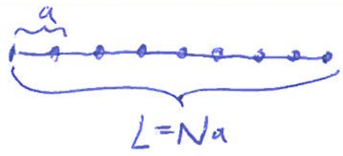
Intensive Week lecture : top. ins., homotopy, Grassmannian, Berry phase, vector bundle, section, invariant, Chern number

Band theory (dim = 1)

- Setting: particle on a ring of length $L = Na$



  - Hilbert space $\mathcal{H} = L^2(S^1)$
  - Translation $t_a : \mathcal{H} \to \mathcal{H}$, $\psi(x) \mapsto \psi(x-a)$
  - $\underbrace{\text{Translation-invariant}}$ Hamiltonian $H : \mathcal{H} \to \mathcal{H}$

    $t_a H = H t_a$ (periodic potential)

- Eigenvalues λ of t_a: $\left((t_a)^N \psi\right)(x) = \psi(x) \Rightarrow (\lambda)^N = 1$

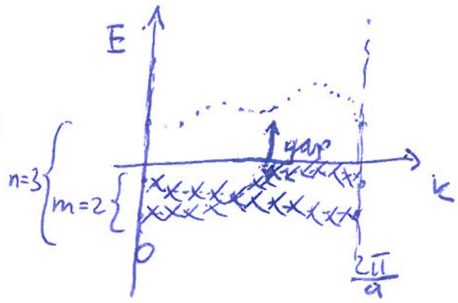
  $$\Rightarrow \lambda = e^{ik_j a} \quad \text{with } k_j = \frac{2\pi}{a}\frac{j}{N},\ j = 1, \dots, N$$

- Decomposition into eigenspaces of t_a: $\mathcal{H} = \bigoplus_{j=1}^{N} \mathcal{H}_{k_j}$

- $t_a H = H t_a \Rightarrow \underbrace{H(k)} := H\big|_{\mathcal{H}_k} : \mathcal{H}_k \to \mathcal{H}_k$, check: $|\psi\rangle \in \mathcal{H}_k \Rightarrow t_a(H|\psi\rangle) = H t_a |\psi\rangle = e^{ika}(H|\psi\rangle)$ ✓

  "Bloch Hamiltonian"

- Spectrum varies with k ⇝ bands:



- Many-body ground state (non-interacting): Fill from bottom with electrons!

  ↪ eigenstates of $H(k)$: $\{|\psi_{1,k}\rangle, \dots, |\psi_{n,k}\rangle\}$ $\Rightarrow |g.s.\rangle = \bigwedge_{j=1,\dots,N} \left(|\psi_{1,k_j}\rangle \wedge \dots \wedge |\psi_{m,k_j}\rangle\right)$

  energies: $E_1(k) \le \dots \le E_n(k)$

  $$= \prod_{j=1,\dots,N} c^\dagger_{1,k_j} \cdots c^\dagger_{m,k_j} |0\rangle$$

or in position space (m=1 here) : $\psi_{g.s.}(x_1, \dots, x_N) = \det\left(\{\psi_{k_i}(x_j)\}\right)$ "Slater determinant"

- Change of basis in $A(k) := \mathrm{span}_{\mathbb{C}}\{|\psi_{1,k}\rangle, \dots, |\psi_{m,k}\rangle\}$ does <u>not</u> change $|g.s\rangle$ (up to a phase)

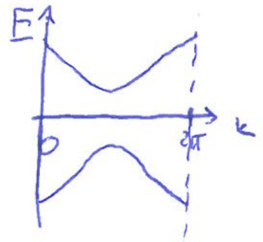
- $L \to \infty$: $k \in S^1$ continuous

- ~~General~~ General dim. d: $k \in \underbrace{S^1 \times \dots \times S^1}_{d \text{ times}} = T^d$ (repeat above $\forall$ dimensions separately)

  ↳ ground state = continuous assignment $\underset{\in T^d}{k} \longmapsto \underset{\in Gr_m(\mathbb{C}^n)}{A(k)}$

- Analogy using $\mathcal{H}_k = \mathbb{R}^2$: (& $a=1$) $\quad H(k) := f(k)\underbrace{\begin{pmatrix} \cos k & \sin k \\ \sin k & -\cos k \end{pmatrix}}_{\text{eigenvalues } \pm 1}$

