,

$$\mathbf{F}_{\mathrm{e}}(\mathbf{r}) = -\partial_{\mathbf{r}} U_{\mathrm{e}}(\mathbf{r}), \tag{1.51}$$

where U_{e} is the **external potential**. Under these conditions, the **energy of the system** may be defined as

$$\begin{aligned} E =& T + U, \\ T =& \sum_i \frac{m_i}{2} \dot{\mathbf{r}}_i^2, \\ U =& \frac{1}{2} \sum_{i \neq j} U(\mathbf{r}_i - \mathbf{r}_j) + \sum_i U_{\mathrm{e}}(\mathbf{r}_i). \end{aligned} \tag{1.52}$$

[15] If the particles differ in their physical properties (charge, etc.) the pair potential functions $U_{ij}(\mathbf{r})$ may explicitly depend on the particle type, and carry an indentifying index. However, that complication does not change our conclusions below, and we will omit the index for simplicity.

Conserved quantity	1 particle	N-particle system
momentum	$\mathbf{F} = 0$	$\mathbf{F}_{i,\mathrm{e}} = 0$
angular momentum	$\mathbf{N} = 0$	$\mathbf{N}_{i,\mathrm{e}} = 0$
energy	$\mathbf{F}$ conservative	$\mathbf{F}_{ij}$ and $\mathbf{F}_{i,\mathrm{e}}$ conservative

Much like in the single particle case, it turns out that

> The energy E of an N-particle system governed by conservative forces is conserved, $d_t E = 0$.

Energy conservation is proven as

$$\begin{aligned} d_t E &= d_t T + d_t U = \\ &= \sum_i m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i + \frac{1}{2} \sum_{i \neq j} \left[\partial_{\mathbf{r}_i} U(\mathbf{r}_i - \mathbf{r}_j) \cdot \dot{\mathbf{r}}_i + \partial_{\mathbf{r}_j} U(\mathbf{r}_i - \mathbf{r}_j) \cdot \dot{\mathbf{r}}_j \right] + \sum_i \partial_{\mathbf{r}_i} U_{\mathrm{e}}(\mathbf{r}_i) \cdot \dot{\mathbf{r}}_i = \\ &= \sum_i \dot{\mathbf{r}}_i \underbrace{\left[m_i \ddot{\mathbf{r}}_i - \sum_{j \neq i} F_{ij}(\mathbf{r}_i - \mathbf{r}_j) - \mathbf{F}_{\mathrm{e}}(\mathbf{r}_i) \right]}_{0}, \end{aligned} \tag{1.53}$$

where in the last equality, we exchanged summation indices $i \leftrightarrow j$ as done before, and noted $\partial_{\mathbf{r}_i} U(\mathbf{r}_j - \mathbf{r}_i) = -\partial_{\mathbf{r}_i} U(\mathbf{r}_i - \mathbf{r}_j)$ (why?).

The table below summarizes the **fundamental conserved quantities of classical mechanics** and the conditions under which they are conserved.

This concludes our survey of the **fundamental structures of classical mechanics**. To summarize,

- ▷ We have introduced **Galilei space time** as the fundamental domain of definition of mechanics,
- ▷ introduced the **Newtonian formulation** of mechanics,
- ▷ explored its fundamental symmetries and invariances, the **Galilei group**,
- ▷ and assured the **general solubility of Newton's equations**.

In the following sections we will illustrate the functioning of the formalism on a number of examples

1.4 Examples and applications

In this section we will discuss various examples of Newton's equations and their solution. While the first three problems can be solved straightforwardly, the solution of the fourth – the Kepler problem – relies on a careful analysis of symmetries and conservation laws.

1.4.1 Motion in a constant force field

Baring the force-free motion, this is the second simplest problem of classical mechanics. We consider a point particle of mass m subject to a constant force $\mathbf{F} = F\mathbf{e}_1$, where we chose a coordinate system such that the force vector is colinear to the unit vector $\mathbf{e}_1$. The equation $d_t\mathbf{p} = \mathbf{F}$ then has the uniqe solution

$$p^1(t) = p^1(0) + Ft, \qquad p^2(t) = p^2(0), \qquad p^3(t) = p^3(0). \tag{1.54}$$

The second half of Newton's equations $d_t\mathbf{q} = m^{-1}\mathbf{p}$ is solved by

$$q^1(t) = \frac{F}{2m}t^2 + \frac{p^1(0)}{m}t + q^1(0), \qquad q^2(t) = \frac{p^2(0)}{m}t + q^2(0), \qquad q^3(t) = \frac{p^3(0)}{m}t + q^3(0). \tag{1.55}$$

Observe the uniqueness of the solutions once the six initial conditions $q_i(0)$ and $p_i(0)$ are specified.

EXAMPLE Conisder a ball of mass m thrown with velocity $\mathbf{v}(0) = (v^1, v^2, v^3)^T$, $v^3 > 0$ from a point at sea level, $q_3(0) = 0$. Assuming the presence of a gravitational force $\mathbf{F} = -mg\mathbf{e}_3$ and ignoring friction, how long will it take the ball to hit the ground? How far will its landing point be away from the point of departure $\mathbf{q}(0) = (q^1(0), q^2(0), 0)^T$?

1.4.2 Harmonic oscillator

The harmonic oscillator

Consider a particle subject to a force $\mathbf{F} = -q^1 m\omega^2\mathbf{e}_1$, where ω is a constant of dimensionality $[\text{time}]^{-2}$. For simplicity, we assume that the particle does not move in $2-$ and $3-$direction, $d_t q^{2,3} = 0$, and set $q^{2,3} = 0$ for simplicity. Since the force acts only in $1-$direction the problem then effectively reduces to a one-dimensional one, and we write $q^1 \equiv q$ for simplicity. Newton's equations read

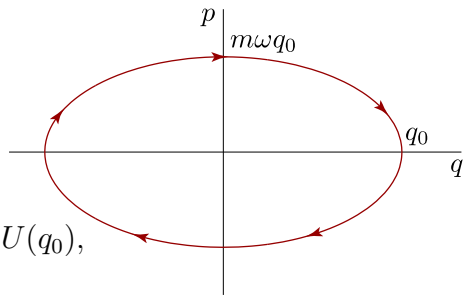
$$\begin{aligned} d_t q &= \frac{p}{m}, \\ d_t p &= -m\omega^2 q. \end{aligned} \tag{1.56}$$

As detailed in the next seection, the solution to these equations is given by

$$\begin{aligned} q(t) &= q(0)\cos(\omega t) + \frac{1}{m\omega}p(0)\sin(\omega t), \\ p(t) &= -m\omega q(0)\sin(\omega t) + \cos(\omega t)p(0). \end{aligned} \tag{1.57}$$

Consider, for example, a particle released at $q(0) = q^0$ at zero velocity, $p(0) = 0$. The particle will begin to move towards the origin, picking up speed in the proccees. At time $t = \pi/2\omega$ it passes the origin at maximal (negative) speed $-q^0\omega$, shoots over towards the point of maximal negative extension $q = -q^0$, which is reached at time $t = \pi/\omega$. The motion then reverses towards the origin, which is passed again at time $t = 3\pi/2\omega$, this time in positive direction. At time $t = 2\pi/\omega$ the particle has reached its point of origin again, and the process repeats.

It is instructive to take a look at the **energy of the particle**. The acting force is conservative and derives from the **potential energy** $U = \frac{m\omega^2}{2}(q^1)^2$, i.e. $\mathbf{F} = -\nabla U = -q^1 m\omega^2 \mathbf{e}_1$. The sum of potential and **kinetic energy** $T = \frac{m}{2}\dot{q}^2$ is given by



$$E = T + U = \frac{m}{2}\omega^2 q_0^2(\sin^2(\omega t) + \cos^2(\omega t)) = \frac{m}{2}\omega^2 q_0^2 = U(q_0), \tag{1.58}$$

and remains conserved. At the point of departure, the energy of the particle is purely potential, $T = 0$. Along its way towards the origin, the energy gets converted into kinetic energy, and becomes purely kinetic at $q = 0$, where $U = 0$. The conversion of potential to kinetic energy periodically repeats, but the sum of the two contributions remains constant.

Solution of harmonic oscillator differential equation

How are the solutions of Eq. (1.56) obtained? We first bring the equations into a more symmetric form by introducing scaled variables,

$$q \equiv \frac{1}{\sqrt{m}\omega} z^1, \qquad p \equiv \sqrt{m} z^2, \tag{1.59}$$

in terms of which

$$\begin{aligned} d_t z^1 &= \omega z^2, \\ d_t z^2 &= -\omega z^1. \end{aligned} \tag{1.60}$$

or

$$d_t \mathbf{z} = \omega \tau \mathbf{z} \tag{1.61}$$

where we defined $\mathbf{z} = (z^1, z^2)^T$, and $\tau \equiv \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ These equations are simple enough to guess the solution

$$\begin{aligned} z^1(t) &= \cos(\omega t) z^1(0) + \sin(\omega t) z^2(0), \\ z^2(t) &= -\sin(\omega t) z^1(0) + \sin(\omega t) z^2(0), \end{aligned}$$

or

$$\mathbf{z}(t) = \begin{pmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{pmatrix} \mathbf{z}(0) = (\cos(\omega t)\mathbb{I} + \sin(\omega t)\tau)\mathbf{z}(0). \tag{1.62}$$

Upon re-introducing the original variables (q, p), we obtain Eq. (1.57). To understand how these solutions are obtained without guessing, we note that we are dealing with a system of linear first order differential equations with constant coefficients. A single equation of

this type, $d_t z = \omega z$ would be solved by an exponential, $z(t) = z(0)\exp(\omega t)$. As we show next, the exponential function plays a key role in the solution of more complex linear first order differential equations as well. By analogy to the formula above, it is now tempting to write a formula such as $\mathbf{z}(t) = \exp(\omega\tau t)\mathbf{z}(0)$, only that we do not know the meaning of the exponential of a *matrix*.

INFO **Functions of matrices** are generally defined by their Taylor series expansion. Consider a function $f(x)$ with expansion

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} x^n \tag{1.63}$$

around some argument 0, where $f^{(n)}$ denotes the nth derivative. For an abitrary square matrix, we then define

$$\boxed{f(A) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} A^n,} \tag{1.64}$$

where $A^n = A \cdot \cdots \cdot A$ is the product of n factors A. Specifically, the **exponential of a matrix** is defined as

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}. \tag{1.65}$$

Using this representation it is straightforward to prove that

$$\begin{aligned} d_t \exp(At) = d_t \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{(n-1)!} = \\ = A \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = A \exp(At). \end{aligned} \tag{1.66}$$

EXERCISE While the result above shows that the matrix exponential bears similarity to the conventional exponential function, there are important differences. Convince yourself that we have

$$\exp(A + B) \neq \exp(A)\exp(B) \tag{1.67}$$

for non-commutative matrices, i.e. matrices with $[A, B] = AB - BA \neq 0$. Why does the equality hold if A, B do commute?

In particular, we conclude that the differential equation (1.60) is indeed solved by

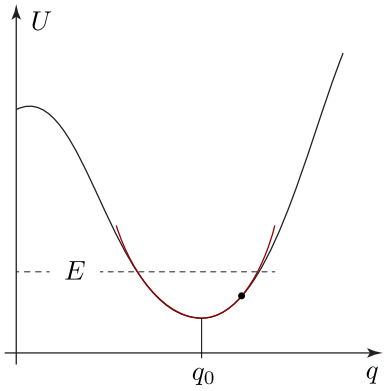
$$\mathbf{z}(t) = \exp(\omega\tau t)\mathbf{z}(0). \tag{1.68}$$

To show the equivalence of the r.h.s. of that equation to (1.62), we observe that $\tau^{2n} = (-)^n \mathbb{I}$ and $\tau^{2n+1} = (-)^n \tau$. Using these formulae, we get

$$\begin{aligned}\exp(\omega\tau t) &= \sum_n \frac{(t\omega\tau)^n}{n!} = \\ &= \sum_l \frac{(t\omega\tau)^{2l}}{2l!} + \sum_l \frac{(t\omega\tau)^{2l+1}}{(2l+1)!} = \sum_l (-)^l \frac{(t\omega)^{2l}}{2l!}\mathbb{I} + \sum_l (-)^l \frac{(t\omega)^{2l+1}}{(2l+1)!}\tau = \\ &= \cos(\omega t)\mathbb{I} + \sin(\omega t)\tau \end{aligned} \tag{1.69}$$

Substituting this result into the formula above, we obtain (1.62).

Harmonic oscillator: a model of low energy physics



The harmonic oscillator is one of the most frequently occurring model systems in physics. To understand why this is so, consider the cartoon of a one-dimensional potential profile shown in the figure. For example, the coordinate might represent the distance between the atoms of a two-atomic molecule, and $U(q)$ the potential energy in dependence on that distance. Imagine that the energy of the particle, $E = T + U$ is only slightly higher than the value of the local minimum $U(q_0)$, i.e. a 'weakly excited' molecule. Since the kinetic energy is positive, only a small region of the potential landscape between the coordinates $q_\pm$ defined by the condition $U(q_\pm) = E$ will be accessible to the particle. If $q_\pm$ is close to q_0, and the potential smooth, we may Taylor approximate $U(q) \simeq U(q_0) + \frac{1}{2}U''(q - q_0)^2$ in that region. The motion therefore is describable in terms of a harmonic oscillator problem centered around q_0 and with characteristic frequency $\omega = \sqrt{U''(q_0)/m}$.

The discussion above illustrates why harmonic oscillator problems frequently appear in the approximation of weakly excited physical problems. This fact, and the exact solubility of the problem make the harmonic oscillator an important paradigm in all areas of physics.

1.4.3 Small oscillations

Linearization of mechanical problems

Imagine a system of coordinates r^i, $i = 1, \ldots, f$, obeying the differential equation

$$m^i \ddot{r}^i = -\partial_{r^i} V(\mathbf{r}), \tag{1.70}$$

where $\mathbf{r} = (r^1, \ldots, r^f)^T$. We assume the system to be close to an 'equilibrium point' $\mathbf{r} = \bar{\mathbf{r}}$ characterized by the simultaneous vanishing of all 'generalized forces' $\partial_{r^i} V(\bar{\mathbf{r}}) = 0$. For smooth

V and $\mathbf{u} \equiv \mathbf{r} - \bar{\mathbf{r}}$ sufficiently small, we may Taylor expand

$$V(\bar{\mathbf{r}} + \mathbf{u}) \simeq V(\bar{\mathbf{r}}) + \frac{1}{2} \sum_{i,j=1}^{f} \partial^2_{r^i r^j} V(\bar{\mathbf{r}}) u^i u^j + \dots \tag{1.71}$$

Due to the vanishing of the first derivative, the Taylor expansion does not contain a first order term.

EXERCISE Consider the $f = 2$ example $V(r^1, r^2) = \sin(r^1)\cos(r^2)$. Where are the equilibrium points of the potential? Taylor expand to second order in r^1 and r^2 around these points. Explain the origin of the factor $1/2$ in (1.71).

Once more assuming that $u^i = r^i - \bar{r}^i$ is small, we have $\ddot{r}^i = \ddot{u}^i$ and $\partial_{r^i} V = \partial_{u^i} V = \sum_j \partial^2_{r^i r^j} V(\bar{\mathbf{r}})\, u^j$. Our differential equations therefore assume the form of a system of second order *linear* equations

$$\ddot{u}^i = -A^i_j u^j, \tag{1.72}$$

where we defined $A^i{}_j = \frac{1}{m_i} \partial^2_{r^i r^j} V(\bar{\mathbf{r}})$ and a summation convention is implied. The equation is linear because the matrix $A \equiv \{A^i{}_j\}$ is independent of the coordinates u^l. The 'linearization' of the full problem is justified as long as the u^l remain sufficiently small. It must be solved subject to the $2f$ **initial conditions** $u^i(0) = u^i_0$ and $\dot{u}^i(0) = v^i_0$, where u^i_0 and v^i_0 determine the initial position and velocity of the variables u^i. We finally notice that it is often convenient to introduce a vector notation $\mathbf{u} \equiv (u^1, \dots, u^f)^T$ in which the linear equation reads

$$\ddot{\mathbf{u}} = -A\mathbf{u}. \tag{1.73}$$

Linearized equations such as (1.70) find **applications** in many different contexts. Consider for example, the Newton equation of a molecule whose atoms are so heavy that a classical description is justified. Assume that the N constituent atoms are at coordinates $\mathbf{r}_i$ and carry mass m_i. Assuming the absence of external forces, the Newton equation reads

$$m_i \ddot{\mathbf{r}}_i = \sum_j \mathbf{F}_{ij}(\mathbf{r}_i - \mathbf{r}_j) = -\nabla_{\mathbf{r}_i} \sum_{j \neq i} U_{ij}(\mathbf{r}_i - \mathbf{r}_j) = -\nabla_{\mathbf{r}_i} V(\mathbf{r}),$$

where U_{ij} is an inter-atomic potential generating the force between i and j, and $V(\mathbf{r}) = \frac{1}{2}\sum_{i \neq j} U_{ij}(\mathbf{r}_i - \mathbf{r}_j)$ the total potential. When written as a set of equations for the $3N$ coordinates r^j_i separately, the equations assume the form of (1.70). Instead of atoms of a molecule, one may consider a system of masses coupled by elastic strings (see the example below), the large number of atoms in a solid, the metal compounds forming the wings of an aricraft, etc. All these systems have in common that they are normally close to a static equilibrium point. Perturbations (an excitation of the masses, heating the solid, exposing the wing to flight conditions) lead to small excitations away from equilibrium, which often can be described in terms of linearized equations. In passing, we note that similar ideas can be applied to 'dynamical systems' outside phyiscs, e.g. the description of the motion of large human crowds, etc.

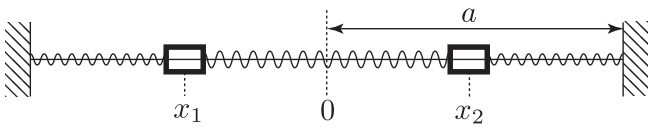


Figure 1.5: Two masses coupled by springs. Discussion, see text

EXAMPLE As a concrete example, consider two masses moving in one dimension and coupled by a system of springs as shown in Fig. 1.5. Describing the position of the masses in terms of two coordinates, $q^{1,2}$, we consider the potential function

$$V(q^1, q^2) = \frac{d}{2}(q^1 + a)^2 + \frac{d}{2}(q^2 - a)^2 + \frac{c}{2}(q^1 - q^2)^2.$$

The first and second term, resp., describe the potential energy stored in the two springs connecting the left and right mass to the walls, and the third term represents the energy contained in the center spring. The equilibrium point of the potential is defined by the condition $\partial_{q^i} V(\bar{q}^1, \bar{q}^2) = 0$, i.e.

$$\begin{aligned} d(\bar{q}^1 + a) + c(\bar{q}^1 - \bar{q}^2) &= 0. \\ d(\bar{q}^2 - a) - c(\bar{q}^1 - \bar{q}^2) &= 0. \end{aligned}$$

These equations are solved by $\bar{q}^2 = -\bar{q}^1 = \frac{da}{d+2c}$. (Discuss the solution in the limiting cases $d/c \to 0$, $c/d \to 0$, and $c = d$.) If we now define $q^i = \bar{q}^i + u^i$ and expand to second order in u^i we obtain

$$V(\bar{q}^1 + u^1, \bar{q}^2 + u^2) = V(\bar{q}^1, \bar{q}^2) + \frac{d+c}{2}((u^1)^2 + (u^2)^2) - cu^1u^2$$

Due to the quadraticity of the potential V in q^i the expansion stops at second order. Notice the absence of terms linear in u^i. Substituting this representation into the Newton equations, we obtain

$$\begin{aligned} \ddot{u}^1 &= -\frac{1}{m}((d+c)u^1 - cu^2), \\ \ddot{u}^2 &= -\frac{1}{m}((d+c)u^2 - cu^1). \end{aligned} \tag{1.74}$$

We define the matrix

$$A \equiv \frac{1}{m}\begin{pmatrix} d+c & -c \\ -c & d+c \end{pmatrix}, \tag{1.75}$$

to write this in the compact form of Eq. (1.72).

Solution of the linear problem

Eq. (1.72) bears similarity with the Newton equations for the harmonic oscillator. Intuitively, too, we may expect a mechanical system weakly perturbed out of equilibrium to perform oscillatory motion. This expectation motivates to search for solutions of the form $u_a^j(t) = e^{i\omega_a t} z_a^j$, whith time independent z_a^j. The subscript a accounts for the fact that we should be

prepared to find several of such solutions. Notice that we temporarily abandon the condition of reality, $u^j \in \mathbb{R}$, in exchange for the convenience of working with the complex exponential function. Real functions will later be obtained by taking the real part of complex solutions.

Substitution of the above ansatz into the linearized equation obtains

$$-\omega_a^2 z_a^j e^{i\omega_a t} = A^j{}_k z_a^k e^{i\omega_a t}.$$

We may simplify the notation by introducing a vector notation $\mathbf{z}_a = (z_a^1, \ldots, z_a^f)^T$ and dividing by the exponential,

$$(A - \omega_a^2)\mathbf{z}_a = 0,$$

where $A = \{A^i{}_j\}$ is an $f \times f$-matrix. The equation tells us that the frequencies ω_a must be chosen such that ω_a^2 is an **eigenvalue** of the matrix A. Linear algebra tells us that A has f complex eigenvalues λ_a. The physics of our problem requires that these eigenvalues be real and positive, $\lambda_a \equiv \omega_a^2 > 0$. Indeed, complex values of ω_a would lead to time dependence $\exp(i\omega_a t) = \exp(i\mathrm{Re}\,\omega_a t - \mathrm{Im}\,\omega_a t)$ divergent at large positive ($\mathrm{Im}\,\omega_a < 0$) or negative ($\mathrm{Im}\,\omega_a > 0$) times. This would be at conflict with our expectation of oscillatory motion. Notice that the components of the eigen*vectors* z_a^i may be complex.

Assuming that we have found f linearly independent eigenvectors with real frequencies, we consider the linear superposition $\mathbf{u}_c(t) \equiv \sum_a w_a \mathbf{z}_a\, e^{i\omega_a t}$. We fix the coefficients $w_a \in \mathbb{C}$ such that the initial conditions

$$\mathbf{u}(0) = \mathrm{Re}\,\mathbf{u}_c(0) = \mathrm{Re}\sum_a w_a \mathbf{z}_a,$$
$$\dot{\mathbf{u}}(0) = \mathrm{Re}\,\dot{\mathbf{u}}_c(0) = \mathrm{Re}\, i\sum_a w_a\, \omega_a \mathbf{z}_a.$$

Notice that we have $2f$ real initial conditions $u^i(0)$ and $\dot{u}^i(0)$ and $2f$ real free parameters contained in the f complex coefficients w_a. The equations above are therefore uniquely solvable. Once the parameters w_a have been identified, we have a solution

$$\boxed{\mathbf{u}(t) = \mathrm{Re}\sum_a w_a \mathbf{z}_a\, e^{i\omega_a t}.} \tag{1.76}$$

EXAMPLE Let us illustrate the procedure on our **masses–coupled–by–springs** example. The eigenvalues and eigenvectors of the matrix (1.75) are readily computed from the secular equation ($\to$ TP0) $\det(\lambda_a - A) = (\lambda_a - (d+c)/m)^2 - (c/m)^2 = 0$, which is solved by $\lambda_a = \frac{d+c}{m} \pm \frac{c}{m}$. We thus obtain the two characteristic frequencies

$$\omega_+ = \sqrt{\lambda_+} = \sqrt{\frac{d+2c}{m}}, \qquad \omega_- = \sqrt{\lambda_-} = \sqrt{\frac{d}{m}}.$$

where we chose the discriminating label $\pm$ rather than $a = 1, 2$. The corresponding eigenvectors $A\mathbf{z}_\pm = \omega_\pm \mathbf{z}_\pm$ are readily obtained as

$$\mathbf{z}_+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \mathbf{z}_- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and from here we obtain the general solution

$$\begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix} = \frac{1}{\sqrt{2}} \mathrm{Re} \left[w_+ \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{i\sqrt{\frac{d+2c}{m}}t} + w_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i\sqrt{\frac{d}{m}}t} \right].$$

Consider a situation in which the bodies are initially elongated to the left $u^i(0) = -\Delta u < 0$, and released at rest $d_t u^i(0) = 0$. This initial condition is realized with the choice $w_+ = 0$ and $w_- = -\sqrt{2}\Delta u$, i.e.

$$\begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix} = -\Delta u \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos\left(\sqrt{\frac{d}{m}} t \right).$$

The two bodies swing 'in phase', i.e. at fixed relative distance Δu. By contrast, consider the situation where $-u^1(0) = u^2(0) = \Delta u$, $\dot{u}^i(0) = 0$, i.e. bodies initially distorted in opposite direction. In this case, we obtain the solution

$$\begin{pmatrix} u^1(t) \\ u^2(t) \end{pmatrix} = -\Delta u \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos\left(\sqrt{\frac{d+2c}{m}} t \right).$$

The bodies now swing in opposite directions and at larger frequency. The two types of motion identified above define the two fundamental '**excitation modes**' of the system. They are fundamental in the sense that they are characterized by a single frequency, $\omega_\pm$. General initial conditions are described by non-trivial linear superpositions of the two fundamental modes and distinguished by more complex temporal behavior.

For the convenience of the reader, we once more **summarize the essential solution steps** of the small oscillation problem:

1. Find the minima of the potential V.[16]

2. Expand to second order in small coordinate deviations to identify the linear approximation of the differential equation and the matrix A, Eq. (1.72).

3. Find the eigenvectors $\mathbf{z}_a$ and eigenfrequencies ω_a characterizing the linear problem.

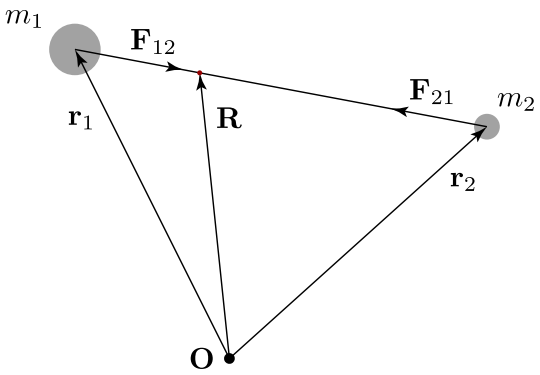
4. Build the linear superposition (1.76) and fix the coefficients w_a such that the initial conditions $\mathbf{u}(0)$ and $\dot{\mathbf{u}}(0)$ are properly resolved.

[16] In general, a potential V will have **stationary points**, i.e. points at which all derivatives $\partial_{r^i} V$ vanish, but V need not be minimal. For example, the potential $V(\mathbf{r}) = (r^1)^2 - (r^2)^2$ has a stationary point at $\mathbf{r} = 0$, which is not a minimum. In such cases, the matrix A has non-positive eigenvalues, which in turn means that the motion around these points is not oscillatory. In the example above, a the 2 coordinate of a mass point would grow indefinitely. The existence of such 'negative eigenmodes' generally means that we are considering a dynamical system at a point of instability. However, the discussion of such problems is beyond the scope of this text.

1.4.4 Two body central force problem

The two-body central force problem (or two body-problem for brevity) describes the dynamics of two bodies of mass m_1 and m_2, resp. interacting by a force $\mathbf{F}$ colinear to the vector $\mathbf{r}$ connecting the two bodies (cf. Fig.) This setup is of importance, in particular, to celestial mechanics, where situations in which two bodies influence each other in a manner only weakly perturbed by the presence of other bodies (think of a satellite moving in earth's gravitational field, the earth-moon system, etc.) frequently occur. However, the problem also plays a role in the description of 'nearly isolated' terrestrial systems, for example in atomic physics (as long as the effects of quantum mechanics do not become overwhelmingly strong.)

Unlike with most other problems of practical importance, the two-body problem is tractable by analytic methods: due to its high degree of symmetry, Newton's equations for the six coordinates fixing the position vectors $\mathbf{r}_{1,2}$ of the participating bodies can be reduced to the differential equation for a single coordinate (viz. the scalar distance r between the bodies.) In the important case of bodies interacting by gravitational or electrostatic forces that effective equation can be solved by analytic means. Much like the harmonic oscillator, the two-body problem, therefore, has a status of an important and tractable model system of classical mechanics.



