BCGS Intensive Week: Lecture II
Tools from quantum information theory

29.7.2014
Thomas Quella

Goal: – Description of topological phases (in condensed matter theory) beyond band models (interactions!)
– Incorporation of symmetries

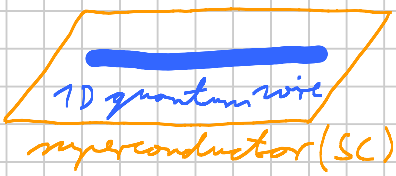
⟶ Two models: I) Kitaev's Majorana chain
II) The AKLT SU(2) spin chain

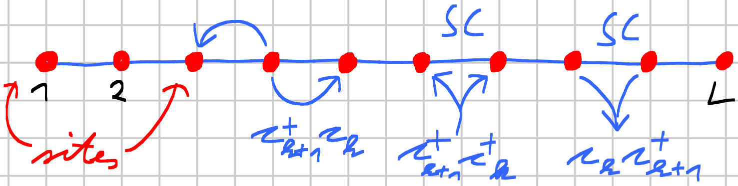
I. Kitaev's Majorana chain

Literature: „Unpaired Majorana fermions in quantum wires"
cond-mat/0010440

I.1 1D Tight binding model of non-interacting spinless fermions

Physical setup ⟶ Mathematical abstraction





Creation/annihilation operators: $\underbrace{\{c_k, c_l^+\}}_{\text{physical (complex) fermions}} = \delta_{kl}$ (no spins!)

Hamiltonian:

$$H = \sum_k \Big\{ \underbrace{-w\left(c_k^+ c_{k+1} + c_{k+1}^+ c_k\right)}_{\text{hopping}} \underbrace{- \mu\left(c_k^+ c_k - \tfrac{1}{2}\right)}_{\text{chemical potential}} + \underbrace{\left(\Delta c_k c_{k+1} + \bar{\Delta} c_{k+1}^+ c_k^+\right)}_{\text{coupling to SC}} \Big\}$$

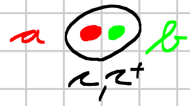
Shall study: Open and periodic boundary conditions (OBC/PBC)

Exact solution: (Fourier transform +) Bogoliubov transformation
for PBC

Here: Focus on two special points in the phase diagram
$\longrightarrow$ nice interpretation in terms of Majorana modes

Majorana mode: „Half a physical fermion" } Schematically

$$c_k = \frac{1}{2}(a_k + i b_k), \qquad c_k^+ = \frac{1}{2}(a_k - i b_k)$$



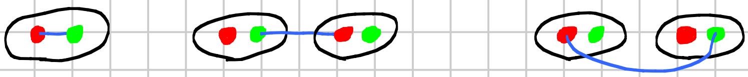
Relation: $\{a_k, a_\ell\} = \{b_k, b_\ell\} = 2\,\delta_{k\ell}, \quad \{a_k, b_\ell\} = 0$

This follows from $a_k = c_k + c_k^+$, $b_k = -i(c_k - c_k^+)$

Useful consequence: $a_k^2 = b_k^2 = 1$

Remark: There is no relation to the Majorana fermions used in relativistic elementary particle physics

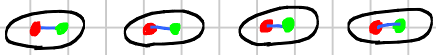
Now: Focus on OBC and write (Δ assumed to be real)



$$H = \frac{i}{2}\sum_k \left\{ -\mu\, a_k b_k + (\Delta + w)\, b_k a_{k+1} + (\Delta - w)\, a_k b_{k+1} \right\}$$

Special points:

i) $\Delta = w = 0$: $\quad H = -\mu \sum_k \left(c_k^+ c_k - \frac{1}{2}\right) = -\frac{i\mu}{2}\sum_k a_k b_k$

Sketch:  $\longrightarrow$ indicates pairing

Groundstate (for $\mu < 0$): vacuum $|0\rangle$ (no fermions)

+ gap $|\mu|$

Useful perspective: $H = \frac{\mu}{2}\sum_k P_k$ with $P_k = -i\, a_k b_k = 1 - 2c_k^\dagger c_k$

<u>Have:</u> i) $P_k^2 = 1$

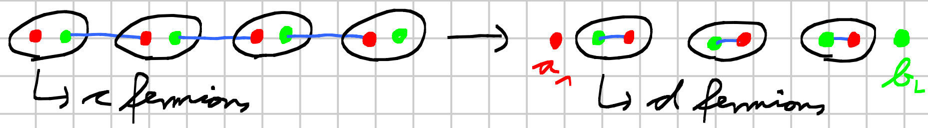
ii) $[P_k, P_l] = 0$

$\Big\}$ can diagonalize P_k simultaneously with eigenvalues $P_k = \pm 1$

Groundstate: $P_k|0\rangle = |0\rangle \quad (P_k = +1)$

ii) $\mu = 0,\ \Delta = w > 0$: $H = iw\sum_{k=1}^{L-1} b_k a_{k+1} = 2w\sum_{k=1}^{L-1}\left(d_k^\dagger d_k - \frac{1}{2}\right)$