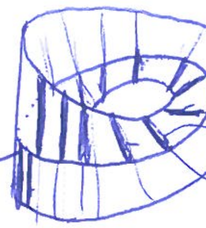
  $f(k)$: periodic and >0



$$\Rightarrow A(k) = \mathrm{span}_{\mathbb{R}}\Big\{\underbrace{\begin{pmatrix} -\sin\frac{k}{2} \\ \cos\frac{k}{2} \end{pmatrix}}_{|\psi_{1,k}\rangle}\Big\}$$

note: $|\psi_{1,2\pi}\rangle = (-)|\psi_{1,0}\rangle$ — Berry phase

↳ ground state:



$A: S^1 \longrightarrow Gr_1(\mathbb{R}^2) = S^1$

or assign slope: $\mathbb{R} \cup \{\infty\} = S^1$

winding number $= 1$ ($= 180°$ twist)

↳ stable against <u>continuous deformations</u>! "homotopies"

↳ <u>topological insulator</u>

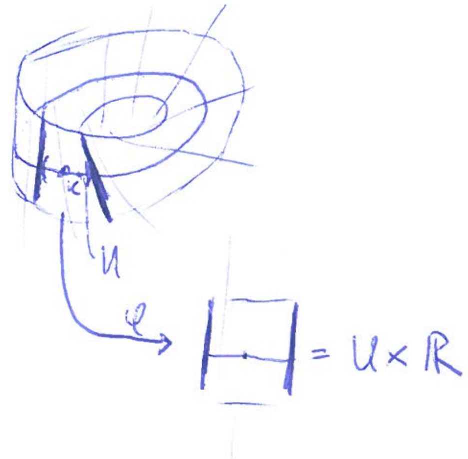
- A_0 & A_1 called <u>homotopic</u> if $\exists$ continuous deformation A_t. ("movie")

- Can view it as vector bundle over S^1.

• Vector bundle over M = vector spaces attached to every $k \in M$, varying <u>continuously</u>!

<u>formal</u>: Surjective $p: E \to M$, $p^{-1}(k)$ vector space,

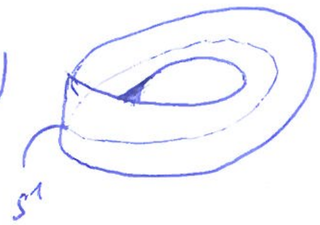
$\forall k \in M$ $\exists$ open neighbourhood U & $\varphi: U \times \mathbb{C}^m \xrightarrow{\sim} p^{-1}(U)$ (homeom.)



• Section of a vector bundle = continuous assignment $\underset{\in M}{k} \mapsto v(k) \in p^{-1}(k)$

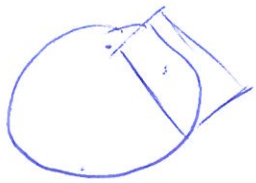
• Vector bundle non-trivial ($\not\cong M \times \mathbb{C}^m$) $\Leftrightarrow$ $\nexists$ m linearly independent, <u>non-zero</u> sections
($\Rightarrow A$ not homotopic to const. map)

↪ Examples: (i)



challenge: draw loop that doesn't cross S^1
↪ bundle non-trivial

(ii)



TS^2: hairy ball theorem: $\nexists$ non-zero $\underbrace{\text{section}}_{=\text{vector field}}$

(iii) Exercises...

- Back to $\mathbb{C}$: $Gr_1(\mathbb{C}^2) = S^2$ (Riemann sphere, by assigning complex slopes)

  ↳ $A: S^1 \longrightarrow S^2$ is loop on S^2

  ↳ can always contract :

- $d=2$: $A: T^2 \longrightarrow S^2$

  ↳ volume form ω w.K $\int_{S^2} \omega = 1$

  ↳ invariant $\boxed{n = \int_{T^2} A^*\omega}$ "mapping degree" (special case of Chern number)

  ↳ in coordinates: $\omega = \frac{1}{8\pi}\varepsilon_{ijk}\, x_i\, dx_j \wedge dx_k$

$$\Rightarrow A^*\omega = \frac{1}{8\pi}\varepsilon_{ijk}\, A_i \frac{\partial A_j}{\partial k_l}\frac{\partial A_k}{\partial k_p}\, dk_l \wedge dk_p$$

$$\Rightarrow n = \frac{1}{4\pi}\int_0^{2\pi}\!\!\int_0^{2\pi} \vec{A}\cdot\left(\frac{\partial \vec{A}}{\partial k_1} \times \frac{\partial \vec{A}}{\partial k_2}\right) dk_1\, dk_2$$

- General problem: Find all homotopy classes of maps $T^d \longrightarrow Gr_m(\mathbb{C}^n)$ subject to arbitrary symmetry constraints (today: no constraints, $m=1$, $n=2$, $d=1,2$)