Reduction of the problem I: momentum conservation

Above, we have shown that the total momentum of an N particle system in the absence of external forces is conserved. We will now use this conservation law to reduce the number of degrees of freedom from $f = 6$ to $f = 3$. To this end, we introduce:

$$\begin{aligned}
\text{Total mass: } & M = m_1 + m_2, \\
\textbf{Reduced mass}\text{: } & m = \frac{m_1 m_2}{M}, \\
\text{Center of mass coordinate: } & \mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{M}, \\
\text{Relative coordinate: } & \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \\
\text{Total momentum: } & \mathbf{P} = M\dot{\mathbf{R}}, \\
\text{Relative momentum: } & \mathbf{p} = m\dot{\mathbf{r}}.
\end{aligned}$$

The condition that the total momentum is conserved $\dot{\mathbf{P}}$ effectively constrains three of the six degrees of freedom. To see this explicitly, we express the coordinates $\mathbf{r}_{1,2}$ in terms of the newly

introduced variables,

$$\begin{aligned}\mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\mathbf{r},\\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\mathbf{r},\end{aligned} \tag{1.77}$$

Defining $\mathbf{F}_{12} \equiv \mathbf{F}$, the equations of motion

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{12}, \qquad m_2\ddot{\mathbf{r}}_2 = -\mathbf{F}_{12}, \tag{1.78}$$

then assume the form,

$$\boxed{m\ddot{\mathbf{r}} = \mathbf{F}, \qquad \mathbf{F}(\mathbf{r}) \equiv F(r)\mathbf{e}_r,} \tag{1.79}$$

where $\mathbf{e}_r$ is a unit vector parallel to $\mathbf{r}$, and the parameterization of the force on the r.h.s. uses that the strength of the force, F, depends only on the distance r between the bodies.

Eq. (1.79) is the Newton equation of a fictitious single particle with coordinate $\mathbf{r}$, mass m, and subject to a force $\mathbf{F}$. The effective problem has $f = 3$ degrees of freedom. Notice that for a problem involving two particles of rather different mass, e.g., $m_1 \gg m_2$, the effective mass $m \simeq m_2$ approximately equals that of the lighter particle.

Reduction of the problem II: energy conservation

The force $\mathbf{F}(\mathbf{r})$ entering the effective one-body problem is a **conservative force**. To see this, we define the function

$$U(r) = -\int_0^r dr\, F(r). \tag{1.80}$$

Interpreting $U(r)$ as a function in space depending only on the distance to origin, we the define the *three-dimensional* function,

$$\begin{aligned}U : \mathbb{R}^3 &\to \mathbb{R},\\ \mathbf{r} &\mapsto U(r), \qquad r = |\mathbf{r}|.\end{aligned}$$

This is the **potential of our conservative problem**: we use the relation $\partial_i r = \partial_i \sqrt{(r^1)^2 + (r^2)^2) + (r^3)^2)} = r^i/r$ to compute the gradient of U as

$$\partial_i U(r) = \partial_r U(r)\partial_i r = -F(r)\frac{r^i}{r} = -F(r)(\mathbf{e}_r)^i = -(\mathbf{F}(\mathbf{r}))^i. \tag{1.81}$$

EXERCISE Recall that for a generic conservative force a potential may be defined as

$$U(\mathbf{r}) = -\int_{\gamma_\mathbf{r}} d\mathbf{s}\cdot\mathbf{F},$$

where the integral is over a straight line $\gamma_\mathbf{r}$ connecting the origin with $\mathbf{r}$. Explain why in the particular case of a central force $\mathbf{F}(\mathbf{r}) = F(\mathbf{r})\mathbf{e}_\mathbf{r}$ the formula for $U(r)$ coincides with the line integral representation.

The conservativeness of the effective one-body problem suggests the introduction of a conserved **energy function**

$$E \equiv T + U = \frac{m\dot{\mathbf{r}}^2}{2} + U(r). \tag{1.82}$$

EXERCISE Before the decomposition of the problem into center of mass motion and relative coordinate, we might have defined its total energy as

$$E_{\rm tot} = \frac{m_1\dot{\mathbf{r}}_1^2}{2} + \frac{m_2\dot{\mathbf{r}}_2^2}{2} + U(|\mathbf{r}_1 - \mathbf{r}_2|), \tag{1.83}$$

where the function U is defined by (1.80). Verify that this energy is conserved along the solutions of the equations (1.78). Show that $E_{\rm tot}$ can be decomposed as

$$E_{\rm tot} = \frac{M\dot{\mathbf{R}}^2}{2} + E,$$

i.e. as the sum of the kinetic energy stored in the center of mass motion and the function E representing the energy stored in the relative motion. Think about the physical meaning of this decomposition.

Although much of our discussion will apply to general potentials, two rather important special cases are,

$$\begin{aligned} \textbf{Coulomb potential}: &\qquad U(r) = k_e\frac{Q_1Q_2}{r}, \\ \textbf{Gravitational potential}: &\qquad U(r) = -\frac{Gm_1m_2}{r}. \end{aligned} \tag{1.84}$$

Here, $k_e \simeq 8.988 \times 10^9 \mathrm{Nm^2C^{-2}}$ is **Coulomb's constant** determining the strength of the Coulomb force between two charged bodies and $G \simeq 6.674 \times 10^{-11}\mathrm{N(m/kg)^2}$ is the universal **gravitational constant** determining the strength of gravitational force acting between two masses. Notice that both potential decay like $\sim r^{-1}$. While the Coulomb potential can be positive or negative depending on the relative sign of the involved charges, the gravitational potential is negative. The force deriving from the gravitational potential $F \sim -\partial_r U \sim -1/r^2$ is of negative sign, too, which means that it is an attractive force. The electrostatic forces deriving from the Coulomb potential are attractive or repulsive for opposite or equally charged bodies, respectively.

To summarize, we have split the $f = 6$ problem into an $f = 3$ sector describing the trivial motion of the center of mass $\dot{\mathbf{R}} = \mathrm{const.}$, and a non-trivial $f = 3$ sector describing the dynamics of the relative coordinate, $\mathbf{r}$ via the conservative problem defined by Eq. (1.79). The $f = 3$ problem is described by $6 = 2 \times 3$ first order differential equation for its coordinates and momenta, reduced down to $5 = 6 - 1$ thanks to the conservation of energy.

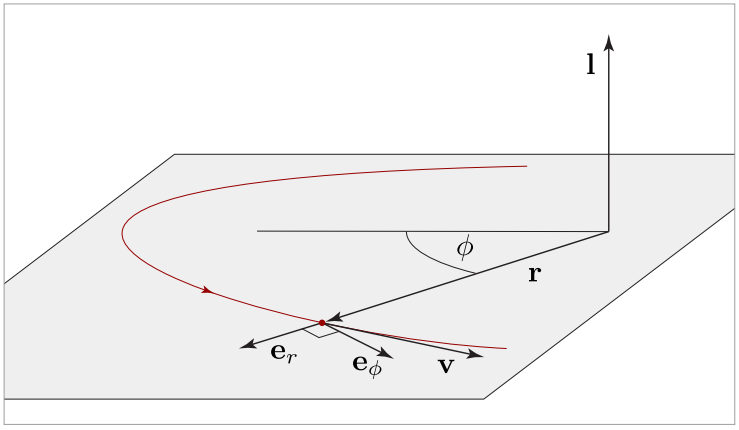


Figure 1.6: Motion of a body under the influence of a central force. Discussion, see text

Reduction of the problem III: angular momentum conservation

We next turn our attention to the fact that the angular momentum $\mathbf{l} = \mathbf{r} \times \mathbf{p}$ relative to the origin, $\mathbf{r} = 0$ is conserved: $d_t \mathbf{l} = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} = 0 + 0$, where $\mathbf{F} \parallel \mathbf{r}$ has been used. Imagine $\mathbf{l}(t) = \mathbf{l}$ as a fixed vector in space. Due to the definition $\mathbf{l} = \mathbf{r}(t) \times \mathbf{p}(t) \propto \mathbf{r}(t) \times \mathbf{v}(t)$, we know that for all times $\mathbf{v}(t) \cdot \mathbf{l} = \mathbf{r}(t) \cdot \mathbf{l} = 0$. In other words, the motion takes place in a *plane* perpendicular to $\mathbf{l}$ (cf. Fig. 1.6.) Counting variables, the conservation of the three components l^i means that we are down to $3 = 12 - 6 - 3$ first order equations before energy conservation is taken into account, and just two with energy conservation — a massive reduction in complexity owed to symmetries. We next derive these effective equations governing the system.

In the plane of motion, we introduce polar coordinates (r, ϕ) centered around $\mathbf{r} = 0$ and relative to an arbitrarily chosen axes $\phi = 0$ (see Fig. 1.6.) Recall that a moving basis ($\rightarrow$ TP0) suitable to the description of problems in polar coordinates is given by the two basis vectors,

$$\mathbf{e}_r = \cos(\phi)\mathbf{e}_1 + \sin(\phi)\mathbf{e}_2, \qquad \mathbf{e}_\phi = -\sin(\phi)\mathbf{e}_1 + \cos(\phi)\mathbf{e}_2, \tag{1.85}$$

where $\mathbf{e}_r \parallel \mathbf{r}$ and $\mathbf{e}_\phi$ is perpendicular to it. Also recall that these vectors are defined relative to a fixed $\mathbf{r} = \mathbf{r}(r, \phi)$. If $\mathbf{r} = \mathbf{r}(t)$ is time dependent, the basis vectors vary in time too, and this variation is described by

$$d_t \mathbf{e}_r = \dot{\phi}\mathbf{e}_\phi, \qquad d_t \mathbf{e}_\phi = -\dot{\phi}\mathbf{e}_r. \tag{1.86}$$

We now represent Newton's equations as

$$\begin{aligned} d_t^2 \mathbf{r} = d_t^2(r\mathbf{e}_r) = d_t\left(\dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi\right) = \ddot{r}\mathbf{e}_r + 2\dot{r}\dot{\phi}\mathbf{e}_\phi + r\ddot{\phi}\mathbf{e}_\phi - r\dot{\phi}^2\mathbf{e}_r = \\ = \left(\ddot{r} - r\dot{\phi}^2\right)\mathbf{e}_r + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\mathbf{e}_\phi = \frac{F(r)}{m}\mathbf{e}_r. \end{aligned} \tag{1.87}$$

This is equivalent to the set of two equations

$$\ddot{r} - r\dot{\phi}^2 = \frac{F(r)}{m},$$
$$2\dot{r}\dot{\phi} + r\ddot{\phi} = 0. \tag{1.88}$$

In the second of these, we re-discover angular momentum conservation. Indeed, $\mathbf{l} = \mathbf{r} \times \mathbf{p} = mr\mathbf{e}_r \times d_t(r\mathbf{e}_r) = mr^2\dot{\phi}\mathbf{e}_z \equiv l\mathbf{e}_z$, where $\mathbf{e}_z$ is a unit vector perpendicular to the plane, and

$$l = mr^2\dot{\phi} \tag{1.89}$$

the conserved magnitude of angular momentum. The constancy of l requires $d_t(r^2\dot{\phi}) = r(2\dot{r}\dot{\phi} + r\ddot{\phi}) = 0$, which is the second equation. We may solve for $\dot{\phi}$ as $\dot{\phi} = \frac{l}{mr^2}$ to reduce the first equation to

$$\boxed{m\ddot{r} - \frac{l^2}{mr^3} = F(r),} \tag{1.90}$$

i.e. an equation for the single variable r. Equation (1.90) is the **radial equation** corresponding to the central force problem. Counting equations, we have three first order equations to solve, Eq. (1.89), and another two (a second order equation counts like two first order equations) for the radial equation. However, at this point, we have not yet used energy conservation!

Solution of the radial equation

We next turn to the solution of the effective radial problem. To this end we represent the angular momentum dependent contribution to the radial equation as a derivative,

$$\frac{l^2}{mr^3} = -\partial_r \frac{l^2}{2mr^2},$$

where the function $l^2/2mr^2$ defines the so-called **centrifugal potential**. This representation suggests the introduction of an **effective potential**

$$U_{\text{eff}} = U(r) + \frac{l^2}{2mr^2}, \tag{1.91}$$

in terms of which the radial equation assumes the form

$$m\ddot{r} = -\partial_r U_{\text{eff}}(r). \tag{1.92}$$

The addition of a centrifugal potential to the radial potential accounts for the fact that two particles at distance r from the origin will perform different motion if their angular momenta relative to the origin are different. For example a satellite whose velocity is near tangential to the surface of earth (finite angular momentum) will move differently from a body at the same distance relative to earth's origin but with no tangential velocity.

Notice that the radial equation has the mathematical structure of a fictitious particle living on the one-dimensional half line parameterized by the coordinate $r > 0$. The fact that the force acting on the particle is generated by a potential U_{eff} suggests the introduction of an **energy** function

$$E = T_r + U_{\text{eff}} \equiv \frac{m\dot{r}^2}{2} + U_{\text{eff}},$$

where the subscript in T_r indicates that this is not the full kinetic energy of the particle, but only the contribution accounting for the radial component of its velocity. A one-dimensional version of the general argument given in section 1.3.4 shows that this energy is conserved along any solution curve of the problem:

$$d_t E(r,\dot{r}) = \partial_{\dot{r}} T_r(\dot{r})\ddot{r} + \partial_r U_{\text{eff}}(r)\dot{r} = \dot{r}(\underbrace{m\ddot{r} + \partial_r U_{\text{eff}}}_{0}) = 0. \tag{1.93}$$

We denote the energy E of the one-dimensional effective problem by the same symbol as the e 'real' energy defined in Eq. (1.82), which suggests that the two quantities equal each other. To verify this statement, compute the kinetic energy appearing in (1.82),

$$T = \frac{m}{2} d_t \dot{\mathbf{r}}^2 = \frac{m}{2}(d_t(r\mathbf{e}_r))^2 = \frac{m}{2}\left(\dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi\right)^2 = \frac{m}{2}\dot{r}^2 + \frac{mr^2\dot{\phi}^2}{2} \overset{(1.89)}{=} \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2}.$$

Substituting this representation into (1.82), we obtain the effective energy of the radial problem, and the added information how the centrifugal contribution to the effective potential originates in kinetic energy.

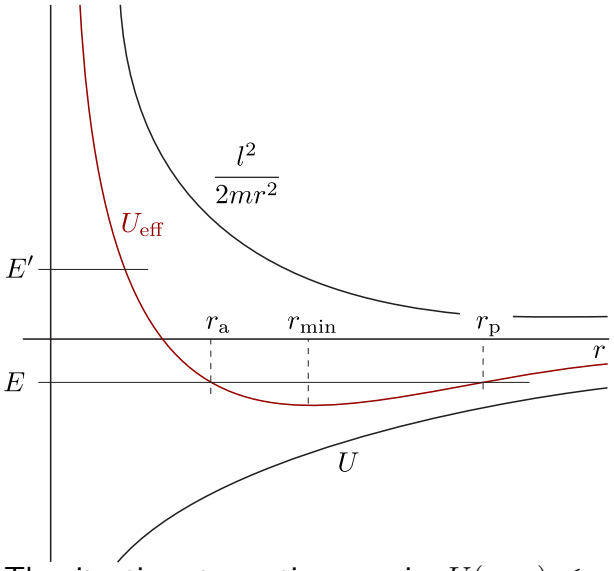
Qualitative discussion of the motion

Before turning to the quantitative solution of the radial problem, let us discuss in qualitative terms what kind of motion we are to expect. Although more general situations can be considered, for our purposes it will be sufficient to focus on the case of an **attractive potential** U, i.e. a potential monotonously increasing potential such that the derivative $F(r) = -\partial_r U(r)$ globally points towards the origin. We require that upon approaching $r \to 0$ the potential behaves as a power law $U(r) \sim -r^{-\alpha}$, where $\alpha < 2$ such that the divergence is slower than that of the centrifugal potential. At infinity, we require vanishing of the potential, $U(r \to \infty) \to 0$. The constraints imposed by these conditions are relatively mild and obeyed, e.g., by the gravitational and Coulomb potential, as well as by various other inter-particle potentials realized in Nature. For the sake of comparison, we will also take a brief look at an **repulsive potential**, i.e. one that is uniformly decreasing, $\partial_r U(r) < 0$.

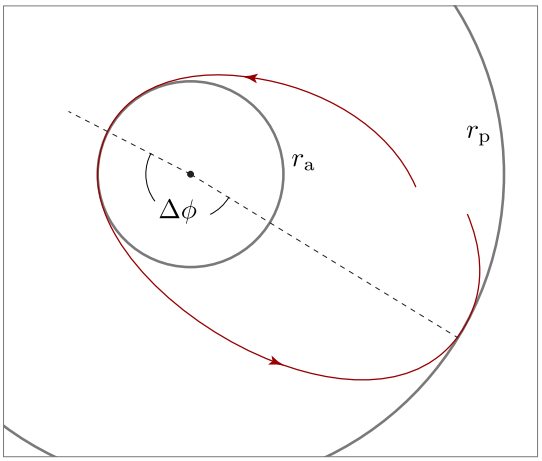
The effective potential obtained in the attractive case then qualitatively looks as shown in the figure. Its distinguishing feature is the existence of a global potential minimum at some finite radial coordinate $r_{\min}$, where $U_{\text{eff}}(r_{\min}) < 0$. To understand how the existence of that minimum affects the radial motion, consider a fixed value of the particle's energy, E. If $E > 0$, then we have an equality $E = U(r)$ at only one definite coordinate r. (Cf. the

line corresponding to the value $E' > 0$ in the figure.) At this coordinate, the kinetic energy vanishes. The particle cannot approach the origin further, because that would require negative kinetic energy. If it is released the point of closest approach it will start moving outwards, gain kinetic energy in the process, and disappear to infinity. We call such type of motion the motion of an **unbound trajectory**.

EXERCISE Discuss what happens if the particle is released at a general coordinate with general positive or negative initial velocity.



The situation at negative energies $U(r_{\text{min}}) \leq E < 0$ is more interesting. We now have two points $r_{\text{a}} < r_{\text{min}} < r_{\text{p}}$ where $E = U_{\text{a,p}}$. The motion is confined to stay between these two points, i.e. we are considering a bound trajectory. The point of closest approach, r_{a}, is called the **aphel** of the motion and r_{p} the **perihel**. Notice how boundedness emerges by conspiracy of attractiveness of the potential, and finiteness of the centrifugal potential. For example, an asteroid of angular momentum relative to earth's center so small that $r_{\text{a}} < R$, where R is earth's radius will hit the surface (or evaporate in the atmosphere), while one with larger angular momentum stands a chance of staying on an earth bound orbit. At the turning points, $\dot{r}(r_{\text{p,a}}) = 0$ which, however, does not mean that the motion stops. Rather, $l = mr^2\dot{\phi}$ means that $\dot{\phi}(r_{\text{p,a}}) = l/mr^2_{\text{p,a}}$. At the aphel the angular velocity $\dot{\phi}$ actually takes its maximum value. Qualitatively we expect the planar motion to



looks as indicated in the figure. If the angle $\Delta\phi$ enclosed between an aphel and the consecutive perihel is an rational multiple of $\Delta\phi = 2\pi\frac{p}{q}$, then the motion will close after q 'bounces', i.e. we have a periodically traversed trajectory.

Quantitative Discussion

Let us now proceed to actually compute the trajectories $r(t)$ performed by the **radial coordinate**, and from there the motion of the three dimensional vector $\mathbf{r}(t)$. Eq. (1.90) is a differential equation of second order. We know ($\rightarrow$ TP0) that equations of nth order are equivalent to *systems* of n DEQs of first order. On the other hand, we have the conservation of energy, $E = \text{const.}$, and this should reduce the problem down to a single first order differential equation. To see how this happens, we solve the energy conservation condition for $\dot{r}$ to obtain

$$E = \frac{m\dot{r}^2}{2} + U_{\text{eff}}(r) \Rightarrow \dot{r} = \sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}.$$

This equation can be solved by the method of 'separation of variables' ($\rightarrow$ TP0) :

$$G_{r_0}(r(t)) \equiv \int_{r_0}^{r(t)} \frac{dr}{\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}} = \int_0^t dt = t,$$

where we assumed an initial condition $r(0) = r_0$. For a given potential function $U_{\text{eff}}(r)$ the integral on the left hand side can be computed if not by analytical then by numerical methods (for an example, see section 1.4.4.) As a result, one obtains a function $G_{r_0}(r(t))$, where the subscript indicates the dependence on the initial condition. We then need to solve (an algebraic operation) the equality $G_{r_0}(r(t)) = t$ for

$$r(t) = G_{r_0}^{-1}(t),$$

where $G_{r_0}^{-1}$ is the inverse function of G_{r_0}, i.e. $G_{r_0}^{-1}(G_{r_0}(s)) = s$.

We now have the information how the radial coordinate changes in time. To fully describe the motion, we also need the **angular variable**. The time dependence of the latter, $\phi(t)$, can be extracted by a similar strategy, viz. by inspection of the conservation law $l = mr^2\dot{\phi}$, or $\dot{\phi} = l/mr^2$. The solution of this equation will obtain a dependence $\phi(r(t))$, i.e. the angle ϕ at a given value of $r(t)$, which we want to compute. To this end, we we write

$$\frac{d\phi}{dt} = \frac{d\phi(r)}{dr}\frac{dr(t)}{dt},$$

which we can re-order to obtain

$$\frac{d\phi}{dr} = \frac{d\phi}{dt}\frac{1}{\frac{dr}{dt}} = \frac{l}{mr^2}\frac{1}{\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}}.$$

This equation is once more solved by separation of variables. We integrate the differential relation

$$d\phi = \frac{l}{m}\frac{dr}{r^2\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}},$$

to obtain

$$\phi(t) - \phi_0 = \int_{\phi_0}^{\phi} d\phi = \int_{r_0}^{r(t)} dr \frac{l}{mr^2}\frac{1}{\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}} \equiv H_{r_0}(r(t)),$$

where the function H_{r_0} is defined by the integral. Our two results,

$$\begin{aligned} r(t) &= G_{r_0}^{-1}(t), \\ \phi(t) &= \phi_0 + H_{r_0}(r(t)), \\ G_{r_0}(r) &= \int_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r'))}}, \qquad H_{r_0}(r) = \frac{l}{m}\int_{r_0}^{r} \frac{dr'}{r'^2}\frac{1}{\sqrt{\frac{2}{m}(E - U_{\text{eff}}(r))}} \end{aligned} \tag{1.94}$$

Eq. (1.94) solves the problem up to a point where a two integrals need to be computed. (In the theory of differential equations, this is called a 'solution up to quadrature'.) For general potentials, these integrals need to be done by numerical methods, or approximately. There exists, however, an important class of model potentials for which the functions G_{r_0} and H_{r_0} can be computed in closed form:

The Kepler problem

As a special case, consider the potential

$$U(r) = -\frac{k}{r},$$

where k is a constant which may be chosen so as to model a gravitational problem, or the Coulomb interaction between oppositely charged particles. Alluding to its relevance to the description of planetary motion – a problem to which Kepler made important contributions before Newton – the two-body system described by $U(r)$ is called the Kepler problem. **??**

For the Kepler problem, the integrals in (1.94) can be computed in closed form. Specifically, we obtain

$$\begin{aligned} \phi(r) = \phi_0 + H_{r_0}(r) &= \phi_0 + \frac{l}{m}\int_{r_0}^{r} \frac{dr'}{r'^2}\frac{1}{\sqrt{\frac{2}{m}(E + \frac{k}{r'} - \frac{l^2}{2mr'^2})}} = \\ &= \phi_0 + \frac{l}{m}\int_{r_0}^{r} \frac{dr'}{r'}\frac{1}{\sqrt{\frac{2}{m}(Er'^2 + kr' - \frac{l^2}{2m})}} = \\ &= \phi_0 + \arccos\left(\frac{p - r}{\epsilon r}\right)\Bigg|_{r_0}^{r}, \end{aligned}$$

where we defined

$$p \equiv \frac{l^2}{km}, \qquad \epsilon \equiv \sqrt{1 + \frac{2El^2}{mk^2}}. \tag{1.95}$$

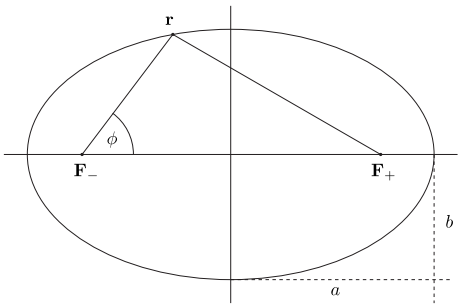
EXERCISE Explore the $\sqrt{}$-definition of the parameter ϵ. Why are energy values E so small that the argument of the square root becomes negative unphysical?

We may solve this equation for r to obtain

$$r(\phi) = \frac{p}{1 + \epsilon \cos(\phi - \tilde{\phi})}, \tag{1.96}$$

where the subtraction of $\tilde{\phi} = \phi_0 - \arccos\left(\frac{p - r_0}{\epsilon r_0}\right)$ makes sure that the initial condition $(r(0), \phi(0)) = (r_0, \phi_0)$ holds. The function $r(\phi)$ is our solution of the problem, we now know how the trajectories look like in the plane of conserved angular momentum. (To understand the time dependence of the trajectories, we would need to discuss the integral G_{r_0} as well, but we will not do this here.)

INFO An **ellipse** in the two dimensional xy plane is defined by two parameters $0 < b \leq a$ as the set of points $E \equiv \left\{\mathbf{r} = (x, y)^T \in \mathbb{R}^2 | \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right\}$. The axes $y = 0$ and $x = 0$ are called the **major and the minor semi-axis** respectively. On the major semi-axis, we define the two so-called **focal points** $\mathbf{F}_\pm$ as points at distance $\pm\sqrt{a^2 - b^2}$ from the origin. It is then not difficult to verify that the ellipse can be described as,



$$E = \{\mathbf{r} \in \mathbb{R}^2 | d(\mathbf{r}, \mathbf{F}_+) + d(\mathbf{r}, \mathbf{F}_-) = 2a\},$$

i.e. the set of all points whose distances to the two focal points sum to the constant $2a$. If you construct an ellipse by the string-and-pin method – take a string, sling it around two pins, and pencil out the curve you get when the string is stretched out – the focal points are where your pins sit. Defining the two parameters (yes, the first one is actually called 'parameter')

$$\textbf{parameter} : p \equiv b\frac{b}{a},$$
$$\textbf{excentricity} : \epsilon \equiv \sqrt{1 - \left(\frac{b}{a}\right)^2},$$

and chosing one of the focal points, say, $\mathbf{F}_-$, as the point of origin of a polar coordinate system, the ellipse may now be described by the curve,

$$r(\phi) = \frac{p}{1 + \epsilon \cos\phi}, \qquad \phi \in [0, 2\pi]. \tag{1.97}$$

Geometrically, the curves $r(\phi)$ describe **ellipses** in such a way that the origin $r = 0$ coincides with one of the focal points of the ellipse (see the info block below), and the axis $\phi = 0$ is aligned with the major semi-axis of it. We have therefore arrived at the important conclusion that the trajectories of $-k/r$-potentials are closed ellipses.

INFO The first observation of ellipsoidal motion in gravitational potentials goes back to Kepler. Observing the motion of Mars in the early 17th century, Kepler noted three empirical results, known as the **three Kepler laws**:

- ▷ The trajectories of planetary motion are ellipses.
- ▷ A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.
- ▷ The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

We have just proven the first law. The second is a straightforward consequence of angular momentum conservation: the area swept out by the radial vector in a short interval of time δt approximately equals the area of a triangle with corner points O, $O+\mathbf{r}(t)$, and $O+\mathbf{r}(t)+\delta t\mathbf{v}(t)$, where O is the origin. Its area is given by $A = \frac{\delta t}{2}|\mathbf{r}(t)\times\mathbf{v}(t)| = \frac{\delta t}{2m}|\mathbf{r}\times\mathbf{p}| = \delta t\frac{l}{2m}$, where l is the conserved angular momentum. In other words, the area equals the time interval δt times a constant, the second law. To prove the third law, we note that the full area of the ellipse, πab, is traversed in time $T \equiv 2\pi abm/l$. So the ratio addressed in the third law is therefore proportional (relative to numerical constants) to

$$\frac{T^2}{a^3} \sim \frac{b^2m^2}{al^2} = \frac{pm^2}{l^2} \overset{(1.95)}{=} \frac{m}{k} = \frac{m}{m_{\rm S}m_{\rm p}G},$$

where in the last equality we used the identification of the proportionality constant $k = m_{\rm S}m_{\rm p}G$ of the gravitational potential, and $m_{\rm S,p}$ are the mass of sun and the planet under consideration, resp. Now, recall that the effective mass of the two body problem, $m = \frac{m_{\rm S}m_{\rm p}}{m_{\rm S}+m_{\rm p}}$, to obtain $T^2/a^3 = 1/(m_{\rm S}+m_{\rm p})G \simeq 1/m_{\rm S}G$. The ratio is indeed 'universal' (independent of the particular planet) if we ignore the difference between the total mass of the respective two-body problem $M = m_{\rm S}+m_{\rm p}$ and the mass of sun $m_{\rm s} \simeq 2\times10^{30}{\rm kg}$. Given that even Jupiter is about three orders of magnitude lighter than sun, the error is indeed very small. To appreciate the accuracy of Kepler's observations, one should also factor in the required observation times (the length of a year on Mars is about twice as long as a terrestrial year!)