Sketch:



Here: $d_k = \frac{1}{2}(b_k + i\, a_{k+1})$

<u>Observations:</u>

- a_1 and b_L are „Majorana zero modes": $[a_1, H] = [b_L, H] = 0$

  $\Rightarrow$ get emergent isolated Majorana fermions

  (gap $\Rightarrow$ finite correlation length)

- H is independent of the <u>delocalized physical</u> fermion

  $f = \frac{1}{2}(b_L + i\, a_1) \hat{=}$ „d_L"

  → vacuum wrt. d

- There are <u>two</u> groundstates $|\tilde{0}\rangle$ and $f^\dagger|\tilde{0}\rangle$ + gap $2w$

<u>Write:</u> $\tilde{P}_k = -i\, b_k a_{k+1} = 1 - 2d_k^\dagger d_k \Rightarrow H = -w\sum_{k=1}^{L-1}\tilde{P}_k$ ($P_k, \tilde{P}_k$ similar properties)

<u>Groundstate:</u> $\tilde{P}_k = +1$ for $k = 1, \dots, L-1$ ($\tilde{P}_L = -i\, b_L a_1$ undetermined)

<u>Claim:</u> i) and ii) realize two topologically distinct phases

<u>Want:</u> Topological invariant

I.2 Symmetry fractionalization

Symmetry of H: $\mathbb{Z}_2$ fermionic parity $Q = (-1)^N$

Write: $Q = e^{\pi i N} = e^{\pi i \sum_k c_k^+ c_k} = \prod_k e^{\pi i c_k^+ c_k} = \prod_k (1 - 2 c_k^+ c_k)$

Note: $Q = \prod_k P_k = (-i)^L \overbrace{a_1 b_1}^{P} a_2 b_2 \cdots a_L b_L = -i\, a_1 b_L \prod_k \tilde{P}_k$

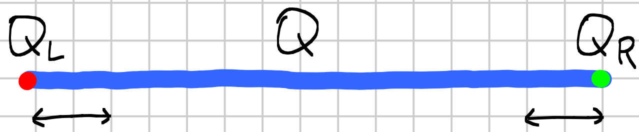
Action on groundstate(s):

case i) $Q = 1$

case ii) $Q = -i a_1 b_L$

Insight: Factorization $Q \sim Q_L \cdot Q_R$ into $\mathbb{Z}_2$-operators Q_L and Q_R at the end of the chain

(assuming that there is a unique groundstate if we close the chain)



finite extension of edge state (mass gap)

case i) Q_L bosonic
case ii) Q_L fermionic

} discrete $\Rightarrow$ invariant under deformation (as long as the gap is preserved)

Prediction: There are two distinct topological classes of spinless superconductors

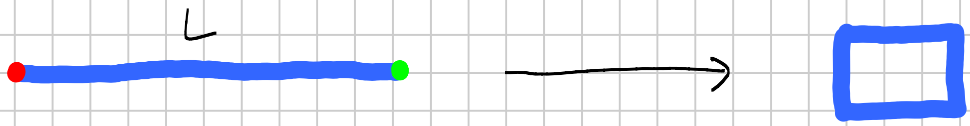
$\longrightarrow$ $\mathbb{Z}_2$ topological invariant $\Gamma = \pm 1$

Question: Signature of topology in the closed system?

1. Groundstate parity / Pfaffian
2. Entanglement spectrum

I.3 From open to closed chains

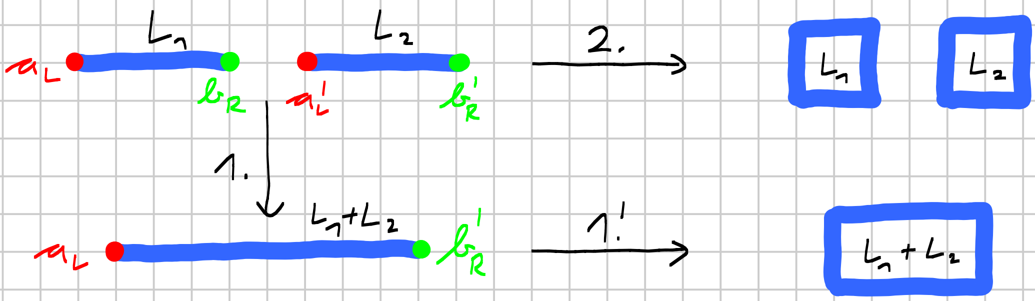
Arbitrary Majorana chain: Length L, OBC, $\Gamma = -1$