EXERCISE A **hyperbola** in the xy-plane is defined as $H \equiv \left\{\mathbf{r} = (x,y)^T \in \mathbb{R}^2 | \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\right\}$. The hyperbola contains two branches, a left one for which $x < 0$ and a right one with $x > 0$. As with the ellipse, we define two focal points, $\mathbf{F}_\pm$ at distance $\pm\sqrt{a^2+b^2}$. Defining the parameter of the hyperbola as $p = b^2/a$ and its excentricity as $\epsilon = \sqrt{1+(b/a)^2}$, show that the hyperbola can again be parameterized by the polar representation (1.97) centered around $\mathbf{F}_-$. Discuss the ranges of ϕ required to parameterize the left/right branch of the hyperbola. Go once more through our solution of the radial equation to confirm that the left branch describes the unbound motion

of a particle at **positive energy** in an attractive potential (think of an asteroid approaching and escaping earth's gravitational center). The right branch describes the motion of a positive energy particle in a repulsive potential, $k < 0$ (two equally charged particles scattering off each other.) Make sure you appreciate that the two branches of the hyperbola really have the shape you would expect for such types of trajectories.

1.5 Generalized forces and the limitations of Newtonian mechanics

The cornerstones of Newtonian mechanics were developed at a time when the understanding of celestial motion stood in the foreground of interest. Planetary motion is distinguished for a number of features which are optimally suited for a description in terms of Newtonian mechanics:

- ▷ Planets and other celestial bodies are often well approximated as point particles,
- ▷ they freely move in space, and
- ▷ interact by the conservative gravitational force.

However, many of the dynamical processes taking place on earth are significantly more complicated. The motion of a bicycle (as stabilized by its rotating wheels) can hardly be approximated as that of a point particles, forces relevant to terrestrial dynamics include time dependent forces, non- conservative forces, friction forces, and others. It turns out that Newtonian mechanics is not optimally prepared to handle many of these complications, its conceptual validity notwithstanding. In later chapters, we will discuss powerful concepts, building on the Newtonian framework, but much better prepared to address complex problems of mechanics. However, before entering this discussion it is worthwhile to introduce the main classes of forces beyond the conservative forces discussed so far.

The most general force $\mathbf{F}(\mathbf{q}, \dot{\mathbf{q}}, t)$ depends on coordinates $\mathbf{q}$, velocities $\dot{\mathbf{q}}$ and time t.

1.5.1 Time dependent forces

Forces $\mathbf{F}(\mathbf{q}, t)$ with explicit time dependence appear when a mechanical system is 'influenced from the outside'. For example, a pendulum might be driven by an external motor. If we focus on the pendulum as our 'system', the motor acts via a time dependent force. The energy of a system affected by time dependent forces is no longer conserved, even if $\mathbf{F}(\mathbf{q}, t) = -\nabla U(\mathbf{q}, t)$ is conservative.

EXERCISE Consider the equation of motion of a **driven harmonic oscillator**

$$m\ddot{q} = -m\omega_0^2 q^2 + A\sin(\omega_e t),$$

where $A\sin(\omega_0 t)$ is a time-periodic external force of frequency ω_e and strength A. Apply a Fourier transformation

$$\tilde{q}(\omega) = \int dt\, e^{i\omega t} q(t),$$

to bring the equation to the algebraic form

$$-m(\omega^2 - \omega_0^2)\tilde{q}(\omega) = \pi A(\delta(\omega - \omega_e) + \delta(\omega + \omega_e)).$$

Show that this equation is solved by

$$\tilde{q}(\omega) = \frac{\pi A(\delta(\omega - \omega_e) + \delta(\omega + \omega_e))}{m(\omega_0^2 - \omega^2)} + 2\pi c_+ \delta(\omega + \omega_0) + 2\pi c_- \delta(\omega - \omega_0),$$

where $c_\pm \in \mathbb{C}$ are free complex constants. Compute the inverse Fourier transform to obtain

$$q(t) = \frac{A}{m}\frac{\sin(\omega_e t)}{\omega_0^2 - \omega_e^2} + \mathrm{Re}\,\left(c_+ e^{i\omega_0 t} + c_- e^{-i\omega_0 t}\right).$$

Discuss why the condition that $q \in \mathbb{R}$ be real effectively limits the freedom in the choice of $c_\pm$ down to two real parameters – as one would expect for a real differential equation of second order. Assume initial conditions $q(0) = q_0$, $\dot{q}(0) = 0$, to obtain the specific solution

$$q(t) = \frac{A}{m}\frac{\sin(\omega_e t)}{\omega_0^2 - \omega_e^2} + q_0 \cos(\omega_0 t).$$

Discuss this result. Why is the divergence of the solution as $\omega_0 \to \pm\omega_e$ called the **resonance catastrophy**?

1.5.2 Velocity dependent forces

There exist three sub-families of forces which depend on velocities: fictitious forces, friction forces, and Lorentz forces. We discuss them in turn.

Fictitious forces

As discussed above, fictitious forces appear upon non-Galilean change of reference frames. They need not depend on velocities (this was the case, e.g., in the example of a uniformly accelerated reference frame discussed in section 1.3.2) but may. This happens, e.g., upon change to a rotating coordinate frame, a situation to be discussed in section xx below.

Friction forces

As we all know, friction forces are of paramount importance to all kinds of terrestrial motion. Friction forces categorically depend on the velocity of the body on which they act, and they always act to impede its motion. The most general form of a frction force $\mathbf{F}_f$ therefore reads

$$\mathbf{F}_f(\mathbf{q}, \mathbf{v}) = -f(\mathbf{q}, \mathbf{v})\mathbf{v}, \tag{1.98}$$

where $f > 0$ is a positive function.

INFO The microscopic mechanism behind friction is **dissipation**, i.e. the flow of energy from a system with few degrees of freedom (the body on which friction acts), into many degrees of freedom. The noticeable consequence is heat, which is generated in any friction process and reflects the increase in energy of the microscopic constitutents causing the friction. (Rub your hands if you find the formulation abstract.)

To make the energy loss incurred by friction manifest, we assume conservativeness of the non-fricition forces, $\mathbf{F} = -\nabla U$, so that the Newton equation reads

$$m\ddot{\mathbf{q}} = -\nabla U(\mathbf{q}) - f(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}}. \tag{1.99}$$

The energy of the particle, $E = T + U$, is then seen to changes as

$$d_t E = \dot{\mathbf{q}}(m\ddot{\mathbf{q}} + \nabla U) = -f\dot{\mathbf{q}} \cdot \dot{\mathbf{q}} < 0.$$

In the negativeness of the derivative shows the energy loss of the particle. The energy gets transferred to the medium causing the friction and turned into heat.

Lorentz force

A third important representative of velocity dependent forces are forces arising when charged particles move through electro-magnetic fields. The corresponding equation of motion reads

$$m\ddot{\mathbf{q}} = q\mathbf{E}(\mathbf{q}, t) + \frac{q}{c}\dot{\mathbf{q}} \times \mathbf{B}(\mathbf{q}, t), \tag{1.100}$$

where q is the particle charge, $\mathbf{E}$ an electric field (which may depend on coordinates and time), and $\mathbf{B}$ the magnetic field. The force created by the magnetic field is called the **Lorentz force**. It acts perpendicular to both, the magnetic field and the instantaneous velocity. The Lorentz force is non-conservative, i.e. it is not generated by a potential. At the same time, a particle acted upon only by Lorentz forces does not change its kinetic energy:

$$d_t T = m\ddot{\mathbf{q}} \cdot \dot{\mathbf{q}} = \frac{q}{c}(\dot{\mathbf{q}} \times \mathbf{B}) \cdot \dot{\mathbf{q}} = 0. \tag{1.101}$$

Rather, the Lorentz force aims to 'bend' the particle motion in a plane perpendicular to the field, without changing the magnitude of its velocity.

EXERCISE Show that for a constant magnetic field $\mathbf{B} = \text{const.}$ the trajectories of charged particles form circles.

Constraint forces

Newton's mechanics as discussed up to this point addresses the motion of particle in free space, acted upon by certain forces. However, only few of terrestrial motions fall into this category!

Think of the motion of a sphere running down a hill, the moving parts of a combustion engine, a turntable of a record player, etc. All these have in common that the motion of a system of interest is effectively *confined*, to dimensions lower than three. Conceptually, there is no mystery about this. For example, the forces keeping a turntable in place, can be understood as 'infinitely strong' forces preventing it from leaving its axis of rotation in infinite space. Forces effecting such kinds of constraints are called **constraint forces**. While constraint forces can be described within the framework of Newtonian mechanics, it is intuitively clear that a description of a turntable as a body in three-dimensional will not be very economical. Rather, one would like to describe it in terms of a single *angular* variable describing the unconfined freedom to rotate it around its axis, i.e. the problem should be effectively one– instead of three- dimensional.

Newtonian mechanics can and has been generalized to cope with the presence of constraint forces. However, it turned out that the description of Newtonian mechanics in terms of Newton's differential equations is not optimally prepared to handle the situation. (Notice that this is a methodological rather than a conceptual point.) This inconvenience led to the development of an alternaitve and in many ways more powerful and flexible formalism, **Lagrange mechanics**. The disucssion of Lagrange mechanics will be the subject of the next chapter.

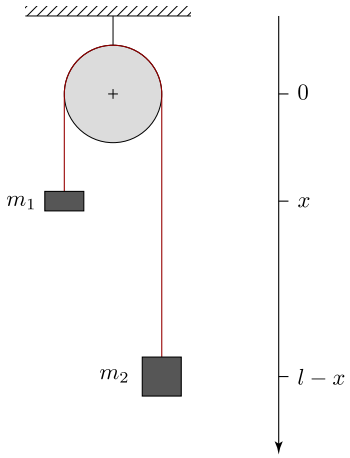
INFO Occasionally, one does need to explicitly include the constraint forces into the description of mechanical systems. For example, the forces keeping the carts of a rollercoaster on its tracks are constraint forces rendering the motion one-dimensional (along the track). By no means do we want the motion to become three-dimensional (a crashing rollercoaster). So we have to make sure that the static construction is strong enough to support the constraint forces required to keep the carts on track, and for that we need to know the forces themselves. The theoretical formalism suitable to compute constraint forces within Newtonian mechanics goes by the name **d'Alambert principle**, and the corresponding equations are knows as **Lagrange equations of the first kind.** However, the discussion of these equations is beyond the scope of this course.

Chapter 2

Lagrangian mechanics

Newton's equations of motion provide a complete description of mechanical motion. Even from today's perspective, their scope is limited only by velocity (at high velocities $v \sim c$, Newtonian mechanics has to be replaced by its relativistic generalization), and classicicity (the dynamics of small bodies is affected by quantum effects.) Otherwise, they remain fully applicable, which is remarkable for a theory that old.

Newtonian theory had its first striking successes in celestial mechanics. But how useful is this theory in a context more worldly than that of a planet in open space? To see the justification of this question, consider the system shown in the figure, a setup known as the **Atwood machine**: two bodies subject to gravitational force are tied to each other by an idealized massless string over an idealized frictionless pulley. What makes this problem different from those considered in celestial mechanics is that the participating bodies (the two masses) are *constrained* in their motion. The question then arises how this constraint can be incorporated into (Newton's) mechanical equations of motion, and how the ensuing equations can be solved. This problem is of profound applied relevance – of the hundreds of motions taking place in, say, the engine of a car practically all are constrained. In the eighteenth century, the era initiating the age of engineering and industrialization, the availability of a formalism capable of efficient formulation of problems subject to mechanical constraints became pressing.



Early solution schemes in terms of Newtonian mechanics relied on the concept of "constraining forces". The strategy there was to formulate a problem in terms of its basal unconstrained variables (e.g., the real space coordinates of the two masses in the figure). In a second step one would then introduce (infinitely strong) **constraining forces** serving to reduce the number of free coordinates (e.g., down to the one coordinate measuring the height of one of the masses in the figure.) However, strategies of this type soon turned out to be operationally suboptimal. The need to find more efficient formulations was motivation for intensive research activity, which eventually culminated in the modern formulations of classical

mechanics, **Lagrangian and Hamiltonian mechanics**.

INFO There are a number of conceptually **different types of mechanical constraints**: constraints that can be expressed in terms of equalities such as $f(\mathbf{q}, \dot{\mathbf{q}}) = 0$ are called **holonomic**. Here, $\mathbf{q} = (q^1, \ldots, q^n)$ are the coordinates of the unconstrained problem, and $\dot{q}$ the corresponding velocities. (The constraint of the Atwood machine belongs to this category: $x^1 + x^2 = l$, where l is a constant, and $x^i, i = 1, 2$ are the heights of the two bodies measured with respect to a common reference height.) This has to be contrasted to **non-holonomic** constraints, i.e. constraints formulated by *in*equalities. (Think of the molecules moving in a piston. Their coordinates obey the constraint $0 \leq q^j \leq L^j, j = 1, 2, 3$, where L_j are the extensions of the piston.)

Constraints explicitly involving time $f(\mathbf{q}, \dot{\mathbf{q}}, t) = 0$ are called **rheonomic**. For example, a particle constraint to move on a moving surface is subject to a rheonomic constraint. Constraints void of explicit time dependence are called **scleronomic**.

In this chapter, we will introduce the concept of variational principles to derive Lagrangian (and later Hamiltonian) mechanics from their Newtonian ancestor. Our construction falls short to elucidate the beautiful and important history developments that eventually led to the modern formulation. Also, it is difficult to motivate in advance. However, within the framework of a short introductory course, these shortcomings are outweighed by the brevity of the derivation. Still, it is highly recommended to consult a textbook on classical mechanics to learn more about the historical developments that led from Newtonian to Lagrangian mechanics.

Let us begin with a few simple and seemingly un–inspired manipulations of Newton's equations $m\ddot{\mathbf{q}} = \mathbf{F}$ of a single particle[1] subject to a conservative force $\mathbf{F} = -\partial_{\mathbf{q}} U(\mathbf{q})$. Also, let's think for a moment of $\mathbf{q}$ and $\dot{\mathbf{q}}$ as $2f$ independent variables (rather than as parameterizations of a curve and its velocity.) Now, notice that the l.h.s. of the equation may be written as $m\ddot{\mathbf{q}} = d_t \partial_{\dot{q}} T$, where $T = T(\dot{\mathbf{q}}) = \frac{m}{2}\dot{\mathbf{q}}^2$ is the particle's kinetic energy. However, T does not depend on the $\mathbf{q}$-'variables', $\partial_{\mathbf{q}} T = 0$. Conversely, the potential does not depend on $\dot{\mathbf{q}}$, $\partial_{\dot{\mathbf{q}}} U(\mathbf{q}) = 0$. Putting these observations together, we note that Newton's equation may be equivalently represented as

$$(d_t \partial_{\dot{\mathbf{q}}} - \partial_{\mathbf{q}}) L(\mathbf{q}, \dot{\mathbf{q}}) = 0, \tag{2.1}$$

where we have defined the **Lagrangian function**,

$$\boxed{L = T - U.} \tag{2.2}$$

But what is this reformulation good for? To appreciate the meaning of the mathematical structure of Eq. (2.1), we need to introduce the purely mathematical concept of

2.1 Variational principles

In standard calculus, one is concerned with functions $F(\mathbf{v})$ that take vectors $\mathbf{v} \in \mathbb{R}^n$ as arguments. Variational calculus generalizes standard calculus, in that one considers "functions"

[1] The generalization to many particles will be obvious.

$F[f]$ taking functions as arguments. Now, "function of a function" does not sound nice. This may be the reason for why the "function" F is actually called a **functional**. Similarly, it is customary to indicate the argument of a functional in angular brackets. The generalization $\mathbf{v} \to f$ is not in the least mysterious: as we have seen in previous chapters, one may discretize a function (cf. the discussion in section **??**) $f \to \{f_i | i = 1, \ldots, N\}$, thus interpreting it as the limiting case of an N-dimensional vector; in many aspects, variational calculus amounts to a straightforward generalization of standard calculus. At any rate, we will see that one may work with functionals much like with ordinary functions.

2.1.1 Definitions

NOTATION Throughout this section, we will denote parameterizations $\boldsymbol{\gamma}(t)$ of curves γ by the same symbol. Assuming cartesian coordinates, an index subscript notation is used, γ_i.

In this chapter we will focus on the class of functionals relevant to classical mechanics, viz. functionals $F[\gamma]$ taking *curves* in (subsets of) $\mathbb{R}^n$ as arguments.[2] To be specific, consider the set of all curves $M \equiv \{\gamma : I \equiv [t_0, t_1] \to U\}$ mapping an interval I into a subset $U \subset \mathbb{R}^n$ of n-dimensional space (see Fig. 2.1.)[3] Now, consider a mapping

$$\begin{aligned} \Phi :& M \to \mathbb{R}, \\ & \gamma \mapsto \Phi[\gamma], \end{aligned} \tag{2.3}$$

assigning to each curve γ a real number, i.e. a (real) **functional** on M.

EXAMPLE The **length of a curve** is defined as

$$L[\gamma] \equiv \int_{t_0}^{t_1} dt \; (\boldsymbol{\gamma}(t) \cdot \boldsymbol{\gamma}(t))^{1/2} \,. \tag{2.4}$$

It assigns to each curve its euclidean length. Some readers may question the consistency of the notation: on the l.h.s. we have the symbol γ (no derivatives), and on the r.h.s. $\dot{\gamma}$ (temporal derivatives.) However, there is no contradiction here. The notation $[\gamma]$ indicates dependence on the curve as a geometric object. By definition, this contains the full information on the curve, including all derivatives. The r.h.s. indicates that the functional L reads only partial information on the curve, viz. that contained in first derivatives.

Consider now two curves $\gamma, \gamma' \in M$ that lie "close" to each other. (For example, we may require that $|\boldsymbol{\gamma}(t) - \boldsymbol{\gamma}'(t)| < \epsilon$ for all t and some positive ϵ.) We are interested in the increment $\Phi[\gamma] - \Phi[\gamma']$. Defining $\gamma' = \gamma + h$, the functional Φ is called differentiable *iff*

$$\Phi[\gamma + h] - \Phi[\gamma] = F\big|_{\gamma}[h] + \mathcal{O}(h^2), \tag{2.5}$$

[2] We have actually met with more general functionals before. For example, the electric susceptibility $\chi[\mathbf{E}]$ is a functional of the electric field $\mathbf{E} : \mathbb{R}^4 \to \mathbb{R}^3$.

[3] The set U may actually be a lower dimensional submanifold of $\mathbb{R}^n$. For example, we might consider curves in a two–dimensional plane embedded in $\mathbb{R}^3$, etc.

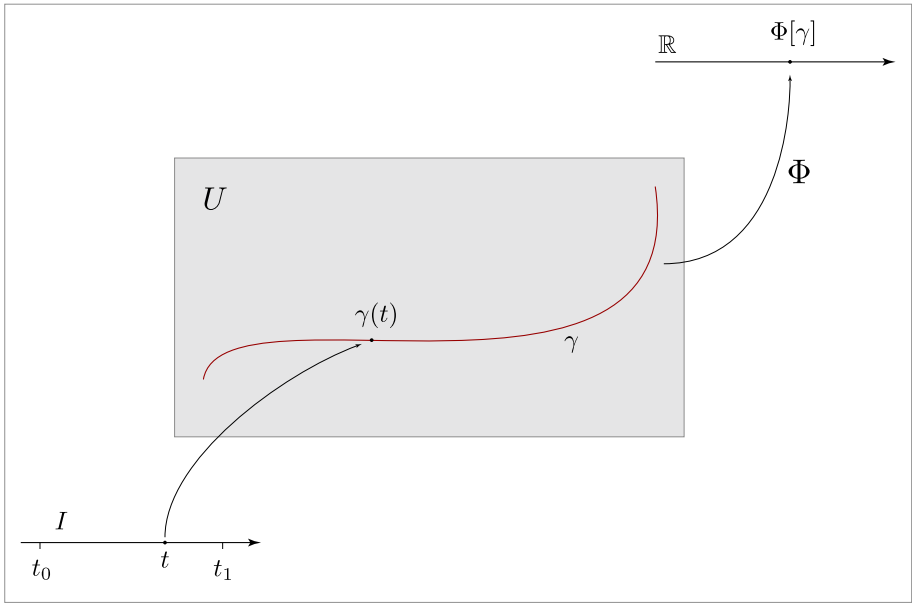


Figure 2.1: On the mathematical setting of functionals on curves (discussion, see text)

where $F\big|_\gamma[h]$ is a *linear* functional of h, i.e. a functional obeying $F\big|_\gamma[c_1h_1 + c_2h_2] = c_1F\big|_\gamma[h_1] + c_2F\big|_\gamma[h_2]$ for $c_1, c_2 \in \mathbb{R}$ and $h_{1,2} \in M$. In (2.5), $\mathcal{O}(h^2)$ indicates residual contributions of order h^2. For example, if $|h(t)| < \epsilon$ for all t, these terms would be of $\mathcal{O}(\epsilon^2)$.

The functional $F|_\gamma$ is called the **differential** of the functional Φ at γ. Notice that $F|_\gamma$ need not depend linearly on γ. The differential generalizes the notion of a derivative to functionals. Similarly, we may think of $\Phi[\gamma+h] = \Phi[\gamma]+F|_\gamma[h]+\mathcal{O}(h^2)$ as a generalized Taylor expansion. The linear functional $F|_\gamma$ describes the behavior of Φ in the vicinity of the reference curve γ. A curve γ is called an **extremal curve** of Φ if $F|_\gamma = 0$.

EXAMPLE Consider the length functional $L[\gamma]$ restricted to all curves $\gamma(t_0) = \gamma_0$, $\gamma(t_1) = \gamma_1$ beginning and ending at common points γ_0 and γ_1, respectively. To obtain the differential of that functional, we use $\boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = \gamma_i\gamma_i$ (summation convention) and consider the variation

$$\begin{aligned} L[\gamma+h] - L[\gamma] &= \int_{t_0}^{t_1} dt\, \left[((\dot\gamma_i+\dot h_i)(\dot\gamma_i+\dot h_i))^{1/2} - (\dot\gamma_i\dot\gamma_i)^{1/2}\right] = \\ &= \int_{t_0}^{t_1} dt\, \left[\frac{\dot\gamma_i\dot h_i}{|\dot{\boldsymbol{\gamma}}|} + \mathcal{O}(h^2)\right] = \\ &= \int_{t_0}^{t_1} dt\, \left[\frac{d}{dt}\left(\frac{\dot\gamma_i}{|\dot{\boldsymbol{\gamma}}|}\right) h_i + \mathcal{O}(h^2)\right], \end{aligned} \tag{2.6}$$

where $|\dot{\boldsymbol{\gamma}}| = (\dot\gamma_i\dot\gamma_i)^{1/2}$, and in the third line we integrated by parts. (Why does the integration by parts not generate a boundary terms?) This identifies the differential of the length functional as

$$F|_\gamma[h] = \int_{t_0}^{t_1} dt\, \frac{d}{dt}\left(\frac{\dot\gamma_i}{|\dot{\boldsymbol{\gamma}}|}\right) h_i. \tag{2.7}$$

The differential vanishes, if for all smooth curves h, $F|_\gamma[h] = 0$. Inspection of the integral representation shows that this is equivalent to the condition

$$\forall t: \qquad \frac{d}{dt}\left(\frac{\dot\gamma_i}{|\dot{\boldsymbol\gamma}|}\right) = 0, \qquad i = 1, \dots, n. \tag{2.8}$$

Argue why the differential vanishes on all curves that are *straight*, e.g. on

$$\boldsymbol\gamma(t) = \frac{1}{t_1 - t_0}\left[-\boldsymbol\gamma_0(t - t_1) + \boldsymbol\gamma_1(t - t_0)\right]. \tag{2.9}$$

Consider another straight connection of the two points,

$$\boldsymbol\gamma'(t) = \boldsymbol\gamma_0 + f(t)(\boldsymbol\gamma_1 - \boldsymbol\gamma_0), \tag{2.10}$$

where $f : [t_0, t_1] \to \mathbb{R}$ is a function with boundary condition $f(t_0) = 0$ and $f(t_1) = 1$. In general, this curve performs accelerated motion, i.e. $\ddot{\boldsymbol\gamma}' \neq 0$. Still it extremizes the length functional. Discuss why.

EXERCISE Re-familiarize yourself with the definition of the derivative $f'(x)$ of higher dimensional functions $f : \mathbb{R}^k \to \mathbb{R}$. Interpret the functional $\Phi[\gamma]$ as the limit of a function $\Phi : \mathbb{R}^N \to R, \{\gamma_i\} \to \Phi(\{\gamma_i\})$ where the vector $\{\gamma_i | i = 1, \dots, N\}$ is a discrete approximation of the curve γ. Think how the definition (2.5) generalizes the notion of differentiability and how $F|_\gamma \leftrightarrow f'(x)$ generalizes the definition of a derivative.

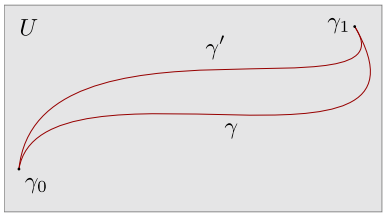
This is about as much as we need to say/define in most general terms. In the next section we will learn how to determine the extremal curves for an extremely important sub–family of functionals.

2.1.2 Euler–Lagrange equations

In the following, we will focus on functionals that afford a "local representation"

$$S[\gamma] = \int_{t_0}^{t_1} dt\, L(\boldsymbol\gamma(t), \dot{\boldsymbol\gamma}(t), t), \tag{2.11}$$

where $L : \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R}$ is a function. The functional S is "local" in that the integral kernel L does not depend on points on the curve at different times. **Local functionals** play an important role in applications. (For example, the length functional (2.4) belongs to this family.) We will consider the restriction of local functionals to the set of all curves $\gamma \in M$ that begin and end at common terminal points: $\boldsymbol\gamma(t_0) \equiv \boldsymbol\gamma_0$ and $\boldsymbol\gamma_{t_1} = \boldsymbol\gamma_1$ with fixed $\boldsymbol\gamma_0$ and $\boldsymbol\gamma_1$ (see the figure.) Again, this is a restriction motivated by the applications below. To keep the notation slim, we will denote the space of all curves thus restricted again by M.



We now prove the following important fact: the local functional $S[\gamma]$ is differentiable and its derivative is given by

$$F\big|_\gamma[h] = \int_{t_0}^{t_1} dt\ (\partial_{\boldsymbol{\gamma}} L - d_t \partial_{\dot{\boldsymbol{\gamma}}} L) \cdot \boldsymbol{h}. \tag{2.12}$$

Here, we are using the shorthand notation $\partial_{\boldsymbol{\gamma}} L \cdot \boldsymbol{h} \equiv \sum_{i=1}^n \partial_{x_i} L(x, \dot{\gamma}, t)\big|_{x_i=\gamma_i} h_i$ and analogously for $\partial_{\dot{\boldsymbol{\gamma}}} \cdot \boldsymbol{h}$. Eq. (2.12) is verified by straightforward Taylor series expansion:

$$\begin{aligned} S[\gamma + h] - S[\gamma] &= \int_{t_0}^{t_1} dt\ \Big(L(\boldsymbol{\gamma} + \boldsymbol{h}, \dot{\boldsymbol{\gamma}} + \dot{\boldsymbol{h}}, t) - L(\boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}}, t)\Big) = \\ &= \int_{t_0}^{t_1} dt\ \Big[\partial_{\boldsymbol{\gamma}} L \cdot \boldsymbol{h} + \partial_{\dot{\boldsymbol{\gamma}}} \cdot \dot{\boldsymbol{h}}\Big] + \mathcal{O}(h^2) = \\ &= \int_{t_0}^{t_1} dt\ [\partial_{\boldsymbol{\gamma}} L - d_t(\partial_{\dot{\boldsymbol{\gamma}}} L)] \cdot \boldsymbol{h} + \partial_{\dot{\boldsymbol{\gamma}}} L \cdot \boldsymbol{h}\big|_{t_0}^{t_1} + \mathcal{O}(h^2), \end{aligned}$$

where in the last step, we have integrated by parts. The surface term vanishes because $\boldsymbol{h}(t_0) = \boldsymbol{h}(t_1) = 0$, on account of the condition $\boldsymbol{\gamma}(t_i) = (\boldsymbol{\gamma}+\boldsymbol{h})(t_i) = \boldsymbol{\gamma}_i, i = 0, 1$. Comparison with the definition (2.5) then readily gets us to (2.12).

Eq. (2.12) entails an important corollary: the local functional S is extremal on all curves obeying the so-called Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}_i} - \frac{\partial L}{\partial \gamma_i} = 0, \quad i = 1, \ldots, N.$$

The reason is that if and only if these N conditions hold, will the linear functional (2.12) vanish on arbitrary curves h. (Exercise: assuming that one or several of the conditions above are violated, construct a curve h on which the functional (2.12) will not vanish.)

Let us summarize what we have got: for a given function $L : \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R} \to \mathbb{R}$, the local functional

$$\begin{aligned} &S : M \to \mathbb{R}, \\ &\gamma \mapsto S[\gamma] \equiv \int_{t_0}^{t_1} dt\, L(\boldsymbol{\gamma}(t), \dot{\boldsymbol{\gamma}}(t), t), \end{aligned} \tag{2.13}$$

defined on the set $M = \{\gamma : [t_0, t_1] \to \mathbb{R}^n | \boldsymbol{\gamma}(t_0) = \boldsymbol{\gamma}_0,\ \boldsymbol{\gamma}(t_1) = \boldsymbol{\gamma}_1\}$ is extremal on curves obeying the conditions

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\gamma}_i} - \frac{\partial L}{\partial \gamma_i} = 0, \quad i = 1, \ldots, n. \tag{2.14}$$

These equations are called the **Euler–Lagrange equations** of the functional S. The function L is often called the **Lagrangian (function)** and S an **action functional.**

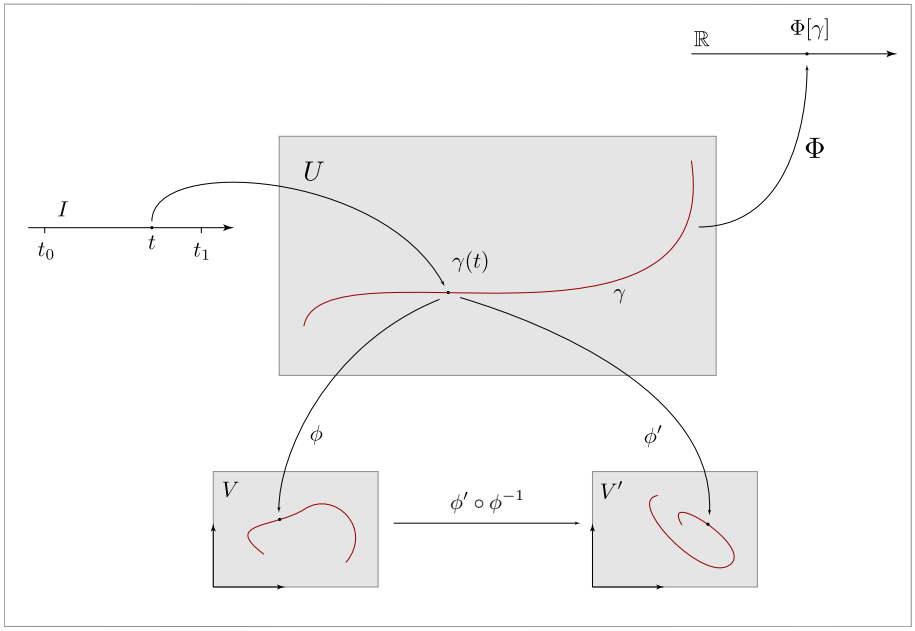


Figure 2.2: On the representation of curves and functionals in different coordinates

At this stage, one may observe a suspicious structural similarity between the Euler-Lagrange equations and our early reformulation of Newton's equations (2.1). However, before shedding more light on this connection, it is worthwhile to illustrate the usage of Euler-Lagrange calculus on an

EXERCISE Compute the Euler-Lagrange equations of the length functional (2.4) to re-establish the results discussed above.

2.1.3 Coordinate invariance of Euler-Lagrange equations

What we actually mean when we write $\frac{\partial L}{\partial \gamma_i(t)}$ is: differentiate the function $L(\gamma, \dot\gamma, t)$ w.r.t. the ith coordinate of the curve γ at time t. However the same curve can be represented in different coordinates! For example, a three dimensional curve γ will have different representations depending on whether we work in cartesian coordinates $\gamma(t) \leftrightarrow (x^1(t), x^2(t), x^3(t))$ or spherical coordinates $\gamma(t) \leftrightarrow (r(t), \theta(t), \phi(t))$.

Yet, nowhere in our derivation of the Euler–Lagrange equations did we make reference to specific properties of the coordinate system. This suggests that the Euler–Lagrange equations assume the same form, Eq. (2.14), in *all* coordinate systems. (To appreciate the meaning of this statement, compare with the Newton equations which assume their canonical form $\ddot x^i = f^i(\mathbf{x})$ only in cartesian coordinates.) Coordinate changes play an extremely important role in analytical mechanics, especially when it comes to problems with constraints. Hence, anticipating that the Euler–Lagrange formalism is slowly revealing itself as a replacement of Newton's formulation, it is well invested time to take a close look at the role of coordinates.

Both a curve γ and the functional $S[\gamma]$ are canonical objects, no reference to coordinates made here. The curve is simply a map $\gamma : I \to U$ and $S[\gamma]$ assigns to that map a number. However, most of the time when we actually need to work with curve, we do so in a system of coordinates. Mathematically, a **coordinate system** of U is a diffeomorphic[4] map:

$$\begin{aligned}\phi :&U \to V,\\ &\boldsymbol{r} \mapsto \boldsymbol{\phi}(\boldsymbol{r}) \equiv \boldsymbol{y} \equiv (y^1, \ldots, y^n),\end{aligned}$$

where the coordinate domain V is an open subset of $\mathbb{R}^m$ and m is the dimensionality of $U \subset \mathbb{R}^m$. (The set U may be of lower dimension than embedding space $\mathbb{R}^n$.) For example, spherical coordinates $]0, \pi[\times]0, 2\pi[\ni (\theta, \phi) \mapsto S^2 \subset \mathbb{R}^3$ assign to each coordinate pair (θ, ϕ) a point on the two–dimensional sphere, etc.[5] Throughout, we will adopt a policy where $\mathbf{y}$ refers to general coordinate vectors (spherical, cyldrical, ...), while $\mathbf{x}$ is reserved for the particular choice of cartesian coordinates.

Given a coordinate system, ϕ, the abstract curve γ defines a curve $\mathbf{y} \equiv \boldsymbol{\phi} \circ \gamma : I \to V; t \mapsto \mathbf{y}(t) = \boldsymbol{\phi}(\gamma(t))$ in coordinate space (see Fig. 2.2.) What we actually mean when we referred to $\gamma_i(t)$ in the previous sections, are the coordinates $y^i(t)$ of γ in the coordinate system $\boldsymbol{\phi}$; we have been following the widespread policy to denote the coordinates y^i (of a given choice) and the geometric object they refer to (γ) by the same symbol $y^i = \gamma_i$. As long as one knows what one is doing, this abbreviated notation does not do harm. Occasionally, however, like in the present discussion, it pays to be more explicit. On the same note, when talking about generic coordinates, it may be expedient to use covariant notation, y^i, i.e. place indices upstairs.

The abstract functional $S[\gamma]$ defines a functional $S_c[\mathbf{y}] \equiv S[\boldsymbol{\phi}^{-1} \circ \mathbf{y}]$ on curves in the coordinate domain. In our derivation of the Euler-Lagrange equations we have been making tacit use of a coordinate representation of this kind. In other words, the Euler–Lagrange equations were actually derived for the representation $S_c[\mathbf{y}]$. (Again it is customary to simply write $S[\gamma]$ if reference to a specific coordinate representation is implied. Another customary notation is $S[\mathbf{y}]$ (omitting the subscript c). Occasionally one writes $S[\boldsymbol{\gamma}]$ where $\boldsymbol{\gamma}$ is meant to be the vector of coordinates of γ in a specific system.)

Now, suppose we are given another coordinate representation of U, that is, a diffeomorphism $\boldsymbol{\phi}' : U \to V'$, $\mathbf{r} \mapsto \boldsymbol{\phi}'(\mathbf{r}) = \mathbf{y}' \equiv (y'^1, \ldots, y'^m)$. A point on the curve, $\boldsymbol{\gamma}(t)$ now has two different coordinate representations, $\mathbf{y}(t) = \boldsymbol{\phi}(\boldsymbol{\gamma}(t))$ and $\mathbf{y}'(t) = \boldsymbol{\phi}'(\gamma(t))$. The **coordinate transformation** between these representations is described by the map

$$\begin{aligned}\boldsymbol{\phi}' \circ \boldsymbol{\phi}^{-1} :&V \to V',\\ &\mathbf{y} \mapsto \mathbf{y}' = \boldsymbol{\phi}' \circ \boldsymbol{\phi}^{-1}(\mathbf{x}).\end{aligned}$$

By construction, this is a smooth and invertible map between open subsets of $\mathbb{R}^m$. For example, if $\mathbf{y} = (r, \theta, \phi)$ are spherical coordinates and $\mathbf{x} = \mathbf{y}'$ Cartesian coordinates, we would have $x^1(\mathbf{y}) = r \sin\theta \cos\phi$, etc.

[4] Loosely speaking, this means invertible and smooth (differentiable.)

[5] We here avoid the discussion of the complications arising when U cannot be covered by a single coordinate "chart" ϕ. For example, the sphere S^2 cannot be fully covered by a single coordinate mapping. (Think why.)

The most important point is that $\mathbf{y}(t)$ and $\mathbf{y}'(t)$ describe the same curve, $\boldsymbol{\gamma}(t)$, only in different representations. Specifically, if the reference curve γ is an extremal curve, the coordinate representations $\mathbf{y} : I \to V$ and $\mathbf{y}' : I \to V'$ will be extremal curves, too. According to our discussion in section 2.1.2 both curves must be solutions of the Euler-Lagrange equations, i.e. we can draw the conclusion:

$$\begin{aligned} &\gamma \text{ extremal } \Rightarrow \\ &\frac{d}{dt}\frac{\partial L}{\partial \dot{y}^i} - \frac{\partial L}{\partial y^i} = 0, \\ &\frac{d}{dt}\frac{\partial L'}{\partial \dot{y}'^i} - \frac{\partial L'}{\partial y'^i} = 0, \end{aligned} \tag{2.15}$$

where $L'(\mathbf{y}', \dot{\mathbf{y}}', t) \equiv L(\mathbf{y}(\mathbf{y}'), \dot{\mathbf{y}}(\mathbf{y}'), t)$ is the Lagrangian in the $\boldsymbol{\phi'}$–coordinate representation. In other words

The Euler–Lagrange equations are coordinate invariant.

They assume the same form in all coordinate systems.

EXERCISE Above we have shown the coordinate invariance of the Euler–Lagrange equations by conceptual reasoning. However, it must be possible to obtain the same **invariance properties by brute force computation**. To show that the second line in (2.15) follows from the first, use the chain rule, $d_t x'^i = \sum_j \frac{\partial x'^i}{\partial x^j} d_t x^j$, and its immediate consequence $\frac{\partial \dot{x}'^i}{\partial \dot{x}^j} = \frac{\partial x'^i}{\partial x^j}$.

EXAMPLE Let us illustrate the coordinate invariance of the variational formalism on the example of the functional "curve length" discussed on p 53. Considering the case of curves in the plane, $n = 2$, we might get the idea to attack this problem in polar coordinates $\phi^{-1}(x) = (r, \varphi)$. The polar coordinate representation of the cartesian Lagrangian $L(x_1, x_2, \dot{x}_1, \dot{x}_2) = (\dot{x}_1^2 + \dot{x}_2^2)^{1/2}$ reads (verify it)

$$L(r, \varphi, \dot{r}, \dot{\varphi}) = (\dot{r}^2 + r^2\dot{\varphi}^2)^{1/2}.$$

It is now straightforward to compute the Euler–Lagrange equations

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} &= \dot{\varphi}(\dots) \stackrel{!}{=} 0, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} &= \dot{\varphi}(\dots) \stackrel{!}{=} 0. \end{aligned}$$

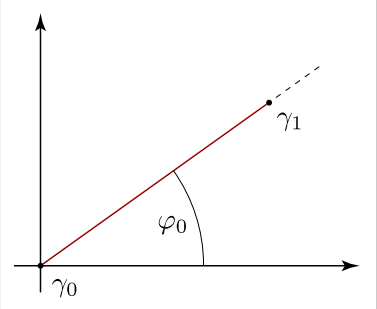
Here, the notation $\dot{\varphi}(\dots)$ indicates that we are getting a lengthy list of terms which, however, are all weighted by $\dot{\varphi}$. Putting the initial point into the origin $\phi(\gamma_0) = (0, 0)$ and the final point somewhere into the plane, $\phi(\gamma_1) = (r_1, \varphi_1)$, we thus conclude that the straight line connectors, $\phi(\gamma(t)) = (r(t), \varphi_0)$ are solutions of the Euler–Lagrange equations. (It is less straightforward to show that these are the *only* solutions.)

With these preparations in store, we are now in a very good position to apply variational calculus to a new and powerful reformulation of Newtonian mechanics.

2.2 Lagrangian mechanics

2.2.1 The idea

Before turning to the general discussion of Lagrangian mechanics, let us illustrate its workings on two simple examples.



EXAMPLE Imagine a mechanical problem subject to constraints. For definiteness, we may consider the system shown on the right – a bead sliding on a rod and subject to the gravitational force, $\mathbf{F}_g$. In principle, we may describe the situation in terms of Newton's equations. These equations must contain an "infinitely strong" force $\mathbf{F}_c$ whose sole function is to keep the bead on the rod. About this force we do not know much (other than that it acts vertically to the rod, and vanishes right on the rod.)
According to the structures outlined above, we may now reformulate Newton's equations as (2.1), where the potential $V = V_g + V_c$ in the Lagrangian $L = T - V$ contains two contributions, one accounting for the gravitational force, V_g, and another, V_c, for the force $\mathbf{F}_c$. We also know that the sought for solution curve $\mathbf{q} : I \to \mathbb{R}^3, t \mapsto \mathbf{q}(t)$ will extremize the action $S[\mathbf{q}] = \int_{t_0}^{t_1} dt\, L(\mathbf{q}, \dot{\mathbf{q}})$. (Our problem does not include explicit time dependence, i.e. L does not carry a time–argument.) So far, we have not gained much. But let us now play the trump card of the new formulation, its invariance under coordinate changes.
In the above formulation of the problem, we are seeking for an extremum of $S[q]$ on the set of *all* curves $I \to \mathbb{R}^3$. However, we know that all curves in $\mathbb{R}^3$ will be subject to the "infinitely strong" potential of the constraint force, unless they lie right on the rod S. The action of those generic curves will be infinitely large and we may remove them from the set of curves entering the variational procedure *from the outset*. This observation suggests to represent the problem in terms of coordinates $(s, \mathbf{q}_\perp)$, where the one–dimensional coordinate s parameterizes the curve, and the $(3-1)$–dimensional coordinate vector $\mathbf{q}_\perp$ parameterizes the space normal to the curve. Knowing that curves with non-vanishing $\mathbf{q}_\perp(t)$ will have an infinitely large action, we may restrict the set of curves under consideration to $M \equiv \{\gamma : I \to S, t \mapsto (s(t), 0)\}$, i.e. to curves in S. On the string, S, the constraint force $\mathbf{F}_c$ vanishes. The above limitation thus entails that the constraint forces will never explicitly appear in the variational procedure; this is an enormous simplification of the problem. Also, our problem has effectively become one–dimensional. The Lagrangian evaluated on curves in S is a function

$$L(\mathbf{q}, \dot{\mathbf{q}}) = L((\mathbf{q}_\perp, s), (\dot{\mathbf{q}}_\perp, \dot{s})) \to L((\mathbf{0}, s), (\dot{\mathbf{0}}, \dot{s})) \equiv L(s, \dot{s})$$

much simpler than the original Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ with its constraint force potential.
To be concrete, we measure the coordinate s from the origin of the rod, whereupon the potential of the gravitational force becomes $V_g = mg \sin\phi s$. The kinetic energy assumes the form $T =$

$\frac{m}{2}\dot{\mathbf{q}}^2 = \frac{m}{2}(\dot{\mathbf{q}}_\perp^2 + \dot{s}^2) \to \frac{m}{2}\dot{s}^2$, so that the effective Lagrangian reads

$$L(s,\dot{s}) = \frac{m}{2}\dot{s}^2 - mg\sin\phi s.$$

From here, we obtain the Euler-Lagrange equation as

$$d_t\frac{\partial L}{\partial \dot{s}} - \frac{\partial L}{\partial s} = m\ddot{s} + mg\sin\phi,$$

or $\ddot{s} = -g\sin\phi$, equivalent to a problem in a constant force $mg\sin\phi$. (Discuss this equation.)

EXAMPLE As a second example, illustrating the convenience of the Lagrangian equation we consider the **Atwood machine** depicted on p47. The Lagrangian of this problem reads

$$L(\mathbf{q}_1,\mathbf{q}_2,\dot{\mathbf{q}}_1,\dot{\mathbf{q}}_2) = \frac{m_1}{2}\dot{\mathbf{q}}_1^2 + \frac{m_2}{2}\dot{\mathbf{q}}_2^2 - m_1gz_1 - m_2gz_2.$$

Here, $\mathbf{q}_i$ is the position of mass i, $i = 1, 2$ and z_i is its height. In principle, the sought for solution $\gamma(t) = (\mathbf{q}_1(t), \mathbf{q}_2(t))$ is a curve in six dimensional space. We assume that the masses are released at rest at initial coordinates $(x_i(t_0), y_i(t_0), z_i(t_0))$. The coordinates y_i and z_i will not change in the process, they are unconstrained (except that have to be equal $(x_1, y_2) = (x_2, y_2)$). Effectively we are thus seeking for solution curves in the two–dimensional space of coordinates (z_1, z_2). The constraint now implies that $x \equiv z_1 = l - z_2$, or $z_2 = l - z$. Thus, our solution curves are uniquely parameterized by the "generalized coordinate" z as $(z_1, z_2) = (z, l - z)$. We now enter with this parameterization into the Lagrangian above to obtain

$$L(z,\dot{z}) = \frac{m_1 + m_2}{2}\dot{z}^2 - (m_1 - m_2)gz + \text{const.}$$

It is important to realize that this function uniquely specifies the action

$$S[z] = \int_{t_0}^{t_1} dt\, L(z(t), \dot{z}(t))$$

of physically allowed curves. The extremal curve, which then describes the actual motion of the two–body system, will be solution of the equation

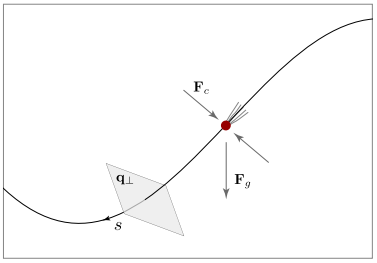
$$d_t\frac{\partial L}{\partial \dot{z}} - \frac{\partial L}{\partial z} = (m_1 + m_2)\ddot{z} + (m_1 - m_2)g = 0.$$

For the given initial conditions $z(t_0) = z_1(t_0)$ and $\dot{z}(t_0) = 0$, this equation is solved by

$$z(t) = z(t_0) - \frac{m_1 - m_2}{m_1 + m_2}\frac{g}{2}(t - t_0)^2.$$

Substitution of this solution into $\mathbf{q}_1 = (x_1, y_1, z)$ and $\mathbf{q}_2 = (x_2, y_2, l - z)$ solves our problem.

2.2.2 Hamilton's principle



After this preparation, we are in a position to formulate a new approach to solving mechanical problems. Suppose, we are given an N–particle setup specified by the following data: (i) a $3N$-dimensional coordinate vector $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ required to register the coordinate vectors $\mathbf{x}_i$ of N particles, a Lagrangian function $L : \mathbb{R}^{6N+1} \to \mathbb{R}, (\mathbf{x}, \dot{\mathbf{x}}, t) \mapsto L(\mathbf{x}, \dot{\mathbf{x}}, t)$ defined in the $6N$–dimensional space of coordinates and velocities, and (ii) a set of constraints limiting the motion of particles to an f–dimensional submanifold of $\mathbb{R}^{3N}$.[6] Mathematically, a (holonomic) set of constraints will be implemented through $3N - f$ equations

$$F_j(\mathbf{x}, t) = 0, \qquad j = 1, \ldots, 3N - f. \tag{2.16}$$

The number f is called the number of **degrees of freedom** of the problem.

Hamiltons's principle states that such a problem is to be solved by a three–step algorithm:

- ▷ Resolve the constraints (2.16) in terms of f parameters $\mathbf{q} \equiv (q_1, \ldots, q_f)$, i.e. find a representation $\mathbf{x}(\mathbf{q})$, such that the constraints $F_j(\mathbf{x}(\mathbf{q})) = 0$ are resolved for all $j = 1, \ldots, 3N - f$. The parameters q_i are called **generalized coordinates** of the problem. The maximal set of parameter configurations $\mathbf{q} \equiv (q^1, \ldots q^f)$ compatible with the constraint defines a subset $V \subset \mathbb{R}^f$ and the map $V \to \mathbb{R}^{3N}, \mathbf{q} \mapsto (\mathbf{x}_1(\mathbf{q}), \ldots, \mathbf{x}_N(\mathbf{q}))$ defines an f–dimensional submanifold of $\mathbb{R}^{3N}$.

- ▷ Reduce the Lagrangian of the problem to an effective Lagrangian

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) \equiv L(\mathbf{x}, \dot{\mathbf{x}}(\mathbf{q}), t).$$

  In practice, this amounts to a substitution of $\mathbf{x}_i(t) = \mathbf{x}_i(\mathbf{q}(t))$ into the original Lagrangian. The effective Lagrangian is a function $L : V \times \mathbb{R}^f \times \mathbb{R} \to \mathbb{R}, (\mathbf{q}, \dot{\mathbf{q}}, t) \mapsto L(\mathbf{q}, \dot{\mathbf{q}}, t)$.

- ▷ Finally formulate and solve the **Euler–Lagrange equations**

$$\frac{d}{dt}\frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{\mathbf{q}}^i} - \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \mathbf{q}^i} = 0, \qquad i = 1, \ldots, f. \tag{2.17}$$

The prescription above is equivalent to the following statement, which is known as **Hamiltons principle**

[6] Loosely speaking, a d–dimensional **submanifold** of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that affords a smooth parameterization in terms of $d < n$ coordinates (i.e. is locally diffeomorphic to open subsets of $\mathbb{R}^d$.) Think of a smooth surface in three–dimensional space $(n = 3, d = 2)$ or a line $(n = 3, d = 1)$, etc.

Consider a mechanical problem formulated in terms of f generalized coordinates $\mathbf{q} = (q^1, \ldots, q^f)$ and a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t) = (T - U)(\mathbf{q}, \dot{\mathbf{q}}, t)$. Let $\mathbf{q}(t)$ be solution of the Euler-Lagrange equations

$$(d_t \partial_{\dot{q}^i} - \partial_{q^i}) L(\mathbf{q}, \dot{\mathbf{q}}, t) = 0, \qquad i = 1, \ldots, f,$$

at given initial and final configuration $\mathbf{q}(t_0) = \mathbf{q}_0$ and $\mathbf{q}(t_1) = \mathbf{q}_1$. This curve describes the physical motion of the system. It is an extremal curve of the **action functional**

$$S[\mathbf{q}] = \int_{t_0}^{t_1} dt\, L(\mathbf{q}, \dot{\mathbf{q}}, t).$$

To conclude this section, let us transfer a number of important physical quantities from Newtonian mechanics to the more general framework of Lagrangian mechanics: the **generalized momentum** associated to a generalized coordinate $q^i i$ is defined as[7]

$$p_i = \frac{\partial L(\mathbf{q}, \dot{\mathbf{q}}, t)}{\partial \dot{q}^i}. \tag{2.18}$$

Notice that for cartesian coordinates, $p_i = \partial_{\dot{q}^i} L = \partial_{\dot{q}^i} T = m\dot{q}^i$ reduces to the familiar momentum variable. We call a coordinate q^i a **cyclic variable** if it does not enter the Lagrangian function, that is if $\partial_{q^i} L = 0$. These two definitions imply that

the generalized momentum corresponding to a cyclic variable is conserved, $d_t p_i = 0$.

This follows from $d_t p_i = d_t \partial_{\dot{q}^i} L = \partial_{q^i} L = 0$.

In general, we call the derivative $\partial_{q^i} L \equiv F_i$ a **generalized force**. In the language of generalized momenta and forces, the Lagrange equations (formally) assume the form of Newton-like equations,

$$d_t p_i = F_i.$$

For the convenience of the reader, the most important quantities revolving around the Lagrangian formalism are summarized in table 2.1.

2.2.3 Lagrange mechanics and symmetries

Above, we have emphasized the capacity of the Lagrange formalism to handle problems with constraints. However, an advantage of equal importance is its flexibility in the choice of

[7] Notice that our momentum variables carry subscript (covariant) indices. What here looks like a simple mnemonic (differentiate scalar w.r.t. superscript variable to get subscript variable) has an underlying reason which becomes transparent within the framework of Hamiltonian dynamics.

quantity	designation or definition
generalized coordinate	q_i
generalized momentum	$p_i = \partial_{\dot{q}_i} L$
generalized force	$F_i = \partial_{q_i} L$
Lagrangian	$L(\mathbf{q}, \dot{\mathbf{q}}, t) = (T - U)(\mathbf{q}, \dot{\mathbf{q}}, t)$
Action (functional)	$S[q] = \int_{t_0}^{t_1} dt\, L(\mathbf{q}, \dot{\mathbf{q}}, t)$
Euler-Lagrange equations	$(d_t \partial_{\dot{q}_i} - \partial_{q_i}) L = 0$

Table 2.1: Basic definitions of Lagrangian mechanics

problem adjusted coordinates: unlike the Newton equations, the Lagrange equations maintain their form in all coordinate systems.

The choice of "good coordinates" becomes instrumental in problems with symmetries. From experience we know that a symmetry (think of z–axis rotational invariance) entails the conservation of a physical variable (the z–component of angular momentum), and that it is important to work in coordinates reflecting the symmetry (z–axis cylindrical coordinates.) But how do we actually *define* the term "symmetry"? And how can we find the ensuing conservation laws? Finally, how do we obtain a system of symmetry–adjusted coordinates? In this section, we will provide answers to these questions.

2.2.4 Noether theorem

Consider a family of mappings, $\mathbf{h}_s$, of the coordinate manifold of a mechanical system into itself,

$$\begin{aligned} \mathbf{h}_s :& V \to V, \\ & \mathbf{q} \to \mathbf{h}_s(\mathbf{q}). \end{aligned} \tag{2.19}$$

Here, $s \in \mathbb{R}$ is a control parameter, and we require that $\mathbf{h}_0 = \mathbf{id}$. is the identity transform. By way of example, consider the map $\mathbf{h}_s(r, \theta, \phi) \equiv (r, \theta, \phi + s)$ describing a rotation around the z-axis in the language of spherical coordinates, etc. For each curve $\mathbf{q} : I \to V, t \mapsto \mathbf{q}(t)$, the map $\mathbf{h}_s$ gives us a new curve $\mathbf{h}_s \circ \mathbf{q} : I \to V, t \mapsto \mathbf{h}_s(\mathbf{q}(t))$ (see the figure).

We call the transformation $\mathbf{h}_s$ a **symmetry (transformation)** of a mechanical system iff

$$S[\mathbf{h}_s \circ \mathbf{q}] = S[\mathbf{q}],$$

for all s and curves $\mathbf{q}$. The action is then said to be *invariant* under the symmetry transformation.

Suppose, we have found a symmetry and its representation in terms of a family of invariant transformations. Associated to that symmetry there is a quantity that is conserved during the dynamical evolution of the system. The correspondence between symmetry and its conservation law is established by a famous result due to Emmy Noether:

Noether theorem (1915-18): Let the action $S[\mathbf{q}]$ be invariant under the transformation $\mathbf{q} \mapsto \mathbf{h}_s(\mathbf{q})$. Let $\mathbf{q}(t)$ be an solution curve of the system (a solution of the Euler-Lagrange equations). Then, the quantity

$$I(\mathbf{q}, \dot{\mathbf{q}}) \equiv \sum_{i=1}^{f} p_i \frac{d}{ds} h_s^i(\mathbf{q})\Big|_{s=0} \tag{2.20}$$

is dynamically conserved:

$$d_t I(\mathbf{q}, \dot{\mathbf{q}}) = 0.$$

Here, $p_i = \partial_{\dot{q}^i} L\big|_{(\mathbf{q},\dot{\mathbf{q}})}$ is the generalized momentum of the ith (untransformed, $s = 0$) coordinate, and the quantity $I(\mathbf{q}, \dot{\mathbf{q}})$ is known as the **Noether momentum**.