Parity: $\overset{\text{operator}}{\overbrace{Q(L)}} = \underbrace{-i\, a_L b_R}\, \overset{\text{number}}{\overbrace{S(L)}} \longrightarrow \overset{\text{number}}{\overbrace{\Pi(L)}} = \varepsilon\, S(L)$

projects to $\varepsilon \in \{\pm 1\}$ when closing the chain

Now combine two such chains in two ways:



$$1.\ Q(L_1+L_2) = (-i\, a_L b_R) S(L_1) (-i\, a_L' b_R') S(L_2)$$
$$= -(-i\, a_L b_R') S(L_1) \underbrace{(-i\, a_L' b_R)}_{\varepsilon} S(L_2)$$

$\hookrightarrow$ nature of edge modes

$$\Rightarrow \Pi(L_1+L_2) = -\varepsilon^2 S(L_1) S(L_2)$$

(no sign for bosonic edge modes)

$$2.\ \Pi(L_1)\Pi(L_2) = \varepsilon^2 S(L_1) S(L_2)$$

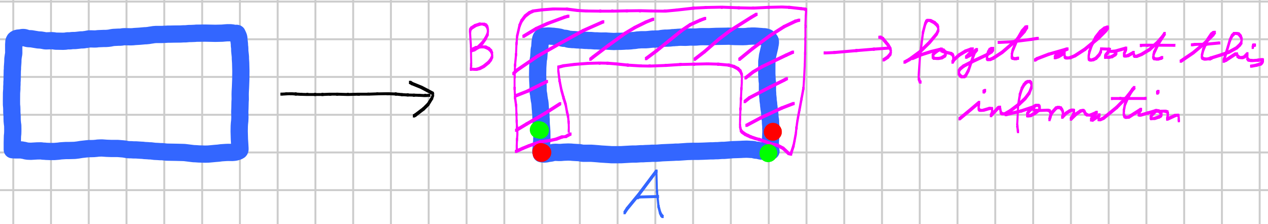
Result: $$\Gamma = \frac{\Pi(L_1+L_2)}{\Pi(L_1)\Pi(L_2)}$$

Remark: For quadratic Hamiltonians there is a simple formula for $\Pi(L)$ as a Pfaffian $\longrightarrow$ Exercises...

I.4 The entanglement spectrum

Goal: From closed to open chains

Closed system **System with virtual boundaries**



This can be formalized in terms of the reduced density matrix of the groundstate $|\psi\rangle$ (we are at zero T)

Def: Density matrix $\rho = |\psi\rangle\langle\psi|$

Reduced density matrix $\rho_A = tr_B(|\psi\rangle\langle\psi|)$

↳ forget about B

Remark: ρ_A contains all relevant information about subsystem A

$$\langle O_A \rangle = tr(O_A \rho) = tr_A(O_A \rho_A)$$

↳ some observable in subsystem A

Associated concepts:

1. Entanglement entropy: $S_E = -tr(\rho_A \ln \rho_A)$
2. Entanglement Hamiltonian: $\rho_A = e^{-H_E}$
3. Entanglement spectrum: Eigenvalues of H_E

All these quantities are easily accessible from the <u>Schmidt decomposition</u>. Namely, there exist sets of orthonormal vectors $|\psi_i^{(A/B)}\rangle$ and constants $e^{-\frac{1}{2}\varepsilon_i} \geq 0$ such that

$$|\psi\rangle = \sum_i e^{-\frac{1}{2}\varepsilon_i} |\psi_i^{(A)}\rangle \otimes |\psi_i^{(B)}\rangle$$

<u>Remark:</u> This formula arises from a singular value decomposition of the (rectangular) matrix $\sum_{i,j} \beta_{ij} \, |e_i^{(A)}\rangle\langle e_j^{(B)}|$ where $|\psi\rangle = \sum_{i,j} \beta_{ij} \, |e_i^{(A)}\rangle \otimes |e_j^{(B)}\rangle$.

In terms of these data one has

$$\rho_A = \sum_i e^{-\varepsilon_i} \, |\psi_i^{(A)}\rangle\langle\psi_i^{(A)}|$$

as well as

$$S_E = -\operatorname{tr}\left[\rho_A \ln \rho_A\right] = \sum_i \varepsilon_i \, e^{-\varepsilon_i}$$

$$H_E = \sum_i \varepsilon_i \, |\psi_i^{(A)}\rangle\langle\psi_i^{(A)}| \quad \Rightarrow \quad \varepsilon_i \text{ is entanglement spectrum}$$

<u>Conjecture:</u> The entanglement spectrum has a two-fold degeneracy for the topologically non-trivial Majorana chain $\longrightarrow$ Exercise