The proof of Noether's theorem is straightforward: the requirement of action–invariance under the transformation $\mathbf{h}_s$ is equivalent to the condition $d_s S[\mathbf{h}_s \circ \mathbf{q}] = 0$ for all values of s. We now consider the action of an extremal curve, i.e. of a solution of the Euler-Lagrange equations, evaluated between *arbitrary* times, t_0 and t_1.

$$S[\mathbf{q}] = \int_{t_0}^{t_1} dt\, L(\mathbf{q}, \dot{\mathbf{q}}, t).$$

We do not require the transformation $\mathbf{h}_s$ to leave the terminal configurations invariant, i.e. $\mathbf{h}_s(\mathbf{q}(t_l)) \neq \mathbf{q}(t_l)$, $l = 0, 1$ in general. We next explore what the condition of action invariance tells us about the Lagrangian of the theory. To this end, we evaluate the invariance condition $d_s S[\mathbf{h}_s \circ \mathbf{q}]$ at $s = 0$, where $\mathbf{h}_0 \circ \mathbf{q} = \mathbf{h}_0(\mathbf{q}) = \mathbf{q}$:

$$\begin{aligned}
0 &\stackrel{!}{=} d_s\big|_{s=0} \int_{t_0}^{t_1} dt\, L(\mathbf{h}_s(\mathbf{q}), \dot{\mathbf{h}}_s(\mathbf{q}), t) \\
&= \int_{t_0}^{t_1} dt\, \left(\partial_{q^i} L\big|_{\mathbf{q}=\mathbf{h}_0(\mathbf{q})} d_s h_s^i + \partial_{\dot{q}^i} L\big|_{\dot{\mathbf{q}}=\dot{\mathbf{h}}_0(\mathbf{q})} d_s \dot{h}_s^i \right) = \\
&= \int_{t_0}^{t_1} dt\, \left(\partial_{q^i} - d_t \partial_{\dot{q}^i} L \right) d_s h_s^i + \partial_{\dot{q}^i} L\, d_s h_s^i \big|_{t_0}^{t_1},
\end{aligned}$$

where in the second equality we integrated by parts to remove the time derivative in $d_s \dot{h}_s$, we used $\mathbf{h}_0(\mathbf{q}) = \mathbf{q}$ and $\dot{\mathbf{h}}_0(\mathbf{q}) = \dot{\mathbf{q}}$, all s-derivatives are exectuted at $s = 0$, i.e. $d_s h_s = d_s|_{s=0} h_s$, and we use the abbreviated notation $h_s^i = (\mathbf{h}_s(\mathbf{q}))^i$. Now, our reference curve is a solution curve, which means that the integrand in the last line vanishes. With $\partial_{\dot{q}^i} L = p_i$, we are led to the conclusion

$$(p_i\, d_s h_s^i)_{t_1} = (p_i\, d_s h_s^i)_{t_0},$$

for arbitrary values t_0, t_1. In other words, $d_t p_i d_s h_s^i$ is temporally constant, which is the statement made by Noether's theorem.

Two practical remarks on this result:

- Since the Noether momentum is computed for $s = 0$, it is often sufficient, to describe the symmetry transformation $\mathbf{h}_s$ for infinitesimal values of the control parameter s. In practice, this means that we fix a pair $(\mathbf{q}, \dot{\mathbf{q}})$ comprising coordinate and velocity of a solution curve. We then consider an **infinitesimal transformation** $h_\epsilon(\mathbf{q})$, where ϵ is infinitesimally small, and $\mathbf{h}_\epsilon = \mathrm{id.} + \mathcal{O}(\epsilon)$ is very close to the identity transformation.

- It is usually convenient to work in **symmetry adjusted coordinates**, i.e. in coordinates where the transformation $\mathbf{h}_s$ assumes its simplest possible form. These are coordinates where $\mathbf{h}_s$ acts by *translation* in one coordinate direction, that is $(\mathbf{h}_s(\mathbf{q}))_i = q_i + s\delta_{ij}$, where j is the affected coordinate direction. Coordinates adjusted to a symmetry are cyclic (think about this point), and the Noether momentum

  $$I = p_i,$$

  reduces to the generalized momentum of the symmetry coordinate.

2.2.5 Examples

In this section, we will discuss two prominent examples of symmetries and their conservation laws.

Translational invariance ↔ conservation of momentum

Consider a mechanical system that is invariant under translation in some direction. Without loss of generality, we choose cartesian coordinates $\mathbf{q} = (q_1, q_2, q_3)$ in such a way that the coordinate q_1 parameterizes the invariant direction. The symmetry transformation $\mathbf{h}_s$ then translates in this direction: $\mathbf{h}_s(\mathbf{q}) = \mathbf{q} + s\mathbf{e}_1$, or $\mathbf{h}_s(\mathbf{q}) = (q^1 + s, q^2, q^3)$. With $d_s\mathbf{h}_s(\mathbf{q}) = \mathbf{e}_1$, we readily obtain

$$I(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \dot{q}^1} = p_1,$$

where $p_1 = m\dot{q}^1$ is the ordinary cartesian momentum of the particle. We are thus led to the conclusion

Translational invariance entails the conservation of cartesian momentum.

EXERCISE Generalize the construction above to a system of N particles in the absence of external potentials. The (interaction) potential of the system then depends only on coordinate *differences* $\mathbf{q}_i - \mathbf{q}_j$. Show that translation in arbitrary directions is a symmetry of the system and that this implies the conservation of the total momentum $\mathbf{P} = \sum_{j=1}^{N} m_j \dot{\mathbf{q}}_j$.

Rotational invariance ↔ conservation of angular momentum

As a second example, consider a system invariant under rotations around the 3-axis. In cartesian coordinates, the corresponding symmetry transformation is described by

$$\mathbf{q} \mapsto h_\phi(\mathbf{q}) \equiv R^{(3)}_\phi \mathbf{q},$$

where the angle ϕ serves as a control parameter (i.e. ϕ assumes the role of the parameter s above), and

$$R^{(3)}_{\phi} \equiv \begin{pmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is an $\mathrm{O}(3)$–matrix generating rotations by the angle ϕ around the 3–axis. The infinitesimal variant of a rotation is described by

$$R^{(3)}_{\epsilon} \equiv \begin{pmatrix} 1 & \epsilon & 0 \\ -\epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\epsilon^2).$$

From this representation, we obtain the Noether momentum as

$$I(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \dot{q}_i} d_\epsilon\big|_{\epsilon=0} (R^{(3)}_\epsilon \mathbf{q})_i = m(\dot{q}_1 q_2 - \dot{q}_2 q_1).$$

This is (the negative of) the 3-component of the particle's angular momentum. Since the choice of the 3–axis as a reference axis was arbitrary, we have established the result:

> Rotational invariance around a symmetry axis entails the conservation of the angular momentum component along that axis.

Now, we have argued above that in cases with symmetries, one should employ adjusted coordinates. Presently, this means coordinates that are organized around the 3–axis: spherical, or cylindrical coordinates. Choosing **cylindrical coordinates** for definiteness, the Lagrangian assumes the form

$$L(r, \phi, \dot{r}, \dot{\phi}, \dot{z}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2) - U(r, z), \tag{2.21}$$

where we noted that the problem of a rotationally invariant system does not depend on ϕ. The symmetry transformation now simply acts by translation, $\mathbf{h}_s(r, z, \phi) = (r, z, \phi + s)$, and the Noether current is given by

$$I \equiv l_3 = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}. \tag{2.22}$$

We recognize this as the cylindrical coordinate representation of the 3–component of angular momentum.

EXAMPLE Let us briefly extend the discussion above to re-derive the symmetry optimized representation of a **particle in a central potential**. We choose cylindrical coordinates, such that (a) the force center lies in the origin, and (b) at time $t = 0$, both $\mathbf{q}$ and $\dot{\mathbf{q}}$ lie in the $z = 0$ plane. Under these conditions, the motion will stay in the $z = 0$ plane. (Exercise: show this from the Euler–Lagrange equations.) and the cylindrical coordinates (r, ϕ, z) can be reduced to the polar coordinates (r, ϕ) of the invariant $z = 0$ plane.

The reduced Lagrangian reads

$$L(r, \dot{r}, \dot{\phi}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - U(r).$$

From our discussion above, we know that $l \equiv l_3 = mr^2\dot{\phi}$ is a constant, and this enables us to express the angular velocity $\dot{\phi} = l_3/mr^2$ in terms of the radial coordinate. This leads us the effective Lagrangian of the radial coordinate,

$$L(r, \dot{r}) = \frac{m}{2}\dot{r}^2 + \frac{l^2}{2mr^2} - U(r).$$

The solution of its Euler–Lagrange equations,

$$m\ddot{r} = -\partial_r U - \frac{l^2}{mr^3},$$

has been discussed in section *

2.3 Application: the rigid body

NOTATION In this section, we will be met with various component-carrying objects that 'are not truly vectors'. We will therefore indiscriminately write components downstairs. (A full discussion of the underlying geometry would take too much time in this text.)

Mechanics is about the motion of *bodies* (as opposed to idealized mass points). The motion of a generic body (think, e.g., of a running dog) is a complicated superposition of the motion of its internal degrees of freedom (the motion of hind legs vs. that of the front legs) and the bodies center of mass. The situation is much simpler with the sub-class of **rigid bodies**, for which the relative motion of internal degrees of freedom is 'frozen out'. Imagine a brick, or a rod of steel thrown into the air. We will observe a motion of the center of masses, and changes in the orientation of the bodies relative to our own. However, to a good approximation, the constituents forming the bodies (their atoms, ultimately) will remain fixed relative to each other.

In this section, we study how the notion of 'rigidity' can be made quantitative, how the motion of rigid bodies may be described in formulas, and how these formulas may eventually be solved.

2.3.1 Definition of the rigid body

Imagine a body as an assembly of N mass points m_i at coordinates $\mathbf{r}^i$. The body is rigid, if the $3N$ coordinates defining it are subject to the set of **holonomic constraints**

$$|\mathbf{r}_i - \mathbf{r}_j| = \text{const.}, \qquad i, j = 1, \ldots, N, \tag{2.23}$$

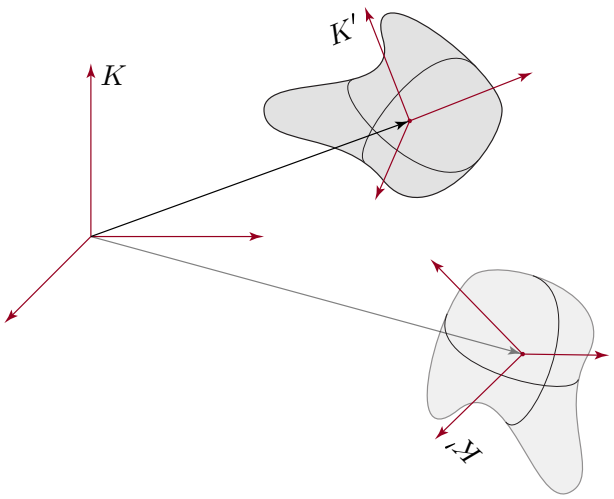


Figure 2.3: On the definition of the six coordinates describing the motion of a rigid body. The shaded drawing describes the position of the body in a different configuration.

i.e. all internal distances between points remain constant. How many coordinates do we need to describe the motion of a body subject to such constraints? The answer follows from Fig. 2.3. Define a point of reference inside the body, for example, its center of mass. Define a system of coordinates K' whose origin is that point. Technically, K' is defined in terms of a choice of origin and that of three basis vectors $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ which we may choose to be orthojormal. Assuming the fixed 'lab system', K from which the body is obeserved to be spanned by an orthonormal system $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the basis vectors $\{\mathbf{e}'_i\}$ are obtained from $\{\mathbf{e}_i\}$ by a three dimensional rotation matrix A.

The position of the body is then uniquely described in terms of (a) the three coordinates required to specify the position of the chosen origin, $\mathbf{R}(t)$ and (b) a generally time dependent rotation matrix $A(t)$. The vector $\mathbf{R}$ is described by three real coordinates, and so is the rotation matrix A. This tells us that

> The motion of a rigid body is described in terms of 6=3+3 coordinates, 3 to specify the coordinates of a fixed reference point within the body, and 3 to specify its orientation.

EXERCISE Recapitulate why a three dimensional rotation matrix $A \in \mathrm{SO}(3)$ is parameterized in terms of three real parameters. Try to find concrete parameterizations and discuss them in terms of their action on the vectors of the standard basis.

2.3.2 Moving coordinate frames

The description of a fixed body's motion crucially involves time dependent mappings between a fixed observer frame K and a body centered moving frame K'. Mathemathically, K and K' are reference systems in the sense of our discussion in section **??**. I.e. K is defined by a

choice $(O, \{\mathbf{e}_i\})$ of an origin and a basis of three dimensional Euclidean space (a set of points $\mathbf{x}$), and the same with K'. Practically, both K and K' describe a given point $\mathbf{x}$ by three component coordinate vectors $\mathbf{q}$ and $\mathbf{q}'$, respectively. We understand the map between the coordinates $\mathbf{q}$ and $\mathbf{q}'$ as a 'dynamical coordinate transformation', i.e. a map that assign to coordinate vectors $\mathbf{q}' \in K' \equiv \mathbb{R}^3$ describing the coordinates of a point $\mathbf{x}$ in the system K' the coordinates $\mathbf{q} \in K \equiv \mathbb{R}^3$ of *the same point* in K. Since K' may move relative to K, the map relating between the two systems may be time-dependent. At any instance of time, the map may involve translation and rotation, i.e. it is an *affine* map.

The mathematical concept required to describe transformations between moving coordinate frames is called a **motion** of K relative to K'. A motion is described by a family of maps,

$$\begin{aligned} D_t : K' &\longrightarrow K, \\ \mathbf{q}' &\longmapsto D_t\mathbf{q}' = \mathbf{q}, \end{aligned} \tag{2.24}$$

where t is a continuous time-like parameter, and is an *affine* map leaving orientation[8] and the norm of vectors invariant. If a motion $R_t \equiv D_t$ leaves the origin invariant $R_t\mathbf{0} = \mathbf{0}$, then it is called a **rotation**. Rotations are *linear* norm and orientation preserving maps, i.e. $R_t \in \mathrm{SO}(3)$ is described by a unit-determinant rotation matrix. A map, *shifting* the coordinate systems, $C_t : K' \to K, \mathbf{q}' \to \mathbf{q}' + \mathbf{R}_t$, where $\mathbf{R}_t \in \mathbb{R}^3$ is a shift vector, is called a **translation**. Intuitively,[9] it is clear that a general motion can be represented as the product of a rotation and a subsequent translation,

$$\begin{aligned} D_t = C_tR_t :&K' \longrightarrow K, \\ &\mathbf{q}' \longmapsto R_t\mathbf{q}' + \mathbf{R}_t = \mathbf{q}. \end{aligned} \tag{2.25}$$

If $R_t = R$ is a time-independent rotation, then D_t is called a **translatory motion**.

Consider now a generally time dependent vector $\mathbf{q}'(t)$ representing the coordinates of a point in the system K'. For example, you may imagine the trajectory $\mathbf{q}'(t)$ of a satellite as seen from the perspective of a space station orbiting earth. We are interested in the coordinates $\mathbf{q}(t) = D_t\mathbf{q}'(t)$ of the same point in the system K, which in our example would be the trajectory of the satellite as seen from, e.g., the surface of earth. However, we not only want to understand the correspondence $\mathbf{q}' \leftrightarrow \mathbf{q}$, but also that of velocities $\dot{\mathbf{q}}' \leftrightarrow \dot{\mathbf{q}}$ and acceleration $\ddot{\mathbf{q}}' \leftrightarrow \ddot{\mathbf{q}}$.

A formal answer to these questions may be obtained by differentiating Eq. (2.25) w.r.t. time. Omitting time arguments for notational clarity, this yields

$$\dot{\mathbf{q}} = R\dot{\mathbf{q}}' + \dot{R}\mathbf{q}' + \dot{\mathbf{R}}. \tag{2.26}$$

To obtain a better understanding of the terms appearing on the right hand side, let us take a look at some particular types of motion:

[8] With this condition, we exclude, e.g., reflections or other orientation changing maps from the set of motions.

[9] Verify this statement from the properties of general affine maps.

Translational motion

For a translational motion, C with $\dot{R} = 0$, the individual terms in the expression can be interpreted as

$\dot{\mathbf{q}}$: Absolute velocity of the point in K,

$R\dot{\mathbf{q}}'$: Velocity relative to the point $\mathbf{R}$,

$\dot{\mathbf{R}}$: Velocity of the system K' relative to K,

$\dot{\mathbf{q}}'$: Velocity of the point as seen in K'.

Rotational motion

We now turn to the somewhat more interesting case of a rotational motion, R, with $\mathbf{R} = 0$, but $d_t R \neq 0$ in general. Let us first consider a point at rest in K', $\dot{\mathbf{q}}' = 0$. We are going to show that there exists a vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$, such that

$$\boxed{\dot{\mathbf{q}} = \boldsymbol{\omega} \times \mathbf{q}.} \tag{2.27}$$

Before interpreting this result, let us prove it. With $\mathbf{q}' = R^{-1}\mathbf{q}$, we have

$$\dot{\mathbf{q}} = \dot{R}\mathbf{q}' = \dot{R}R^{-1}\mathbf{q} \equiv A\mathbf{q}, \tag{2.28}$$

where we defined the matrix $A = \dot{R}R^{-1} : K \to K$. The most important property of A is its anti-symmetry, $A^T = -A$. This follows from $R^{-1} = R^T$ (the defining property of a rotation matrix), and $0 = d_t \mathbb{1} = d_t(RR^T) = \dot{R}R^T + R\dot{R}^T$. With this, we obtain $A^T = (R^{-1})^T\dot{R}^T = R\dot{R}^T = -\dot{R}R^T = -A$. A 3×3 antisymmetric matrix affords the representation

$$A = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{2.29}$$

in terms of three real parameters ω_i. Defining the vector $\boldsymbol{\omega}$ as $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$, it is then straightforward to verify that $A\mathbf{q} = \boldsymbol{\omega} \times \mathbf{q}$.

Notice that the $\boldsymbol{\omega}$ is not, in fact, an 'ordinary' vector. Our discussion shows that it is more closely related to an antisymmetric matrix, i.e. a *tensor* of first co- and contravariant degree. The anomaly of ω is also reflected in its physical dimension, which is $[\boldsymbol{\omega}] = (\text{time})^{-1}$. It does not make sense, e.g., to compare the length of ω with that of a conventional vector.

The heuristic interpretation of Eq. (2.27) is that $\mathbf{q}$ **precesses** at a frequency $\omega \equiv |\boldsymbol{\omega}|$ around an axis defined by $\hat{\boldsymbol{\omega}} = \boldsymbol{\omega}/\omega$. This is best seen by considering a infinitesimally small time evolution,

$$\mathbf{q}(t + \delta) \simeq \mathbf{q}(t) + \delta\dot{\mathbf{q}}(t) = \mathbf{q}(t) + \delta\omega\hat{\boldsymbol{\omega}} \times \mathbf{q}. \tag{2.30}$$

The increment $\boldsymbol{\omega} \times \mathbf{q}$ is perpendicular to both $\mathbf{q}$ and $\boldsymbol{\omega}$ which is characterizing for rotational motion around an axis $\| \boldsymbol{\omega}$ (why?). Adding infinitesimal increments, we conclude that after a time $\Delta t = 2\pi/\omega$, we are back to the starting vector $\mathbf{q}(t + \Delta t) = \mathbf{q}(t)$ which identifies the rotational frequency as ω.

If we now drop the condition of constancy of $\mathbf{q}'$, we obtain

$$\dot{\mathbf{q}} = \boldsymbol{\omega} \times \mathbf{q} + R\dot{\mathbf{q}}', \tag{2.31}$$

where the individual terms are,

$\dot{\mathbf{q}}$: Absolute velocity of the point in K,

$\boldsymbol{\omega} \times \mathbf{q}$: Contribution to $\dot{\mathbf{q}}$ originating in the rotational motion of the body.

$R\dot{\mathbf{q}}'$: Contribution to the velocity originating in the non-constancy of the point relative to the system K'.

General case

The transformation formulas can be generalized to the description of a general motion $K' \xrightarrow{D} K$ if we decompose the latter as $D = CR$ into a rotation followed by a translation, $K' \xrightarrow{R} K'' \xrightarrow{C} K$. Our transformation formulas above tell us,

$$\begin{aligned} \mathbf{q}'' &= R\mathbf{q}', & \mathbf{q} &= \mathbf{q}'' + \mathbf{R}, \\ \dot{\mathbf{q}}'' &= \boldsymbol{\omega} \times \mathbf{q}'' + R\dot{\mathbf{q}}', & \dot{\mathbf{q}} &= \dot{\mathbf{q}}'' + \dot{\mathbf{R}}. \end{aligned} \tag{2.32}$$

Combining these formulas, we obtain

$$\dot{\mathbf{q}} = \boldsymbol{\omega} \times (\mathbf{q} - \mathbf{R}) + R\dot{\mathbf{q}}' + \dot{\mathbf{R}}, \tag{2.33}$$

as the most general transformation formula between moving systems. It is often convenient to express all terms on the r.h.s. through quantities in K'. To this end, we define

$$\boldsymbol{\omega}' \equiv R^{-1}\boldsymbol{\omega}, \tag{2.34}$$

i.e. the rotation vector as seen in K'. Using that for orthogonal matrices (check it), $R\mathbf{v} \times R\mathbf{w} = R(\mathbf{v} \times \mathbf{w})$, we then obtain

$$\dot{\mathbf{q}} = (R\boldsymbol{\omega}') \times (R\mathbf{q}') + R\dot{\mathbf{q}}' + \dot{\mathbf{R}} = R(\boldsymbol{\omega}' \times \mathbf{q}') + R\dot{\mathbf{q}}' + \dot{\mathbf{R}}, \tag{2.35}$$

and from there

$$\boxed{\dot{\mathbf{q}} = R(\boldsymbol{\omega}' \times \mathbf{q}' + \dot{\mathbf{q}}') + \dot{\mathbf{R}}.} \tag{2.36}$$

INFO Eq. (2.36) is the starting point for the quantitative discussion of **fictitious forces**. One may differentiate the formula once more w.r.t. time, to obtain $\ddot{\mathbf{q}} = \ldots$, where on the r.h.s. we

have a lengthy expression involving time derivatives of the constituents $\boldsymbol{\omega}', R$ and $\mathbf{R}$ describing the transition between coordinate systems. Even in the absence of forces in K, $\ddot{\mathbf{q}} = 0$, the acceleration $\ddot{\mathbf{q}}' \neq 0$ in general. Specifically, for a rotational motion one finds that the acceleration in K' is governed by a centrifugal force, and the more subtle action of the Coriolis force. For the discussion of these forces, we refer to the literature.

2.3.3 Lagrangian of the rigid body

In this section we will express the Lagrangian of a rigid body in terms of its six generalized coordinates. To this end, let us introduce a coordinate system K' fixed in the body. For convenience, we choose the center of mass as its origin, $\mathbf{R} = M^{-1}\sum_i \mathbf{r}_i m_i$, where $\mathbf{r}_i$ are the lab coordinates of the N masses m_i defining the body. Importantly, these points have vanishing velocity $\dot{\mathbf{r}}_i'$ inside K'. Using this fact, and Eq. (2.33), the kinetic energy of the system then assumes the form

$$T = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2}\sum_i m_i(\boldsymbol{\omega}\times(\mathbf{r}_i - \mathbf{R}) + \dot{\mathbf{R}})^2 = \frac{1}{2}\sum_i m_i(\boldsymbol{\omega}\times\mathbf{q}_i + \dot{\mathbf{R}})^2, \tag{2.37}$$

where we defined $\mathbf{q}_i = \mathbf{r}_i - \mathbf{R}$ as the coordinate vectors relative to the point $\mathbf{R}$ (but observed in K, not in K' where they are constant!). This expression can be simplified as

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i((\boldsymbol{\omega}\times\mathbf{q}_i)\cdot(\boldsymbol{\omega}\times\mathbf{q}_i) + 2\dot{\mathbf{R}}\cdot(\boldsymbol{\omega}\times\mathbf{q}_i) + \dot{\mathbf{R}}^2) = \\ &= \frac{1}{2}\sum_i m_i\, q_{ia}(\omega^2\delta_{ab} - \omega_a\omega_b)q_{jb} + M\dot{\mathbf{R}}^2, \end{aligned} \tag{2.38}$$

where we used that $\sum_i m_i\dot{\mathbf{q}}_i = d_t\sum_i m_i(\mathbf{r}_i - \mathbf{R}) = 0$. This expression affords the alternative representation

$$\boxed{T = \frac{1}{2}\boldsymbol{\omega}^T I\boldsymbol{\omega} + \frac{M}{2}\dot{\mathbf{R}}^2,} \tag{2.39}$$

where

$$\boxed{I_{ab} \equiv \sum_i m_i(q_{ic}q_{ic}\delta_{ab} - q_{ia}q_{ib}),} \tag{2.40}$$

defines the so-called **tensor of inertia**. The kinetic energy $T = T(\dot{\mathbf{R}}, \boldsymbol{\phi}, \dot{\boldsymbol{\phi}})$ is a function of the center of mass velocity, and of the rotation vector $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{\phi}, \dot{\boldsymbol{\phi}})$ which in turn depends on the three angles (and their derivatives) required to specify the rotational motion of the body. We may now subtract a potential energy $U = U(\mathbf{R}, \boldsymbol{\phi})$ which may depend on the orientation and the position of the body, to obtain the **Lagrangian of the rigid body**,

$$\boxed{L(\mathbf{R}, \dot{\mathbf{R}}, \boldsymbol{\phi}, \dot{\boldsymbol{\phi}}) = T(\dot{\mathbf{R}}, \boldsymbol{\phi}, \dot{\boldsymbol{\phi}}) - U(\mathbf{R}, \dot{\mathbf{R}}, \boldsymbol{\phi}, \dot{\boldsymbol{\phi}}),} \tag{2.41}$$

expressed as a function of the six generalized coordinates $(\mathbf{R}, \boldsymbol{\phi})$ and their time derivatives. The Lagrangian (2.41) conveniently contains contributions describing the translational and the rotational motion of the system.

2.3.4 Angular momentum

The general description of the motion of a rigid body can be very complicated. In the following, we will focus on the rotational parts of the dynamics, i.e. that what discriminates the motion of a body from that of a mass point. To this end, we will focus on what is called a **top**. A top is a rigid body fixed at one point, usually the center of mass, in space. This means that the Lagrangian of a top assumes the form

$$L = \frac{1}{2}\boldsymbol{\omega}^T I \boldsymbol{\omega} - U, \tag{2.42}$$

where both T and U depend on angular coordinates only. We will aim to characterize the motion of the system in terms of its **angular momentum** $\mathbf{L}$ relative to the fixed origin. An advantage of emphasizing the angular momentum is that it is a main player in the equations of motion of the system. In particular we know that $d_t\mathbf{L} = 0$ is conserved if no external torque $\mathbf{N}$ is applied.

The angular momentum must, of course, be closely related to the rotation vector $\boldsymbol{\omega}$. Indeed, a little calculation shows,

$$\mathbf{L} \equiv \sum_i m_i \mathbf{q}_i \times \dot{\mathbf{q}}_i = \sum_i \mathbf{q}_i \times (\boldsymbol{\omega} \times \mathbf{q}_i) = \sum_i m_i(\omega_a q_{ib} q_{ib} - q_{ia}\omega_b q_{ib})\mathbf{e}_a = I_{ab}\omega_b \mathbf{e}_a,$$

or

$$\boxed{\mathbf{L} = I\boldsymbol{\omega}.} \tag{2.43}$$

Alternatively, we may consider the angular momentum $\mathbf{L}' = R^{-1}\mathbf{L}$ in the in the body centered system. Comparison with (2.43) shows that

$$\boxed{\mathbf{L}' = I'\boldsymbol{\omega}',} \tag{2.44}$$

where $I' = R^{-1}IR$ is the transformed tensor of inertia. A quick check shows that I' is defined as in (2.40), only that we have to replace $q_i \to q_i'$. Both representations of the angular momentum have their own advantages: in K, we have the relatively simple equation of motion $d_t\mathbf{L} = \mathbf{N}$, the angular momentum changes according to the externally applied torque. However, I is generally time dependent in K which means that the relation between $\mathbf{L}$ and the directly observable rotation axis $\boldsymbol{\omega}$ can be complicated. In K', the tensor of inertia I' is fixed, and this means that the connection $\mathbf{L}' \leftrightarrow \boldsymbol{\omega}'$ is simple. However, even in the absence of external torque, $\mathbf{L}'$ will be time varying in general.

Our strategy in the following is to first pick a K' basis, that brings the constant I' into a maximally simple form. Within this framework, we will then study the equations of motion for $\mathbf{L}'$ and $\boldsymbol{\omega}'$. In a final step, we may then transform back to quantities $\mathbf{L}$ or $\boldsymbol{\omega}$ defined in the laboratory system.

2.3.5 Tensor of inertia

In K', the tensor of inertia, Eq. (2.40) is a symmetric 3×3-matrix, which means that it can be diagonalized in terms of an orthonormal basis $\{\mathbf{e}'_a\}$. The three perpendicular directions specified by the three basis vectors are called **principal axes** of the body, and the eigenvalues of I' are the **principal moments of inertia**. Let us denote these quantities by I_a. We then know that in the principal basis, the inertia tensor assumes the form $I' = \mathrm{diag}(I_1, I_2, I_3)$. On the other hand, the representation (2.40) remains valid. Comparison of the two representations then entails the identification

$$I_1 = \sum_i m_i((q'_2)^2 + (q'_3)^2), \tag{2.45}$$

and analogously for I_2, I_3. For bodies showing a high degree of symmetry, the principal axes generally coincide with the symmetry axes. For example, the principal axes of a cuboid are pierce through the centers of its faces, one of the principal axes of a cylinder runs through the cylinder axes, the two others are perpendicular to it (and perpendicular to each other). The second example shows that the principal axes need not be uniquely determined. In such cases, the corresponding principal moments are degenerate, e.g., $I_2 = I_3$ for a cylinder with $\mathbf{e}'_3$ as its symmetry axis. Also notice that the numbers I_a assume large/small values if the body is wide/of narrow shape in the complementary directions. For example, a body shaped like a thin disk would have its dominant moment of inertia assigned to the principal axis perpendicular to the disk.

EXERCISE Verify these statements from the definitions above.

Let us now assume that the principal axes define the basis system in K'. The components of the angular momentum and the kinetic energy then assume the simple form

$$\begin{aligned} L'_a &= I_a \omega'_a, \\ T &= \frac{1}{2} \sum_a (\omega'_a)^2 I_a. \end{aligned} \tag{2.46}$$

2.3.6 Euler Equations

We are now in a position to derive equations of motion describing the dynamics of a rigid body in terms of the motion of the vector $\boldsymbol{\omega}$. These equations follow from rate change of angular momentum $d_t \mathbf{L} = \mathbf{N}$, where $\mathbf{N}$ is the external torgue acting on the body[10].

We know that $d_t \mathbf{L} \overset{(2.36)}{=} R(\omega' \times \mathbf{L}' + d_t \mathbf{L}')$. With $\mathbf{N}' = R^{-1}\mathbf{N}$, this leads to

$$\omega' \times \mathbf{L}' + d_t \mathbf{L}' = \mathbf{N}',$$

[10] We here assume that torque to be given, in concrete applications, i.e. a body subject to a gravitational force, it has to be calculated as $\mathbf{N} = \sum_i \mathbf{q}_i \times \mathbf{F}_i$, where $\mathbf{F}_i$ the extenal force acting on the ith component of the body. In this expression, $\mathbf{q}_i = \mathbf{q}_i(\mathbf{R}, \boldsymbol{\phi})$ can be expressed in terms of the six coordinates desribing the body's position.

or, using (2.46),

$$I_a d_t \omega'_a + \epsilon_{abc} \omega'_b I_c \omega'_c = N'_a. \tag{2.47}$$

Writing out the equations for each component separately, we obtain the set of **Euler equations**

$$\begin{aligned} I_1 d_t \omega'_1 &= (I_2 - I_3)\omega'_2 \omega'_3 + N'_1, \\ I_2 d_t \omega'_2 &= (I_3 - I_1)\omega'_3 \omega'_1 + N'_2, \\ I_3 d_t \omega'_3 &= (I_1 - I_2)\omega'_1 \omega'_2 + N'_3. \end{aligned} \tag{2.48}$$

These are three coupled nonlinear equations which cannot be solved in general.

2.3.7 Free symmetric top

The Euler equations define the starting point of the theory of tops, a complex and highly developed sub-field of classical mechanics. To gain some intuition into the type of phenomena deriving from the Euler equations, we here discuss a particularly simple example, the free ($\mathbf{N} = 0$) symmetric ($I_1 = I_2$) top. The former condition means that $\mathbf{L} = \text{const.}$. The latter condition means that we are discussing a body with cylindrical symmetry, such that the moments of inertia associated to the axes perpendicular to the symmetry axis, $\mathbf{e}'_3$ coalesce, $I_1 = I_2 \equiv I'$. The third Euler equation, then tells us that $\omega'_3 \equiv \omega_\parallel = \text{const.}$ Defining

$$\Omega \equiv \frac{I - I_3}{I} \omega_\parallel,$$

we find that the first two equations are solved by

$$\begin{aligned} \omega'_1 &= \omega_\perp \cos(\Omega t + \phi), \\ \omega'_2 &= \omega_\perp \sin(\Omega t + \phi), \end{aligned} \tag{2.49}$$

where ϕ is some phase, and $\omega_\perp$ a constant. What this means is that the vector $\boldsymbol{\omega}'$ performs a **precession** around the symmetry axis of the body. The cone swept out by the precession goes by the name **body cone** (cf. Fig. 2.4.)

To obtain an idea how the motion will look from the outside, i.e. the perspective of the space centered system, we need to transform $\boldsymbol{\omega}' \to \boldsymbol{\omega}$. This task is most efficiently performed by exploiting symmetries and conservation laws. First, a bit of vector algebra (do it!) shows that the three vectors $(\mathbf{L}', \boldsymbol{\omega}', \mathbf{e}'_3)$ are linearly dependent; they lie in a plane. This means, that the transformed vectors, $(\mathbf{L}, \boldsymbol{\omega}, \mathbf{e}_3)$, too, must lie in a plane (why?). In the space-system, $\mathbf{L}$, is constant. We also know that $|\boldsymbol{\omega}| = |\boldsymbol{\omega}'| = \text{const.}$. Further, energy conservation implies the constancy of the kinetic energy,

$$T = \frac{1}{2}\mathbf{L} \cdot \boldsymbol{\omega} = \frac{1}{2}|\mathbf{L}||\boldsymbol{\omega}| \cos(\theta) = \text{const.}, \tag{2.50}$$

where θ is the angle enclosed by $\mathbf{L}$ and $\boldsymbol{\omega}$. The constancy of this angle means, that in the space system the rotation vector $\boldsymbol{\omega}$ will precess around the constant angular momentum vector.

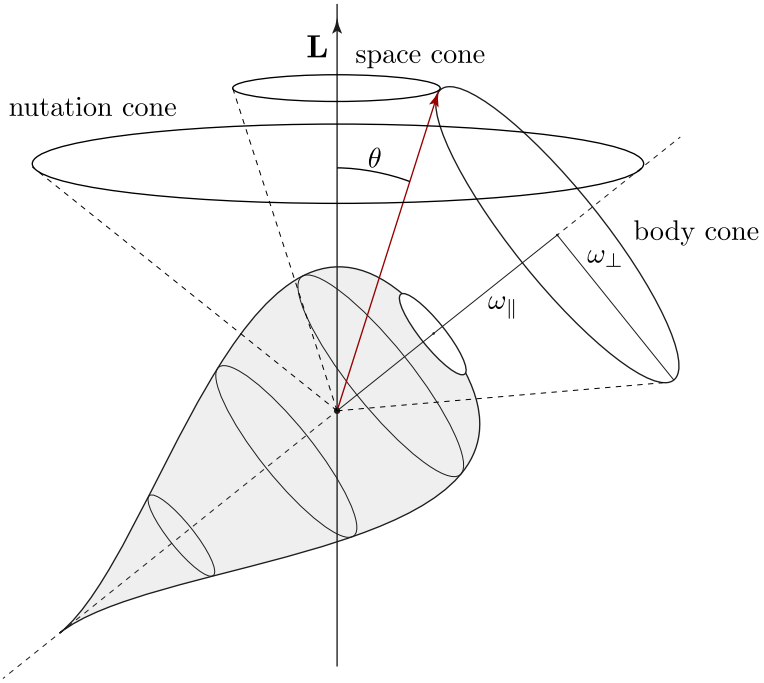
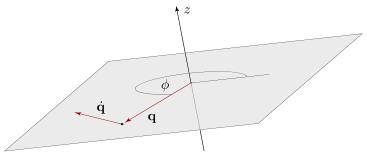


Figure 2.4: On the motion of a free symmetric top. Discussion, see text.

We thus conclude that in the space system, the two vectors $\boldsymbol{\omega}$ and $\mathbf{e}_3$ precess synchronously around the constant angular momentum axis. The cones traced out by $\boldsymbol{\omega}$ and $\mathbf{e}_3$ are called **space cone** and **nutation cone**, resp.



Chapter 3

Hamiltonian mechanics

The Lagrangian approach has introduced a new degree of flexibility into mechanical theory building. Still, there is room for further development. To see how, notice that in the last chapter we have characterized the state of a mechanical system in terms of the $2f$ variables $(q_1, \ldots, q_f, \dot{q}_1, \ldots, \dot{q}_f)$, the generalized coordinates and velocities. It is clear that we need $2f$ variables to specify the state of a mechanical system. But are coordinates and velocities necessarily the best variables? The answer is: often yes, but not always. In our discussion of symmetries in section 2.2.3 above, we have argued that in problems with symmetries, one should work with variables q_i that transform in the simplest possible way (i.e. additively) under symmetry transformations. If so, the corresponding momentum $p_i = \partial_{\dot{q}_i} L$ is conserved. Now, if it were possible to express the velocities uniquely in terms of coordinates and momenta, $\dot{\mathbf{q}} = \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})$, we would be in possession of an alternative set of variables $(\mathbf{q}, \mathbf{p})$ such that in a situation with symmetries a some of our variables stay constant, and hence assume the simplest possible form.

In this chapter, we show that the reformulation of mechanics in terms of coordinates and momenta as independent variables is an option. The resulting description is called **Hamiltonian mechanics**. Salient features of the new approach include:

- ▷ It exhibits a maximal degree of flexibility in the choice of problem adjusted coordinates.
- ▷ The variables $(\mathbf{q}, \mathbf{p})$ live in a mathematical space, so called "phase space", that carries a high degree of mathematical structure (more than an ordinary vector space.) This structure turns out to be of great use in the solution of complex problems. Relatedly,
- ▷ Hamiltonian theory is the method of choice in the solution of advanced mechanical problems. For example, the theory of (conservative) chaotic systems is almost exclusively formulated in the Hamiltonian approach.
- ▷ Hamiltonian mechanics is the gateway into quantum mechanics. Virtually all concepts introduced below have a direct quantum mechanical extension.

However, this does not mean that Hamiltonian mechanics is "better" than the Lagrangian theory. The Hamiltonian approach has its specific advantages. However, in some cases it

may be preferable to stay on the level of the Lagrangian theory. At any rate, Lagrangian and Hamiltonian mechanics form a pair that represents the conceptual basis of many modern theories of physics, not just mechanics. For example, electrodynamics (classical and quantum) can be formulated in terms of a Lagrangian and an Hamiltonian theory, and these formulations are strikingly powerful.

3.1 Foundations of Hamiltonian mechanics

The Lagrangian $L = L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is a function of coordinates and velocities. What we are after is a function $H = H(\mathbf{q}, \mathbf{p}, t)$ that is a function of coordinates and momenta, where the momenta and velocities are related to each other as $p_i = \partial_{\dot{q}_i} L = p_i(\mathbf{q}, \dot{\mathbf{q}}, t)$. Once in possession of the new function H, there must be a way to express the key information carriers of the theory – the Euler Lagrange equations – in the language of the new variables. In mathematics, there exist different ways of formulating variable changes of this type, and one of them is known as

3.1.1 Legendre transform

Reducing the notation to a necessary minimum, the task formulated above amounts to the following problem: given a function $f(x)$ (here, f assumes the role of L and x represents $\dot{q}_i$), consider the variable $z = \partial_x f(x) = z(x)$ (z represents the momentum p_i.) If this equation is invertible, i.e. if a representation $x = x(z)$ exists, find a partner function $g(z)$, such that g carries the same amount of information as f. The latter condition means that we are looking for a *transformation*, i.e. a mapping between functions $f(x) \to g(z)$ that can be inverted $g(z) \to f(x)$, so that the function $f(x)$ can be reconstructed in unique terms from $g(z)$. If this latter condition is met, it must be possible to express any mathematical operation formulated for the function of f in terms of a corresponding operation on the function g (cf. the analogous situation with the Fourier transform.)

Let us now try to find a transformation that meets the criteria above. The most obvious guess might be a direct variable substitution: compute $z = \partial_x f(x) = z(x)$, invert to $x = x(z)$, and substitute this into f, i.e. $g(z) \stackrel{?}{=} f(x(z))$. This idea goes in the right direction but is not quite good enough. The problem is that in this way information stored in the function f may get lost. To illustrate this point, consider the function

$$f(x) = C \exp(x),$$

where C is a constant. Now, $z = \partial_x f(x) = C \exp(x)$, which means $x = \ln(z/C)$ substitution back into f gets us to

$$g(z) = C \exp(\ln(z/C)) = z.$$

The function g no longer knows about C, so information on this constant has been irretrievably lost. (Knowledge of $g(z)$ is not sufficient to re-construct $f(x)$.)

However, it turns out that a slight extension of the above idea does the trick. Namely, consider the so called **Legendre transform** of $f(x)$,

$$\boxed{g(z) \equiv f(x(z)) - zx(z).} \tag{3.1}$$

We claim that $g(z)$ defines a transformation of functions. To verify this, we need to show that the function $f(x)$ can be obtained from $g(z)$ by some suitable inverse transformation. It turns out that (up to a harmless sign change) the Legendre transform is self-inverse; just apply it once more and you get back to the function f. Indeed, let us define the variable $y \equiv \partial_z g(z) = \partial_x f\big|_{x(z)} \partial_z x(z) - z\partial_z x(z) - x(z)$. Now, by definition of the function $z(x)$ we have $\partial_x f\big|_{x(z)} = z(x(z)) = z$. Thus, the first two terms cancel, and we have $y(z) = -x(z)$. We next compute the Legendre transform of $g(z)$:

$$h(y) = f(x(z(y)) - x(z(y))z(y) - z(y)y.$$

Evaluating the relation $y(z) = -x(z)$ on the specific argument $z(y)$, we get $y = y(z(y)) = -x(z(y))$, i.e. $f(x(z(y)) = f(-y)$, and the last two terms in the definition of $h(y)$ are seen to cancel. We thus obtain

$$h(y) = f(-y),$$

the Legendre transform is (almost) self–inverse, and this means that by passing from $f(x)$ to $g(z)$ no information has been lost.

The abstract definition of the Legendre transform with its nested variable dependencies can be somewhat confusing. However, when applied to concrete functions, the transform is actually easy to handle. Let us illustrate this on the example considered above: with $f(x) = C\exp(x)$, and $x = \ln(z/C)$, we get

$$g(z) = C\exp(\ln(z/C)) - \ln(z/C)z = z(1 - \ln(z/C)).$$

(Notice that $g(z)$ "looks" very different from $f(x)$, but this need not worry us.) Now, let us apply the transform once more: define $y = \partial_z g(z) = -\ln(z/C)$, or $z(y) = C\exp(-y)$. This gives

$$h(y) = C\exp(-y)(1 + y) - C\exp(-y)y = C\exp(-y),$$

in accordance with the general result $h(y) = f(-y)$.

The **Legendre transform of multivariate functions** $f(\mathbf{x})$ is obtained by application of the rule to all variables: compute $z_i \equiv \partial_{x_i} f(\mathbf{x})$. Next construct the inverse $\mathbf{x} = \mathbf{x}(\mathbf{z})$. Then define

$$g(\mathbf{z}) = f(\mathbf{x}(\mathbf{z})) - \sum_i z_i x_i(\mathbf{z}). \tag{3.2}$$

3.1.2 Hamiltonian function

Definition

We now apply the construction above to compute the Legendre transform of the Lagrange function. We thus apply Eq. (3.2) to the function $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ and identify variables as $\mathbf{x} \leftrightarrow \dot{\mathbf{q}}$

and $\mathbf{z} \leftrightarrow \mathbf{p}$. (Notice that the coordinates, $\mathbf{q}$, themselves play the role of spectator variables. The Legendre transform is in the variables $\dot{\mathbf{q}}$!) Le us now formulate the few steps it takes to pass to the Legendre transform of the Lagrange functions:

1. Compute the f variables
$$p_i = \partial_{\dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}, t). \tag{3.3}$$
2. Invert these relations to obtain $\dot{q}_i = \dot{q}_i(\mathbf{q}, \mathbf{p}, t)$
3. Define a function
$$\boxed{H(\mathbf{q}, \mathbf{p}, t) \equiv \sum_i p_i \dot{q}_i(\mathbf{q}, \dot{\mathbf{q}}, t) - L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)).} \tag{3.4}$$

   Technically, this is the negative of L's Legendre transform. Of course, this function carries the same information as the Legendre transform itself.

The function H is known as the **Hamiltonian function** of the system. It is usually called "Hamiltonian" for short (much like the Lagrangian function) is called "Lagrangian". The notation above emphasizes the variable dependencies of the quantities $\dot{q}_i, p_i$, etc. One usually keeps the notation more compact, e.g. by writing $H = \sum_i p_i \dot{q}_i - L$. However, it is important to remember that in both L and $\sum_i \dot{q}_i p_i$, the variable $\dot{q}_i$ has to be expressed as a function of $\mathbf{q}$ and $\mathbf{p}$. It doesn't make sense to write down formulae such as $H = \ldots$, if the right hand side contains $\dot{q}_i$'s as fundamental variables!

Hamilton equations

Now we need to do something with the Hamiltonian function. Our goal will be to transcribe the Euler–Lagrange equations to equations defined in terms of the Hamiltonian, hoping that these equations contain operational advantages over the Euler–Lagrange equations.

Now, the Euler-Lagrange equations probe changes (derivatives) of the function L. It will therefore be a good idea to explore what happens if we ask similar questions to the function H. Let us then compute

$$\partial_{q_i} H(\mathbf{q}, \mathbf{p}, t) = p_j \partial_{q_i} \dot{q}_j(\mathbf{q}, \mathbf{p}) - \partial_{q_i} L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})) - \partial_{\dot{q}_j} L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})) \partial_{q_i} \dot{q}_j(\mathbf{q}, \mathbf{p}).$$

The first and the third term cancel, because $\partial_{\dot{q}_j} L = p_j$. Now, iff the curve $\mathbf{q}(t)$ is a solution curve, the middle term equals $-d_t \partial_{\dot{q}_i} L = -d_t p_i$. We thus conclude that the $(\mathbf{q}, \mathbf{p})$-representation of solution curves must obey the equation

$$\dot{p}_i = -\partial_{q_i} H(\mathbf{q}, \mathbf{p}, t).$$

Similarly,

$$\partial_{p_i} H(\mathbf{q}, \mathbf{p}, t) = \dot{q}_i(\mathbf{q}, \mathbf{p}) + p_j \partial_{p_i} \dot{q}_j(\mathbf{q}, \mathbf{p}, t) - \partial_{\dot{q}_j} L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p})) \partial_{p_i} \dot{q}_j(\mathbf{q}, \mathbf{p}, t).$$

The last two terms cancel, and we have

$$\dot{q}_i = \partial_{p_i} H(\mathbf{q}, \mathbf{p}, t).$$

Finally, let us compute the *partial* time derivative $\partial_t H$.[1]

$$\partial_t H(\mathbf{q}, \mathbf{p}, t) = p_i \partial_t \dot{q}_i(\mathbf{q}, \mathbf{p}, t) - \partial_{\dot{q}_i} L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t)) \partial_t \dot{q}_i(\mathbf{q}, \mathbf{p}, t) - \partial_t L(\mathbf{q}, \dot{\mathbf{q}}, t).$$

Again, two terms cancel and we have

$$\partial_t H(\mathbf{q}, \mathbf{p}, t) = -\partial_t L(\mathbf{q}, \dot{\mathbf{q}}(\mathbf{q}, \mathbf{p}, t), t), \tag{3.5}$$

where the time derivative on the r.h.s. acts only on the third argument $L(\,.\,,\,.\,,t)$.

Let us summarize where we are: The solution curve $\mathbf{q}(t)$ of a mechanical system defines a $2f$–dimensional "curve", $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$. Eq. (3.3) then defines a $2f$–dimensional curve $(\mathbf{q}(t), \mathbf{p}(t))$. The invertibility of the relation $\mathbf{p} \leftrightarrow \dot{\mathbf{q}}$ implies that either representation faithfully describes the curve. The derivation above shows that

> The solution curves $(\mathbf{q}, \mathbf{p})(t)$ of mechanical problems obey the so–called **Hamilton equations**
>
> $$\begin{aligned} \dot{q}_i &= \quad \partial_{p_i} H, \\ \dot{p}_i &= -\partial_{q_i} H. \end{aligned} \tag{3.6}$$

Some readers may worry about the numbers of variables employed in the description of curves: in principle, a curve is *uniquely* represented by the f variables $\mathbf{q}(t)$. However, we have now decided to represent curves in terms of the $2f$–variables $(\mathbf{q}, \mathbf{p})$. Is there some redundancy here? No there isn't and the reason can be understood in different ways. First notice that in Lagrangian mechanics, solution curves are obtained from second order differential equations in time. (The Euler–Lagrange equations contain terms $\ddot{q}_i$.) By contrast, the Hamilton equations are first order in time.[2] The price to be payed for this reduction is the introduction of a second set of variables, $\mathbf{p}$. Alternatively, we may note that the solution of the (2nd order differential) Euler–Lagrange equations requires the specification of $2f$ **boundary conditions**. These can be the $2f$ conditions stored in the specification $\mathbf{q}(t_0) = \mathbf{q}_0$ and $\mathbf{q}(t_1) = \mathbf{q}_1$ of an initial and a final point,[3] or the specification $\mathbf{q}(t_0) = \mathbf{q}_0$ and $\dot{\mathbf{q}}(t_0) = \mathbf{v}_0$ of an initial configuration and

[1] Here it is important to be very clear about what we are doing: in the present context, $\dot{q}$ is a variable in the Lagrange function. (We could also name it v or z, or whatever.) It is considered a free variable (no time dependence), unless the transformation $\dot{q}_i = \dot{q}_i(\mathbf{q}, \mathbf{p}, t)$ becomes explicitly time dependent. Remembering the origin of this transformation, we see that this may happen if the function $L(\mathbf{q}, \mathbf{p}, t)$ contains explicit time dependence. This happens, e.g., in the case of rheonomic constraints, or time dependent potentials.

[2] Ordinary differential equations of nth order can be transformed to systems of n ordinary differential equations of first order. The passage $L \to H$ is example of such an order reduction.

[3] But notice that conditions of this type do not always uniquely specify a solution – think of examples!

velocity. By contrast[4]

> the Hamilton are uniquely solvable once an initial configuration $(\mathbf{q}(t_0), \mathbf{p}(t_0)) = (\mathbf{q}_0, \mathbf{p}_0)$ has been specified.

Either way, we need $2f$ boundary conditions, and hence $2f$ variables to uniquely specify the state of a mechanical system.

EXAMPLE To make the new approach more concrete, let us formulate the **Hamilton equations of a point particle in cartesian coordinates**. The Lagrangian is given by

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{m}{2}\dot{\mathbf{q}}^2 - U(\mathbf{q}, t),$$

where we allow for time dependence of the potential. Thus

$$p_i = m\dot{q}_i,$$

which is inverted as $\dot{q}_i(\mathbf{p}) = \frac{p_i}{m}$. This leads to

$$H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^{3} p_i \frac{p_i}{m} - L(\mathbf{q}, \mathbf{p}/m, t) = \frac{\mathbf{p}^2}{2m} + U(\mathbf{q}, t).$$

The Hamilton equations assume the form

$$\begin{aligned} \dot{\mathbf{q}} &= \frac{\mathbf{p}}{m}, \\ \dot{\mathbf{p}} &= -\partial_{\mathbf{q}} U(\mathbf{q}, t). \end{aligned}$$

Further, $\partial_t H = -\partial_t L = \partial_t U$ inherits its time dependence from the time dependence of the potential. The two Hamilton equations are recognized as a reformulation of the Newton equation. (Substituting the time derivative of the first equation, $\ddot{\mathbf{q}} = \dot{\mathbf{p}}/m$, into the second we obtain the Newton equation $m\ddot{\mathbf{q}} = -\partial_{\mathbf{q}} U$.)

Notice the Hamiltonian function $H = \frac{p^2}{2m} + U(\mathbf{q}, t)$ equals the *energy* of the particle! In the next section, we will discuss this connection in more general terms.

EXAMPLE Let us now solve these equations for the simple example of the one-dimensional **harmonic oscillator**. In this case, $V(q) = \frac{m\omega^2}{2} q^2$ and the Hamilton equations assume the form

$$\begin{aligned} \dot{q} &= \frac{p}{m}, \\ \dot{p} &= -m\omega^2 q. \end{aligned}$$

For a given initial configuration $\mathbf{x}(0) = (q(0), p(0))$, these equations afford the unique solution

$$\begin{aligned} q(t) &= q(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t, \\ p(t) &= p(0)\cos\omega t - m\omega \sin\omega t. \end{aligned} \tag{3.7}$$

[4] Formally, this follows from a result of the theory of ordinary differential equations: a system of n first order differential equations $\dot{x}_i = f_i(x_1, \ldots, x_n), i = 1, \ldots, n$ affords a unique solution, once n initial conditions $x_i(0)$ have been specified.

Physical meaning of the Hamilton function

The Lagrangian $L = T - U$ did not carry immediate physical meaning; its essential role was that of a generator of the Euler–Lagrange equations. However, with the Hamiltonian the situation is different. To understand its physical interpretation, let us compute the full time derivative $d_t H = d_t H(\mathbf{q}(t), \mathbf{p}(t), t)$ of the Hamiltonian evaluated on a solution curve $(\mathbf{q}(t), \mathbf{p}(t))$, (i.e. a solution of the Hamilton equations (3.6)):

$$d_t H(\mathbf{q}, \mathbf{p}, t) = \sum_{i=1}^{f} \left(\frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial q_i} \dot{q}_i + \frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial p_i} \dot{p}_i \right) + \frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial t}$$
$$\overset{(3.6)}{=} \frac{\partial H(\mathbf{q}, \mathbf{p}, t)}{\partial t}.$$

Now, above we have seen (cf. Eq. (3.5)) that the Hamiltonian inherits its *explicit* time dependence $\partial_t H = -\partial_t L$ from the explicit time dependence of the Lagrangian. In problems without explicitly time dependent potentials (they are called **autonomous problems**), the Hamiltonian stays constant on solution curves.

We have thus found that the function $H(\mathbf{q}, \mathbf{p})$, (the missing time argument indicates that we are now looking at an autonomous situation) defines a constant of motion of extremely general nature. What is its physical meaning? To answer this question, we consider an general N–body system in an arbitrary potential. (This is most general conservative autonomous setup one may imagine.) In cartesian coordinates, its Lagrangian is given by

$$L(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{N} m_j \dot{\mathbf{q}}_j^2 - U(\mathbf{q}_1, \ldots, \mathbf{q}_N),$$

where U is the N–body potential and m_j the mass of the jth particle. Proceeding as in the example on p 80, we readily find

$$H(\mathbf{q}, \mathbf{p}) = \sum_{j=1}^{N} \frac{\mathbf{p}_i^2}{2m_i} + U(\mathbf{q}_1, \ldots, \mathbf{q}_N). \tag{3.8}$$

The expression on the r.h.s. we recognize as the energy of the system. We are thus led to the following important conclusion:

> The Hamiltonian $H(\mathbf{q}, \mathbf{p})$ of an autonomous problem (a problem with time–independent potentials) is dynamically conserved: $H(\mathbf{q}(t), \mathbf{p}(t)) = E = \text{const.}$ on solution curves $(\mathbf{q}(t), \mathbf{p}(t))$.

However, our discussion above implies another very important corollary. It has shown that

> The Hamiltonian $H = T + U$ is given by the sum of potential energy, $U(\mathbf{q}, t)$, and kinetic energy, $T(\mathbf{q}, \mathbf{p}, t)$, expressed as a function of coordinates and momenta.

We have established this connection in cartesian coordinates. However, the coordinate invariance of the theory implies that this identification holds in general coordinate systems. Also, the identification $H = T + U$ did not rely on the time–independence of the potential, it extends to potentials $U(\mathbf{q}, t)$.

INFO In principle, the identification $H = T + U$ provides an **option to access the Hamiltonian without reference to the Lagrangian**. In cases where the expression $T = T(\mathbf{q}, \mathbf{p}, t)$ of the kinetic energy in terms of the coordinates and momenta of the theory is known, we may just add $H(\mathbf{q}, \mathbf{p}, t) = T(\mathbf{q}, \mathbf{p}, t) + U(\mathbf{q}, t)$. In such circumstances, there is no need to compute the Hamiltonian by the route $L(\mathbf{q}, \dot{\mathbf{q}}, t) \overset{\text{Legendre}}{\longrightarrow} H(\mathbf{q}, \mathbf{p}, t)$. Often, however, the identification of the momenta of the theory is not so obvious, and one is better advised to proceed via Legendre transform.

3.2 Phase space

NOTATION To keep the notation simple, we will suppress reference to an optional explicit time dependence of the Hamilton functions throughout this section, i.e. we just write $H(\dots)$ instead of $H(\dots, t)$.

In this section, we will introduce phase space as the basic "arena" of Hamiltonian mechanics. We will start from an innocent definition of phase space as a $2f$–dimensional coordinate space comprising configuration space coordinates, $\mathbf{q}$, and momenta, $\mathbf{p}$. However, as we go along, we will realize that this working definition is just the tip of an iceberg: the coordinate spaces of Hamiltonian mechanics are endowed with a lot of mathematical structure, which is the ultimate reason for the power of Hamiltonian mechanics.

3.2.1 Phase space and structure of the Hamilton equations

The Hamilton equations are coupled equations for coordinates $\mathbf{q}$, and momenta $\mathbf{p}$. As was argued above, the coordinate pair $(\mathbf{q}, \mathbf{p})$ contains sufficient information to uniquely encode the state of a mechanical system. This suggests to consider the $2f$–component objects

$$\mathbf{x} \equiv \begin{pmatrix} \mathbf{q} \\ \mathbf{p} \end{pmatrix} \tag{3.9}$$

as the new fundamental variables of the theory. Given a coordinate system, we may think of $\mathbf{x}$ as element of a $2f$–dimensional vector space. However, all what has been said in section 2.1.3 about the curves of mechanical systems and their coordinate representations carries over to the present context: in abstract terms, the pair "(configuration space points, momenta)" defines a mathematical space known as **phase space**, Γ. The formulation of coordinate invariant descriptions of phase space is a subject beyond the scope of the present course.

However, locally it is always possible to parameterize Γ in terms of coordinates,[5] and to describe its elements through $2f$–component objects such as (3.9). This means that, locally, phase space can be identified with a $2f$–dimensional vector space. However, different coordinate representations correspond to different coordinate vector spaces, and sometimes it is important to recall that the identification (phase space) $\leftrightarrow$ (vector space) may fail globally.[6]

Keeping the above words of caution in mind, we will temporarily identify phase space Γ with a $2f$–dimensional vector space. We will soon see that this space contains a very particular sort of "scalar product". This additional structure makes phase space a mathematical object far more interesting than an ordinary vector space. Let us start with a rewriting of the Hamilton equations. The identification $x_i = q_i$ and $x_{f+i} = p_i$, $i = 1, \ldots, f$ enables us to rewrite Eq. (3.6) as,

$$\dot{x}_i = \partial_{x_{i+f}} H,$$
$$\dot{x}_{i+f} = -\partial_{x_i} H,$$

where $i = 1, \ldots, f$. We can express this in a more compact form as

$$\dot{x}_i = I_{ij} \partial_{x_j} H, \qquad i = 1, \ldots, 2f, \tag{3.10}$$

or, in vectorial notation,

$$\dot{\mathbf{x}} = I \partial_{\mathbf{x}} H. \tag{3.11}$$

Here, the $(2f) \times (2f)$ matrix I is defined as

$$I = \begin{pmatrix} & \mathbf{1}_f \\ -\mathbf{1}_f & \end{pmatrix}, \tag{3.12}$$

and $\mathbf{1}_f$ is the f-dimensional unit matrix. The matrix I is sometimes called the **symplectic unity**. Note its anti-symmetry $I^T = -I$.

3.2.2 Variational principle

As in Lagrangian mechanics, the solutions of the Hamilton equations $\mathbf{x}(t)$ can be interpreted as extremal curves of an action functional. This connection will be a gateway to the further development of the theory.

Let us consider the set $M = \{\mathbf{x} : [t_0, t_1] \to \Gamma | \mathbf{q}(t_0) = \mathbf{q}_0, \mathbf{q}(t_1) = \mathbf{q}_1\}$ of all phase space curves beginning and ending at a common configuration space point, $\mathbf{q}_{0,1}$ and define the local functional

$$\begin{aligned} S :& M \to \mathbb{R}, \\ & \mathbf{x} \mapsto S[\mathbf{x}] = \int_{t_0}^{t_1} dt \left(\sum_{i=1}^{f} p_i \dot{q}_i - H(\mathbf{q}, \mathbf{p}) \right). \end{aligned} \tag{3.13}$$

[5] Which means that Γ has the status of a $2f$–dimensional manifold.

[6] For example, there are mechanical systems whose phase space is a two–sphere, and the sphere is not a vector space. (However, locally, it can be represented in terms of vector–space coordinates.)

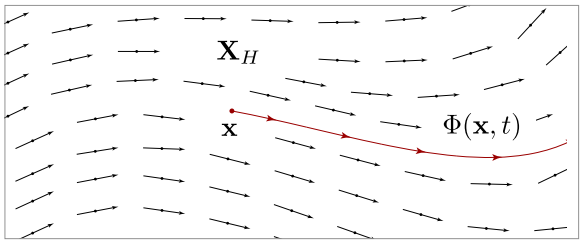


Figure 3.1: Visualization of the Hamiltonian vector field and its flow lines

We now claim that the extremal curves of this functional are solutions of the Hamilton equations (3.6). First note that the statement should not surprise us. According to Eq. (3.4), the integrand is (the negative of) the Lagrangian, expressed in terms of (q_i, p_i) as independent variables. Alluding to the geometric interpretation of variational principles, we should expect that the extrema of the functional in this new representation define physical trajectories. To see this in explicit terms, define the function $F(\mathbf{x}, \dot{\mathbf{x}}) \equiv \sum_{i=1}^{f} p_i \dot{q}_i - H(\mathbf{q}, \mathbf{p})$, in terms of which $S[\mathbf{x}] = \int_{t_0}^{t_1} dt\, F(\mathbf{x}, \dot{\mathbf{x}})$. Now, according to the general discussion of section 2.1.2, the extremal curves are solutions of the Euler Lagrange equations

$$(d_t \partial_{\dot{x}_i} - \partial_{x_i})\, F(\mathbf{x}, \dot{\mathbf{x}}) = 0, \qquad i = 1, \ldots, 2f.$$

Evaluating these equations for $i = 1, \ldots, f$ and $i = f+1, \ldots, 2f$, respectively, we obtain the first and second set of the equations (3.6).

3.2.3 Hamiltonian flow

For any $\mathbf{x}$, the quantity $I\partial_{\mathbf{x}} H(\mathbf{x}) = \{I_{ij}\partial_{x_j} H(\mathbf{x})\}$ is a vector in phase space. This means that the map

$$\boxed{\begin{aligned} &\mathbf{X}_H : \Gamma \to \mathbb{R}^n, \\ &\qquad \mathbf{x} \mapsto I\partial_{\mathbf{x}} H \equiv \mathbf{X}_H(\mathbf{x}), \end{aligned}} \tag{3.14}$$

defines a vector field in phase space, the so-called **Hamiltonian vector field**. The form of the **Hamilton equations**

$$\boxed{\dot{\mathbf{x}} = \mathbf{X}_H(\mathbf{x}),} \tag{3.15}$$

suggests an interpretation in terms of the "flow lines" of the Hamiltonian vector field: at each point in phase space, the field $\mathbf{X}_H$ defines a vector $\mathbf{X}_H(\mathbf{x})$. The Hamilton equations state that the solution curve $\mathbf{x}(t)$ is tangent to that vector, $\mathbf{x}(t) = \mathbf{X}_H(\mathbf{x})$. One may visualize the situation in terms of the streamlines of a fluid. Within that analogy, the value $\mathbf{X}_H(\mathbf{x})$ is a measure of the local current flow. If one injected a drop of colored ink into the fluid, its trace would be a representation of the curve $\mathbf{x}(t)$.

For a general vector field $\mathbf{v} : U \to \mathbb{R}^N, \mathbf{x} \mapsto \mathbf{v}(\mathbf{x})$, where $U \subset \mathbb{R}^N$, one may define a parameter–dependent map

$$\begin{aligned} \Phi : U \times \mathbb{R} &\to U, \\ (\mathbf{x}, t) &\mapsto \Phi(\mathbf{x}, t), \end{aligned} \tag{3.16}$$

through the condition $\partial_t \Phi(\mathbf{x}, t) \stackrel{!}{=} \mathbf{v}(\Phi(\mathbf{x}, t))$ and $\Phi(\mathbf{x}, 0) = \mathbf{x}$. The map Φ is called the **flow of the vector field $\mathbf{v}$.** Specifically, the flow of the Hamiltonian vector field $\mathbf{X}_H$ is defined by the prescription

$$\Phi(\mathbf{x}, t) \equiv \mathbf{x}(t), \tag{3.17}$$

where $\mathbf{x}(t)$ is a curve with initial condition $\mathbf{x}(t = 0) = \mathbf{x}$. Equation (3.17) is a proper definition because for a given $\mathbf{x}(0) \equiv \mathbf{x}$, the solution $\mathbf{x}(t)$ is uniquely defined. Further, $\partial_t \Phi(\mathbf{x}, t) = d_t \mathbf{x}(t) = \mathbf{X}_H(\mathbf{x}(t))$ satisfies the flow condition. The map $\Phi(\mathbf{x}, t)$ is called the **Hamiltonian flow**.

There is a corollary to the uniqueness of the curve $\mathbf{x}(t)$ emanating from a given initial condition $\mathbf{x}(0)$:

Phase space curves never cross.

For if they did, the crossing point would be the initial point of the two out-going stretches of the curve, and this would be in contradiction to the uniqueness of the solution for a given initial configuration. There is, however, one subtle caveat to the statement above: although phase space curves do not cross, they may actually touch each other in common terminal points. (For an example, see the discussion in section 3.2.5 below.)

We conclude this section with a general remark on the concept of **phase space vector (fields)**. We have introduced phase space as a set of 'vectors' $\mathbf{x} = (\mathbf{q}, \mathbf{p})$. Actually, however, the concept of vectors is not very useful in this context. We rarely 'add' $\mathbf{x} + \mathbf{y}$, etc. Instead, $\mathbf{q}$ are coordinates parameterizing a smooth configuration space 'surface' or a configuration space *manifold*.

> INFO A manifold is the mathematical abstraction of a smooth geometric object. A manifold can be locally, but not necessarily globally identified with a subset of $\mathbb{R}^n$. For example, a sphere like the surface of earth is a manifold. Locally, we can parameterize it in terms of two coordinates and in this way identify it with a subset of the plane. (This is what a map visualizable on a sheet of paper does.) However, globally it is different from a subset of the plane.

The same holds true for the momenta of the theory, and so we should think of phase space as a $2f$-dimensional manifold, Γ. Phase space vector fields lie tangent to this manifold. To each point $\mathbf{x}$ in Γ, we may attach a **tangent vector space** $T_{\mathbf{x}}\Gamma$, i.e. the set of vectors tangent to the manifold at this particular point. For example, vectors tangent to earth at the northpole define a two-dimensional plane isomorphic to, but different from vectors tangent to earth at a point on the equator. The vectors $\mathbf{X}_H(\mathbf{x}) \in T_{\mathbf{x}}\Gamma$ lie tangent to phase space at $\mathbf{x}$. This interpretation will become useful below, when we discuss further geometric structures in phase space.

3.2.4 Liouville Theorem

Besides the non-crossing of flow lines, the phase space flow has one more general property which is indirectly responsible for many properties of mechanical systems: it is area conserving. Intuitively this means that if you could paint an area element in two-dimensional phase space and monitored its fate under the flow, it would in general change its position and shape, but not its geometric area. The statement carries over to higher dimensional phase spaces.

To formulate and prove it in informal terms[7], consider a phase space and inside it a certain subset M. (Think of simple geometric structures such as cubes or spheres, etc.). Choosing coordinates $\mathbf{x} = \{q_i, p_i\}$, we define its volume by $V = \int_M dx$, where the subscript indicates that the integration extends over M, and the integration measure is defined as $dx = \prod_i dp_i dq_i$. Will this volume depend on the choice of coordinates? The answer is 'no', as long as we restrict ourselves to the *canonical* coordinate transformations defined in section 3.5.

Intuitively, you may think of the construction as the continuum limit of a swarm of infinitely many phase space points concentrated in M. We next ask what will happen to this swarm under the Hamiltonian flow. The answer is that individual points $\mathbf{x}$ will evolve to Eq. (3.17), and M into $\Phi(M,t) = \{\mathbf{x}(t)|\mathbf{x} \in M\}$. What is the volume of the evolved set? To investigate this question, think of the evolution $\rho : \mathbf{x} \mapsto \mathbf{x}(t) \equiv \mathbf{y}$ as a smooth and invertible (why?) map. Under it, M maps to $\rho(M) = \{\mathbf{y} = \rho(\mathbf{x})|\mathbf{x} \in M\}$, and the phase space volume of the evolved set is given by $V(\epsilon) \equiv \int_{\rho(M)} d\mathbf{y}$, with the same measure $d\mathbf{y} = \prod dq_i dp_i$. (In the computation of this volume, $\mathbf{y}$ is simply an integration variable, denoted differently from $\mathbf{x}$ in preparation of the next step of the construction.)

To study the evolution of the thus defined volume, it is sufficient to consider infinitesimal times $t = \epsilon$. The volume at finite times can then be obtained by iteration of the procedure. The evolution under infinitesimal times can be read off from Eq. (3.11): $\mathbf{x}(\epsilon) \approx \mathbf{x}(0) + \epsilon I \partial_{\mathbf{x}} H|_{\mathbf{x}(0)}$. We thus have $\mathbf{x} = \mathbf{x}(0)$ and $\mathbf{y} \equiv \mathbf{x}(\epsilon)$. Let us now forget for a momentum about the mechanical context and make use of a theorem of multi-dimensional integration: For a smooth $\rho : \mathbf{x} \to \mathbf{y}$ mapping an integration domain M to $\rho(M)$, we have

$$\int_{\rho(M)} dy f(\mathbf{y}) = \int_M dx \left|\frac{\partial y}{\partial x}\right| f(\mathbf{y}(\mathbf{x})),$$

where $\left|\frac{\partial y}{\partial x}\right|$ is a shorthand for the determinant of the Jacobi-matrix, i.e. the matrix J defined by the partial derivatives $\{J_{ij} = \frac{\partial y^i}{\partial x^j}\}$. The statement holds for arbitrary functions, including $f = 1$. In this case, the left hand side produces $V(\epsilon)$, and what the right hand side does depends on the determinant factors.

So, we need to investigate the Jacobi matrix, which for our present case is close to the unit matrix. With $\mathbf{y} = \mathbf{x} + \epsilon I \partial_{\mathbf{x}} H$, we have

$$J_{ij} = \frac{\partial y^i}{\partial x^j} = \delta^i_j + \epsilon \frac{\partial}{\partial x^j} I_{ik} \frac{\partial}{\partial x^k} H(\mathbf{x})$$

[7] By 'informal', we mean that we will not spell out all technical points in full detail. The emphasis is on conveying the idea of the construction.

We may abbreviate this result as $J = \mathbb{1} + \epsilon IR$, where R is the symmetric matrix of partial derivatives $R_{jk} = \partial^2_{jk} H$. Now, for a matrix $A = \mathbb{1} + \epsilon B$, we have $\det(A) = 1 + \epsilon \operatorname{tr}(B) + \mathcal{O}(\epsilon^2)$ (why?). Applied to our construction, $\det(J) \approx 1 + \epsilon \operatorname{tr}(AI)$ Remembering that $I^T = -I$ is anti-symmetric, $\operatorname{tr}(AI) = \operatorname{tr}(I^T A^T) = -\operatorname{tr}(IA) = -\operatorname{tr}(AI)$, where we used elementary properties of the trace. This proves $\det J = 1$, within the required ϵ-accuracy, and in this way the constancy of the phase space volume.

In our discussion of stability theory and of chaotic flows below we will understand the far-reaching consequences of this finding.

3.2.5 Phase space portraits

Graphical representations of phase flow, so-called **phase space portraits** are very useful in the qualitative characterization of mechanical motion. The only problem with this is that phase flow is defined in $2f$ dimensional space. For $f > 1$, we cannot draw this. What *can* be drawn, however, is the projection of a curve onto any of the $2f(2f-1)/2$ coordinate planes (x_i, x_j), $1 \le i < j \le 2f$. See the figure for the example of a coordinate plane projection out of $3d$–dimensional space. However, it takes some experience to work with those representations and we will not discuss them any further.

However, the concept of phase space portraits becomes truly useful in **problems with one degree of freedom**, $f = 1$. In this case, the conservation of energy on individual curves $H(q,p) = E = \text{const.}$ enables us compute an explicit representation $q = q(p,E)$, or $p = p(q,E)$ just by solving the relation $H(q,p) = E$ for q or p. This does not "solve" the mechanical problem under consideration. (For that one would need to know the time dependence at which the curves are traversed.) Nonetheless, it provides a far–reaching characterization of the motion: for any phase space point (q,p) we can construct the curve running through it *without solving differential equations*. The projection of these curves onto configuration space $(q(t), p(t)) \mapsto q(t)$ tells us something about the evolution of the configuration space coordinate.

Let us illustrate these statements, on an example where a closed solution *is* possible, the harmonic oscillator. From the Hamiltonian $H(q,p) = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 = E$, we find

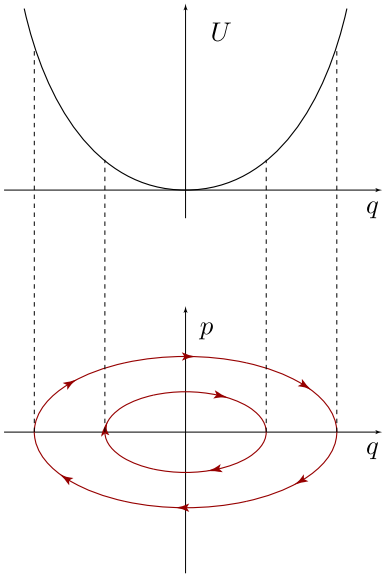
$$p = p(q,E) = \pm\sqrt{2m\left(E - \frac{m\omega^2}{2} q^2\right)}.$$

Examples of these (ellipsoidal) curves are shown in the figure. It is straightforward to show (do it!) that they are tangential to the Hamiltonian vector field

$$\mathbf{X}_H = \begin{pmatrix} m^{-1} p \\ -m\omega^2 q \end{pmatrix}.$$

At the turning points, $p = 0$, the energy of the curve $E = \frac{m\omega^2}{2} q^2$, equals the potential energy.

Now, the harmonic oscillator may be an example a little bit too basic to illustrate the usefulness of phase space portraits. Instead, consider the problem defined by the potential shown in Fig. 3.2. The Hamilton equations can now no longer be solved in closed form. However, we may still compute curves by solution of $H(p,q) = E \Rightarrow p = p(q,E)$, and the results look as shown qualitatively for three different values of E in the figure. These curves give a far–reaching impression of the motion of a point particle in the potential landscape. Specifically, notice the threshold energy E_2 corresponding to the local potential maximum. At first sight, the corresponding phase space curve appears to violate the criterion of the non-existence of crossing points (cf. the "critical point" at $p = 0$ beneath the potential maximum.) In fact, however, this isn't a crossing point. Rather the two incoming curves "terminate" at this point: coming from the left, or right it takes infinite time for the particle to climb the potential maximum.



EXERCISE Show that a trajectory at energy $E = V^* \equiv V(q^*)$ equal to the potential maximum at $q = q^*$ needs infinite time to climb the potential hill. To this end, use that the potential maximum at q^* can be locally modelled as an **inverted harmonic oscillator** potential, $V(q) \simeq V^* - C(q-q^*)^2$, where C is a positive constant. Consider the local approximation of the Hamilton function $H = \frac{p^2}{2m} - C(q-q^*)^2 + \text{const.}$ and formulate and solve the corresponding equations of motion (hint: compare to the standard oscillator discussed above.) Compute the time it takes to reach the maximum if $E = V(q^*)$.

Eventually it will rest at the maximum in an unstable equilibrium position. Similarly, the outgoing trajectories "begin" at this point. Starting at zero velocity (corresponding to zero initial momentum), it takes infinite time to accelerate and move downhill. In this sense, the curves touching (not crossing!) in the critical point represent idealizations that are never actually realized. Trajectories of this type are called **separatrices**. Separatrices are important in that they separate phase space into regions of qualitatively different type of motion (presently, curves that make it over the hill and those which do not.)

EXERCISE Take a few minutes to familiarize yourself with the phase space portrait and to learn how to "read" such representations. Sketch the periodic potential $V(q) = \cos(2\pi q/a)$, $a = \text{const.}$ and its phase space portrait.

3.2.6 Poisson brackets

Consider a (differentiable) function in phase space, $g : \Gamma \to \mathbb{R}, \mathbf{x} \mapsto g(\mathbf{x})$. A natural question to ask is how $g(\mathbf{x})$ will evolve, as $\mathbf{x} \equiv \mathbf{x}(0) \mapsto \mathbf{x}(t)$ traces out a phase space trajectory. In other words, we ask for the time evolution of $g(\mathbf{x}(t))$ with initial condition $g(\mathbf{x}(0)) = g(\mathbf{x})$.

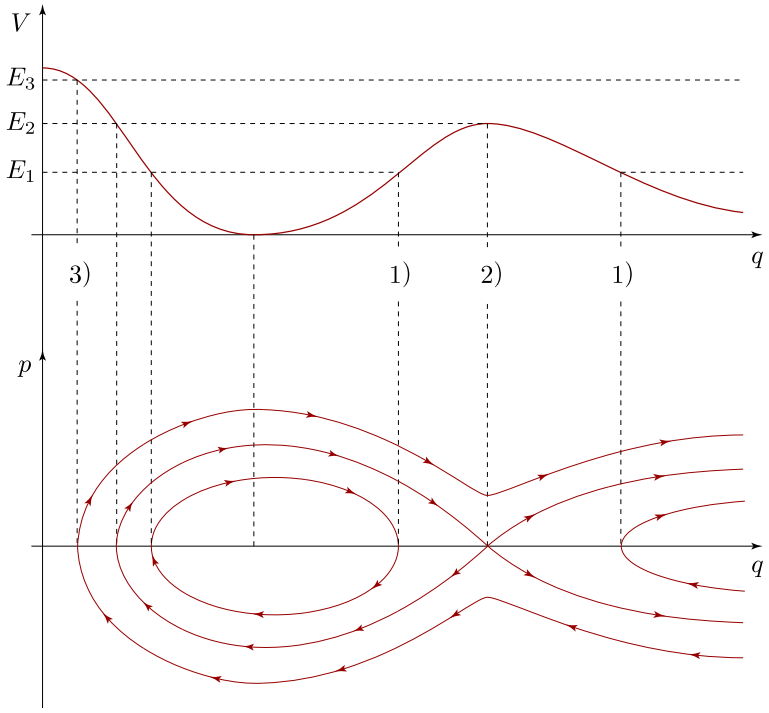


Figure 3.2: Phase space portrait of a "non–trivial" one-dimensional potential. Notice the "critical point" at the threshold energy 2).

The answer is obtained by straightforward time differentiation:

$$\frac{d}{dt}g(\mathbf{x}(t)) = \frac{d}{dt}g(\mathbf{q}(t),\mathbf{p}(t)) = \sum_{i=1}^{f}\left(\frac{\partial g}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial g}{\partial p_i}\frac{dp_i}{dt}\right)_{(\mathbf{q}(t),\mathbf{p}(t))} = \\ = \sum_{i=1}^{f}\left(\frac{\partial g}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial g}{\partial p_i}\frac{\partial H}{\partial q_i}\right)_{(\mathbf{q}(t),\mathbf{p}(t))},$$

where the terms $\partial_t g$ account for an optional explicit time dependence of g. The characteristic combination of derivatives governing this expressions appears frequently in Hamiltonian mechanics. It motivates the introduction of a shorthand notation,

$$\boxed{\{f,g\} \equiv \sum_{i=1}^{f}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right).} \tag{3.18}$$

This is expression is called the **Poisson bracket** of two phase space functions f and g. In the invariant notation of phase space coordinates $\mathbf{x}$, it assumes the form

$$\boxed{\{f,g\} = (\partial_{\mathbf{x}} f)^T\, I\, \partial_{\mathbf{x}} g,} \tag{3.19}$$

where I is the symplectic unit matrix defined above. The time evolution of functions may be concisely expressed in terms of the Poisson bracket of the two functions H and g:

$$\boxed{d_t g = \{g, H\}.} \tag{3.20}$$

We note that the Poisson brackets satisfy a number of algebraic properties:

- ▷ $\{f, g\} = -\{g, f\}$, (skew symmetry),
- ▷ $\{cf + c'f', g\} = c\{f, g\} + c'\{f', g\}$, (linearity),
- ▷ $\{c, g\} = 0$,
- ▷ $\{ff', g\} = f\{f', g\} + \{f, g\}f'$, (product rule),
- ▷ $\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0$ (Jacobi identity),

where, $c, c' \in \mathbb{R}$ are constants. The first three of these relations are immediate consequences of the definition and the fourth follows from the product rule. The proof of the Jacobi identity amounts to a straightforward if tedious exercise in differentiation.

Finally notice how Eq. (3.19) resembles a scalar product. To make this observation more concrete, we observe that for phase space vectors, which we recall live in the tangent spaces $T_{\mathbf{x}}\Gamma$, we may define an **inner product**:

$$\begin{aligned} \omega : T_{\mathbf{x}}\Gamma \times T_{\mathbf{x}}\Gamma &\to \mathbb{R}, \\ (\mathbf{X}, \mathbf{Y}) &\mapsto X_i I_{ij} Y_j = X^T I Y \equiv \omega(\mathbf{X}, \mathbf{Y}). \end{aligned}$$

This inner product is skew symmetric $\omega(\mathbf{X}, \mathbf{Y}) = -\omega(\mathbf{Y}, \mathbf{X})$, and non-degenerate. Further, to each function $f : \Gamma \to \mathbb{R}$, we assign a vector field $\mathbf{x} \mapsto T_{\mathbf{x}}\Gamma$ according to the following recipe: multiply the vector of derivatives $\partial_{x^i} f(\mathbf{x})$ with the symplectic unity, I, to obtain $\mathbf{x} \mapsto \mathbf{X}_f(\mathbf{x})$ with components $X_{f,i} = I_{ij}\partial_j f(\mathbf{x})$.[8] It is then straightforward to verify that

$$\{f, g\}(\mathbf{x}) = \omega(\mathbf{X}_f(\mathbf{x}), \mathbf{X}_g(\mathbf{x})). \tag{3.21}$$

To understand the meaning of these vector fields and their scalar product, note that

$$\{f, g\} = \partial_i f I_{ij} \partial_j g = \partial_i f X_{g,i} = df(\mathbf{X}_g), \tag{3.22}$$

where we omitted the arguments $\mathbf{x}$ for clarity. By anti-symmetry, $\{f, g\} = -dg(\mathbf{X}_f)$. Combining the above formulas, we have

$$\omega(\mathbf{X}_f, \mathbf{X}_g) = df(\mathbf{X}_g) = -dg(\mathbf{X}_f).$$

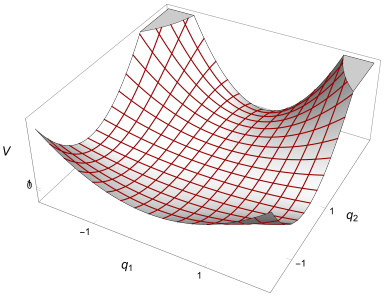
In other words, the scalar product measures how much the function f *changes* in $\mathbf{X}_g$-direction, vice versa. For example, the fact that the function H does not change along the Hamiltonian

[8] Without further discussion beyond the scope of this introduction it is not obvious why we make I a part of the definition. Just accept it for the time being.

flow can be expressed as $0 = \omega(\mathbf{X}_H, \mathbf{X}_H) = dH(\mathbf{X}_H)$, where the first equality follows from anti-symmetry. Since $\mathbf{X}_H$ is tangent to the flow, Eq. (3.15), the second equality states the conservation of H along it.

The structures introduced in this section define the starting point of what is called *formal mechanics*, an amazingly rich web of structures revolving around Hamiltonian flows in phase space. However, for reasons not exclusively related to time, we will not delve into this subject in abstract generality. Indeed, from a perspective of modern physics, one of the prime applications of formal mechanics is the description of chaotic dynamics[9]. We take this as an incentive to reverse the order of the presentation: A brief discussion to chaotic dynamics will motivate the introduction of advanced concepts of Hamiltonian dynamics tailored to its description.

3.3 Application: Hamiltonian chaos



Looking back, the majority of examples discussed so far in this text described oscillatory, or free motion. Focusing on these is both by their simplicity, and the fact that close to configurations of mechanical equilibrium (think of a guitar string in rest) motion tends to be oscillatory (guitar in a state of vibration). More generally, however, we encounter different types of mechanical motion, which, broadly speaking, can be categorized into two groups. The first is motion that is 'predictable' in the sense that a small perturbation, such as a small deviation in initial conditions, will lead to predictable consequences. Such dynamics is callled **integrable**, alluding to the solubility of the underlying equations of motion. (The practical solution can still be complicated, though.) The complementary class is **chaotic** in that a small perturbation will lead to consequences exponentially divering in time. No matter how small the perturbation, eventually it will lead to divergences reflecting instabilities in the equations of motion. Chaos is a typical (but not necessary) phenomenon of **non-integrability**, i.e. the absence of analytical solutions of equations of motion.

The majority of mechanical systems is neither fully integrable, nore fully chaotic, but lie somewhere in-between. Even the mathematically and physically precise definition of chaos is far from straightforward, and a subject of ongoing research. As a simple example, illustrating the principle, consider the two-dimensional **Henon-Heiles oscillator**, defined by the Hamilton function

$$H = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2^2 - \frac{1}{2}q_2^3 \tag{3.23}$$

Where we have set dimensionful constants, such as masses or frequencies, to unity for simplicity. This Hamiltoninan was proposed in 1964 as a simple model for motion of stars close

[9] Another application field is quantum dynamics close to classical limits, where (semi)classical concepts of classical mechanics can be applied.

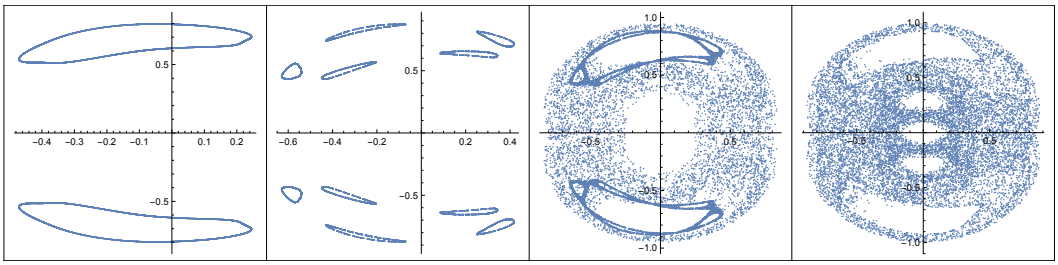


Figure 3.3: Poincaré plots defined by the condition $q_2(t) = 0$ for inital conditions $q_1(0) = 0.5, p_1(0) = p_2(0) = 0$, and $q_2(0) = 0.8, 0.88, 0.8825, 0.95$ from left to right. Note how the transition from regular (left) to ergodic (right) motion occurs over a narrow parameter window.

to a galactic center [10]. For energies close to the local minimum of its potential shown in the figure, the system is linearizable in the sense of our discussion of Section 1.4.3, and in this limit becomes a two-dimensional harmonic oscillator. However, for larger energies, the terms beyond quadratic order become important, and the motion begins to show signatures of chaos.

3.3.1 Poincaré Plots

To visualize this phenomenon, we introduce the concept of a **surface of section** aka **Poincaré plot**. Surfaces of section provide potent vizualizations of dynamics in the lowest dimensional phase spaces where chaotic motion may occur, namely the four dimensional spaces corresponding to systems with two degrees of freedom, $f = 2$. (We will see momentarily, that systems with just one degree of freedom are never chaotic.) Since energy is conserved in Hamiltonian dynamics, we have one conservation law $E(q_1, q_2, p_1, p_2)$, reducing the number of independent variables on a surface of conserved energy down to three. Now let's assume we monitor the motion $\mathbf{x}(t)$ on this surface, and every time one of the coordinates assumes a specific value, say $q_2(t) = 0$, we graph $(q_1(t), p_1(t))$, or any other pair of the remaining three variables in a plane. If we wait long enough, we generate a cloud of points, defining the surface of section.

How should we expect these clouds to look like? Imagine the fixation $q_2 = 0$ like cutting phase space with a knife. The three-dimensional plane (or hypersurface) defined in this way may or may not intersect with the three-dimensional manifold on which the dynamical motion takes place. *If* they intersect, and *if*, in the limit of infinitely long tracking $t \to \infty$, the motion fully covers the energy surface [11] the Poincaré plot should yield a uniformly filled two-dimensional area. However, if there is another conservation laws besides energy, the motion will be confined to a two-dimensional surface, and the Poincaré plot, if not empty, will show intersection *curves*.

Figure ?? shows Poincaré plots for the Henon-Heiles oscillator for differently chosen initial conditions. On the left, we observe a closed intersection curve, which should remind us

[10] Hénon, M.; Heiles, C. (1964). "The applicability of the third integral of motion: Some numerical experiments". The Astronomical Journal. **69**: 73–79.

[11] We call this the case of **ergodic motion**, i.e. the Hamiltonian flow uniformly covering the energetically accessible regions in phase space.

of the closed phase space curves identified earlier for the harmonic oscillator (more on this connection will follow). However, only a slight change in initial conditions radically changes the situation towards ergodic forms of motion. Also notice that the transition includes structure, notably the breakup of closed curves into smaller closed curves visible in the second panel. Notice that similar constructions can, in principle, we defined for higher dimensional surfaces. However, in that case (think about it) the interpretation of the ensuing portraits will be far less straightforward, and different techniques for diagnosing ergodicity or chaos are applied.

3.3.2 Linear stability theory

Chaos is associated with dynamical instability. The study of such phenomena generically begins with the analysis of fixed points, and of their stability. A fixed point $\mathbf{x}^*$ of a Hamiltonian system is a point in phase space which remains stationary, think of a particle resting at the bottom of a potential well. What happens, if we distort the configuration a little, $\mathbf{x}^* \to \mathbf{x}^* + \mathbf{y}$? Will the trajectory defined by the distorted initial condition run away from $\mathbf{x}^*$, or stay close to it, and if so, how? In the following, we learn how to formulate and answer such questions, in generality.

In mathematics, a system of coupled first order differential equations,

$$d_t\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0,$$

$\mathbf{f}(\mathbf{x}) = (f^1(\mathbf{x}), \ldots, f^n(\mathbf{x}))$, $\mathbf{x} \in \mathbb{R}^n$, is called a **dynamical system**. With $\mathbf{f}(\mathbf{x}) = \mathbf{X}_H(\mathbf{x})$, our Hamilton equations define a dynamical system, however, the concept is so general, and important, that we do not specialize to the Hamiltonian case just yet.

A fixed point is defined by the vanishing of the r.h.s., $\mathbf{f}(\mathbf{x}^*) = 0$. In order to understand what happens in the vicinity of it, we substitute the configuration $\mathbf{x}(t) = \mathbf{x}^* + \mathbf{y}(t)$ into the equation, assume smoothness of $\mathbf{f}$ around its zero and Taylor expand to first order. The components of $x^i(t)$ then evolve according to

$$d_t x^i(t) = d_t(x^{*i} + y^i(t)) = f^i(\mathbf{x}^* + \mathbf{y}(t)) \approx f^i(\mathbf{x}^*) + \frac{\partial f^i(\mathbf{x}^*)}{\partial x^j} y^j(t),$$

or,

$$d_t\mathbf{y}(t) = A\mathbf{y}(t), \qquad A^i{}_j = \left.\frac{\partial f^i(\mathbf{x}^*)}{\partial x^j}\right|_{\mathbf{x}=\mathbf{x}^*},$$

where we used $d_t\mathbf{x}^* = \mathbf{f}(\mathbf{x}^*) = 0$, and $\frac{\partial f^i(\mathbf{x}^*)}{\partial x^j} = \left.\frac{\partial f^i}{\partial x^j}\right|_{\mathbf{x}=\mathbf{x}^*}$ We observe that the deviation vector evolves according to a *linear* first order differential equation, governed by a matrix A, which in turn is determined by the derivatives of $\mathbf{f}$ at the fixed point.

The linear equation is solved by[12]

$$\mathbf{y}(t) = \exp(At)\mathbf{y}(0).$$

[12] Recall that the matrix exponential is defined by the formal Taylor series, $\exp(B) = \sum_n \frac{1}{n!}B^n$.

To understand the different classes of behavior, we assume A to be diagonalizable with eigvenvectors bv_j and -values λ_j. An expansion $\mathbf{y} = y^j \mathbf{v}_j$ in the eigenbasis then leads to the solution

$$\mathbf{y}(t) = \sum_j e^{\lambda_j t} y^j(0) \mathbf{v}_j.$$

Recall that eigenvectors and -values are generically complex, even if the underlying matrix is real. Taking this fact into account, we may anticipate the following types of near-fixed point evolution:

- **Oscillatory motion** around the fixed point: all eigenvalues purely imaginary, $\mathrm{Re}(\lambda_j) = 0$;
- **Damped oscillatory motion**: eigenvalues have finite negative real part, $\mathrm{Re}(\lambda_j) < 0$;
- **Attenuated motion back to the fixed point**: eigenvalues real and negative, $\mathrm{Re}(\lambda_j) < 0$, $\mathrm{Im}(\lambda_j) = 0$;
- **Instability**: there exist eigenvalues with positive real part, $\mathrm{Re}(\lambda_j) > 0$, for at least one i.

In the last case, a deviation in the direction $\mathbf{v}_j$ will grow exponentially, $\sim \mathbf{v}_j e^{\lambda_j t}$, hence the fixed point is **unstable** and the system will flow away from it. In such cases, the condition that the deviation $\mathbf{y}$ is small holds only for short time scales and different solution methods must be applied to describe the dynamics at longer time scales. Let us illustrate the above cases on a few examples:

EXAMPLE Consider the one dimensional **damped oscillator**, described by the equations of motion

$$\begin{aligned} d_t q &= \frac{1}{m} p, \\ d_t p &= -m\omega^2 q - 2\gamma p. \end{aligned}$$

For $\gamma = 0$, these are the Hamilton equations of motions of an harmonic oscillator of frequency ω, with $H = p^2/2m + m\omega^2 q^2/2$. The term proportional to γ describes friction, i.e. a de-accelarating force proportional to velocity or momentum. The equations are already linear, and the resting configuration $(q, p) = (0, 0)$ is a fixed point. With

$$A = \begin{pmatrix} & \frac{1}{m} \\ -m\omega^2 & -2\gamma \end{pmatrix}$$

and $\mathbf{x} = (q, p)$, the equations of motion assume the form $d_t \mathbf{x} = A\mathbf{x}$. The eigenvalues of this matrix are obtained as (do it!)

$$\lambda_\pm = -\gamma \pm i\sqrt{\omega^2 - \gamma^2}.$$

Even without solving for the eigenvectors and specific solutions, we can anticipate two different type of motion: for $\omega > \gamma$, we have complex eigenvalues. The negative real part, γ, desccribes

damping, and the imaginary part, $\sqrt{\omega^2-\gamma^2}$ oscillation with a frequency modified by damping. The net result is damped oscillatory motion. (Think of the sound of a musical instrument fading due to attenuation.) For strong friction, $\gamma > \omega$, the eigenvalues are negative and real, and the one with smaller real part $-\gamma+\sqrt{\gamma^2-\omega^2} \approx \frac{\omega^2}{2\gamma}$ describes an exponential approach $\sim \exp(-t\omega^2/2\gamma)$ towards the resting fixed point. (Think of a pendulum immersed in syrup.)

EXAMPLE Consider the Hamiltonian $H(q,p) = \frac{1}{2m}p^2 + V(q)$ with the **washboard potential** $V(q) = -U\cos(aq)$, at constant $U > 0$. The Hamilton equations of motion have fixed points at $p^* = 0$ and $q^* = n\pi/a$. For even n they correspond to particles resting at minima of the potential, for odd n at its hills. Linearization of the equations of motion for $\mathbf{x} = \mathbf{x}^* + \mathbf{y}$ leads to (verify it!)

$$d_t\mathbf{y} = \begin{pmatrix} 0 & \frac{1}{m} \\ -Ua^2(-1)^n & 0 \end{pmatrix} \mathbf{y}$$

with eigenvalues $z_\pm = \pm\sqrt{(Ua^2/m)(-1)^{n+1}}$. For the minima, they are purely imaginary and of opposite sign, corresponding to oscillatory motion with frequency $\omega = a\sqrt{U/m}$. These are the harmonic oscillators defining the dynamics close to the minima of the potential. For odd n, we have a likewise sign-opposite real eigenvalues $\lambda_\pm = a\sqrt{U/m}$, corresponding to accelerated motion away from the unstable hill-fixed points. (Write down solutions for concretely specified initial conditions and discuss their coordinate and momentum dependence.)

Are all these scenarios realized in Hamiltonian dynamics? The discussion of the second example gives us a hint that this may not be the case. We saw that both for oscillatory and unstable motion, the eigenvalues characterizing the problem came in sign inverted pairs $\pm\lambda$. To understand if this is a general structure, let us go back to the definition of the Hamiltonian vector field in Eq. (3.15). Consider the Hamilton equations $\dot{x}_i = I_{ij}\partial_j H(\mathbf{x}) \equiv X_i(\mathbf{x})$ linearized in $\mathbf{y}$ around a fixed point $\mathbf{x}^*$. With $X_i(\mathbf{x}^*+\mathbf{y}) \approx X_i(\mathbf{x}^*) + \partial_k X_i(\mathbf{x}^*)y_k$, we obtain

$$\dot{y}_i = A_{ik}y_k, \qquad A_{ik} = I_{ij}\partial^2_{jk}H(\mathbf{x}^*),$$

i.e. the same matrix structure that featured before in our discussion of the Liouville theorem in section 3.2.4. With the abbreviated notation $A = IR$, and the symmetric $R_{jk} = \partial^2_{jk}H(\mathbf{x}^*)$, we need to understand the eigenvalues of a matrix A possessing a curious symmetry: With $I^T = -I$, and $I^TI = \mathbb{1}$,

$$A^T = R^TI^T = I^TIR(-I) = -I^TAI.$$

From this relation, we obtain a condition on A's eigenvalues: suppose that λ is one of them, i.e. $\det(\lambda - A) = 0$. Using properties of the determinant, we obtain

$$\det(\lambda - A) = \det(\lambda - A)\underbrace{\det(II^T)}_{1} = \det(I^T(\lambda - A^T)I) = \det(\lambda + A) = \det(-\lambda - A),$$

where in the final step we used that the evenness of A's matrix dimension. In other words, $-\lambda$ is an eigenvalue too. This finding is closely related to our earlier observation of volume conservation under Hamiltonian flow. For example, consider a parallelogram in a two-dimensional phase space aligned along the directions of eigenvectors corresponding to eigenvalues $\pm\lambda$. In the course of time, its edges will shrink and grow according to the factors $\exp(\pm\lambda t)$. However, the area $\propto \exp(\lambda t)\exp(-\lambda t)$ remains constant.

3.4 Integrability

Linear stability theory can diagnose instabilities in the vicinity of fixed points of dynamical motion, however, such instabilities need not be associated to chaos. A case in point is the motion in the washboard potential discussed in the previous example: while the hilltops of the potential define unstable fixed points, there is no chaoticity in the dynamics.

The best avenue towards understanding actual chaos begins with an analysis of its absense, the case of **integrable motion**. We will begin with a somewhat technical definition of integrability, then discuss its physics, and in a third step use this discussion as a platform to venture into regions where integrability is absence, and chaos may await.

We call a mechanical system with f degrees of freedom **integrable** if there exist f independent and mutually involutive functions $f_i(\mathbf{x})$ which are dynamically conserved in its motion, $f_i(\mathbf{x}(t)) = \text{const}$. There are two keywords in this definition which require explanation: first, two functions f and g are involutive in the sense of classical mechanics, if $\{f, g\} = 0$. Second, much as with linearly dependent vectors, functions f_i are dependent functions, if coefficient *functions* $c_i(\mathbf{x})$ can be found such that $\sum_i c_i(\mathbf{x}) df_i(\mathbf{x}) = 0$, i.e. if the differentials, which are linear maps, and hence can be added and subtracted, can be linearly combined to zero.

What is the meaning of these conditions and of the definition of integrability? The condition of independence serves to prevent overcounting. For example, consider a mechanical system for which three components of angular momentum are conserved, $L_i(\mathbf{x}) = \text{const}$. Now, obviously any function of these components $F(L_1, L_2, L_3)$, will likewise be conserved. However, $dF(\mathbf{x}) = \sum_i \partial_{L_i} F(\mathbf{x}) dL_i(\mathbf{x})$, in violation of independence. In this formula $\partial_{L_i} F(\mathbf{x})$ assume the role of the coefficient functions. According to Eq. (3.21), the condition $\{f_i, f_j\} = 0$ means $\omega^2(\mathbf{X}_{f_i}, \mathbf{X}_{f_j}) = 0$.

In order to understand the physical meaning of the definition, we first need to prove the following mathematical consequence of the definition: we first note that the mechanical motion takes place on an f-dimensional manifold, T, defined by the constancy $f_i(\mathbf{x}) = \text{const}$. This is not hard to understand. Phase space is parameterized by $2f$ coordinates, f of these are constraint by the f equations $f_i(\mathbf{x}) = \text{const.}$, and so we are left with a $2f - f = f$ dimensional generalized surface, or manifold of motion. Let us assume, that T is compact, i.e. we exclude unbounded motion in any direction. (Think of pendula, or planetary motion.) We now use the above conditions to demonstrate that the vector fields $\mathbf{X}_{f_i}$ are tangent to T and linearly independent. Tangency means, that the functions f_i do not change in $\mathbf{X}_j$-direction, for if they did, $\mathbf{X}_j$ would 'point away' from T. In other words, we require $df_i(\mathbf{X}_{f_j}) = \{f_i, f_j\} = 0$, where Eq. (3.22) was used, i.e. the tangency condition is a consequence of the involution property. To prove the linear independence, we assume the opposite, i.e. the existence of coefficient (functions) such that $\sum_i c_i(\mathbf{x}) \mathbf{X}_i(\mathbf{x}) = 0$. In particular, the scalar product with any other phase space vector $\mathbf{X}$ would vanish, $\omega^2(c_i \mathbf{X}_i, \mathbf{X}) = c_i df_i(\mathbf{X}) = 0$. However, this cannot be, due to the assumed independence.

We still don't know what all these conditions are useful for. However, this follows from an important, and deep, statement we now import from mathematics: informally, it states that a compact f-dimensional surface supporting f linearly independent continuous tangent vector fields is diffeomorphic to an f-dimensional *torus*. What does *this* mean? Try to imagine an

f-dimensional closed and finite surface, it will help to focus on the case $f = 2$. Examples include spheres, or deformations of spheres, tori (the abstractions of rings with one hole), or surfaces supporting multiple holes. Of these, only the torus satisfies the above condition. To understand this, imagine your surface covered with hair, and trying to comb it such that there are no singularities (this would be the analog of a globally continuous tangent vector field.) On a sphere, this is not an option, there will remain a discontinuous 'vortex', a fact that will be confirmed by any barber. However, a torus, and only the torus, can be combed not just in one, but in two directions, namely the directions aligned with its two ring like bounding circles:

INFO An f-dimensional torus $T \simeq S^1 \times \cdots \times S^1$ is a manifold diffeomorphic to the product of f-circles. As such, it can be parameterized in terms of f coordinates ϕ_i playing the roles of angles.

Note that 'diffeormorphic' to a torus means that our motion manifold need not actually look like a torus. However, topologically, it is one. Heuristically, this includes wild deformations of tori made of knead. If you now imagine taking a knife and cutting your torus, you will end up with circles or unions of circles in the plane of intersection. This is precisely what we observed earlier for the Poincaré plots of the Henon-Heiles oscillator. Close to the minimum of its potential, $q_1 = q_2 = 0$, we can safely forget about the terms beyond quadratic order in q_i (why?). The ensuing two-dimensional harmonic oscillator has rotational symmetry in the (q_1, q_2) plane, which defines a conservation law, namely the constancy of planar angular momentum. Along with the conservation with energy, this makes for two conserved quantities for an $f = 2$ system, integrability. The Poincaré plots defined in its four-dimensional surface reveal circular structures as intersection manifolds. For a simpler example of an $f = 1$ torus, consider the ellipsoidal (diffeomorphic to circles) phase space curves of the harmonic oscillator.

EXERCISE Consider the $f = 6$ two-body central force problem. Is it integrable in the sense of the above definition?

3.5 Canonical transformations

Having identified the torus geometry of the generalized surfaces in phase space on which integrable motion takes place, we next ask how to describe it in quantitative terms. This is, in the first place, a question concerning a good choice of coordinates.

In the previous chapter, we have emphasized the flexibility of the Lagrangian approach in the choice of problem adjusted coordinates: any diffeomorphic transformation $q_i \mapsto q_i' = q_i'(\mathbf{q}), i = 1, \ldots, f$ leave the Euler–Lagrange equations form–invariant. Now, in the Hamiltonian formalism, we have twice as many variables $x_i, i = 1, \ldots, 2f$. Of course, we may consider "restricted" transformations, $q_i \mapsto q_i'(\mathbf{q})$, where only the configuration space coordinates are transformed (the transformation of the p_i's then follows.) However, things get more interesting, when we set out to explore the full freedom of coordinate transformations "mixing" configuration space coordinates, and momenta. This would include, for example, a mapping

$q_i \mapsto q_i' \equiv p_i, p_i \mapsto p_i' = q_i$ (a diffeomorphism!) that exchanges the roles of coordinates and momenta.

This freedom raises tempating new perspectives. For example, an f-dimensional torus can be parameterized by f angles ϕ_i. We also know that an integrable system possesses f conserved quantities p_i. Wouldn't it be nice to use $X_i \equiv (\phi_i, p_i)$ as a new set of $2f$ generalized coordinates? The issue with such choices is that many of them will not be 'canonical'. This is another way of saying that the Hamilton equations of motion transformed to the new set of variables will not assume the form $\dot{\mathbf{X}} = I\partial_{\mathbf{X}} H'(\mathbf{X})$, with a transformed function H'.

3.5.1 Definition

Since much of what we are doing in Hamiltonian dynamics makes explicit reference to the form of the Hamilton equation, this is a price we are not ready to pay. Instead, we will restrict ourselves to transformations that are canonical in the sense that the form of the equations of motion remains conserved: We call a coordinate transformation "canonical", if the following condition is met:

A diffeomorphism $\mathbf{x} \mapsto \mathbf{X}(\mathbf{x})$ defines a **canonical transformation** if there exists a function $H'(\mathbf{X})$ such that the representation of solution curves (i.e. solutions of the equations $\dot{\mathbf{x}} = I\partial_{\mathbf{x}} H$) in the new coordinates $\mathbf{X} = \mathbf{X}(\mathbf{x})$ solves the "transformed Hamilton equations"

$$\dot{\mathbf{X}}' = I\partial_{\mathbf{X}} H'(\mathbf{X}). \tag{3.24}$$

What may appear strange about this definition is that it does not tell how the "new" Hamiltonian H' relates to the old one, H. In the most general formulation of the Hamiltonian mechanics, where we allow for coordinate transformations $\mathbf{X}(\mathbf{x}, t)$ carrying explicit time dependence[13], finding $H'(\mathbf{X})$ can be a difficult task. However, we will here restrict ourselves to maps $\mathbf{X}(\mathbf{x})$. Referring to our discussion below, in these cases we simply have $H'(\mathbf{X}) = H(\mathbf{x}(\mathbf{X}))$, the new Hamiltonian is the old one expressed in the new coordinates.

Before exploring the ways by which canonical transformations can be constructively obtained, let us ask how a 'good canonical transformation' might facilitate our description of (for now integrable) mechanical systems. Supposed we had managed to find a canonical transformation from our original phase space coordinates $\mathbf{x} = (\mathbf{q}, \mathbf{p})$ to a new set $\mathbf{X} = (\boldsymbol{\phi}, \mathbf{I})$, such that the f coordinate I_i are dynamically conserved. In other words, $I_i(t) = \text{const.}$ along phase space trajectories. Since the transformation is canonical, $d_t I_i(t) = -\partial_{\phi_i} H'(\mathbf{X}) = 0$, meaning that the Hamiltonian expressed in new coordinates does not depend on the ϕ-variables. The time dependence of the latter is then given by

$$\dot{\phi}_i(t) = \partial_{I_i} H'(\mathbf{I}) \equiv \omega_i = \text{const.},$$

[13] Imagine a system with constraints explicitly depending on time, such as the motion of a bead on a time-dependent loop. In this case, one might want to consider such generalizations.

where we used the constancy of all I_j. This is trivially solved as $\phi_i(t) = \omega_i t + \phi_i(0)$. Now remember that our motion manifold is a torus, i.e. an object described by f angular coordinates. The discussion above suggests that these f coordinates are just angles ϕ_i required to parameterize the f-dimensional torus on which integrable motion takes place.

3.5.2 Canonical transformations: why?

Why would one start thinking about generalized coordinate transformations mixing configuration space coordinates and momenta, etc.? In previous sections, we had argued that coordinates should relate to the symmetries of a problem. Symmetries, in turn, generate conservations laws (Noether), i.e. for each symmetry one obtains a quantity, I, that is dynamically conserved. Within the framework of phase space dynamics, this means that there is a function $I(\mathbf{x})$ such that $d_t I(\mathbf{x}(t)) = 0$, where $\mathbf{x}(t)$ is a solution curve. We may think of $I(\mathbf{x})$ as a function that is constant along the Hamiltonian flow lines.

Now, suppose we have $s \leq f$ symmetries and, accordingly, s conserved functions $I_i, i = 1, \ldots, f$. Further assume that we find a *canonical* transformation to phase space coordinates $\mathbf{x} \mapsto \mathbf{X}$, where $\mathbf{P} = (I_1, \ldots, I_s, P_{s+1}, \ldots, P_f)$, and $\mathbf{Q} = (\phi_1, \ldots, \phi_s, Q_{s+1}, \ldots, Q_f)$. In words: the first s of the new momenta are the functions I_i. Their conjugate configuration space coordinates are called ϕ_i.[14] The new Hamiltonian will be some function of the new coordinates and momenta, $H'(\phi_1, \ldots, \phi_s, Q_{s+1}, \ldots, Q_f, I_1, \ldots, I_s, P_{s+1}, \ldots, P_f)$. Now let's take a look at the Hamilton equations associated to the variables (ϕ_i, I_i):

$$d_t \phi_i = \partial_{I_i} H',$$
$$d_t I_i = -\partial_{\phi_i} H' \stackrel{!}{=} 0.$$

The second equation tells us that H' must be independent of the variables ϕ_i:

> The coordinates, ϕ_i, corresponding to conserved momenta I_i are **cyclic coordinates**,
>
> $$\frac{\partial H'}{\partial \phi_i} = 0.$$

At this point, it should have become clear that coordinate sets containing conserved momenta, I_i, and their coordinates, ϕ_i are tailored to the optimal formulation of mechanical problems.

The power of these coordinates becomes fully evident in the extreme case where $s = f$, i.e. where we have as many conserved quantities as degrees of freedom. In this case

$$H' = H'(I_1, \ldots, I_f),$$

[14] It is natural to designate the conserved quantities as "momenta": in section 2.2.4 we saw that the conserved quantity associated to the "natural" coordinate of a symmetry (a coordinate transforming additively under the symmetry operation) is the Noether momentum of that coordinate.

and the Hamilton equations assume the form

$$\begin{aligned} d_t\phi_i &= \quad \partial_{I_i} H' \equiv \omega_i \\ d_t I_i &= -\partial_{\phi_i} H' = 0. \end{aligned}$$

These equations can now be trivially solved:[15]

$$\begin{aligned} \phi_i(t) &= \omega_i t + \alpha_i, \\ I_i(t) &= I_i = \text{const.} \end{aligned}$$

A very natural guess would $H'(\mathbf{X}, t) \stackrel{?}{=} H(\mathbf{x}(\mathbf{X}), t)$, i.e. the "old" Hamiltonian expressed in new coordinates. However, we will see that this ansatz can be too restrictive.

Progress with this situation is made once we remember that solution curves $\mathbf{x}$ are extrema of an action functional $S[\mathbf{x}]$ (cf. section 3.2.2). This functional is the $\mathbf{x}$–representation of an "abstract" functional. The same object can be expressed in terms of $\mathbf{X}$–coordinates (cf. our discussion in section 2.1.3.) as $S'[\mathbf{X}] = S[\mathbf{x}]$. The form of the functional S' follows from the condition that the $\mathbf{X}$–representation of the curve be solution of the equations (3.24). This is equivalent to the condition

$$S'[\mathbf{X}] = \int_{t_0}^{t_1} dt \left(\sum_{i=1}^{f} P_i \dot{Q}_i - H'(\mathbf{Q}, \mathbf{P}, t) \right) + \text{const.},$$

where "const." is a contribution whose significance we will discuss in a moment. Indeed, the variation of $S'[\mathbf{X}]$ generates Euler–Lagrange equations which we saw in section 3.2.2 are just the Hamilton equations (3.24).

[15] The problem remains solvable, even if $H'(I_1, \ldots, I_f, t)$ contains explicit time dependence. In this case, $\dot{\phi}_i = \partial_{I_i} H'(I_1, \ldots, I_f, t) \equiv \omega_i(t)$, where $\omega_i(t)$ is a function of *known* time dependence. (The time dependence is known because the variables I_i are constant and we are left with the externally imposed dependence on time.) These equations – ordinary first order differential equations in time – are solved by $\phi_i(t) = \int_{t_0}^{t} dt' \, \omega_i(t')$.

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