II. The AKLT model

Literature: „Valence bond ground states in isotropic quantum antiferromagnets", Comm. Math. Phys. '88

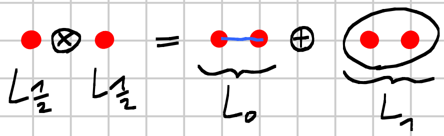
II.1 Valence bond construction (for a spin-1 chain)

Strategy:
- Start with a „nice" state $|\psi\rangle$
- Construct a „parent Hamiltonian" that has $|\psi\rangle$ as a groundstate

Representation of SU(2): $L_j \leftrightarrow$ spin j $\leftrightarrow$ Casimir $C_j = j(j+1)$

Idea: Realize L_1 as a subspace of $L_{\frac{1}{2}} \otimes L_{\frac{1}{2}} = L_0 \oplus L_1$

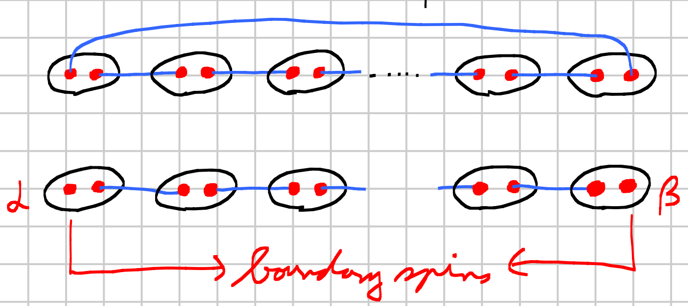
Schematically:



We will consider states of the form

PBC:

OBC:

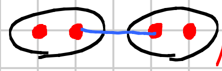


compare with Majorana chain!

Observation: OBC lead to fractional ($j = \frac{1}{2}$!) boundary spins

Question: Can such a groundstate be realized physically?

II.2 The parent Hamiltonian

Focus on two (physical) sites... α  β $= |\psi_\alpha^\beta\rangle$

Physical layer: $L_1 \otimes L_1 = L_0 \oplus L_1 \oplus L_2$

Auxiliary layer: $L_{\frac{1}{2}} \otimes L_{\frac{1}{2}} = L_0 \oplus L_1 \longrightarrow$ spanned by $|\psi_\alpha^\beta\rangle$

$|\psi_\alpha^\beta\rangle$ are the unique groundstates of $H = P_2$

↳ projector onto $S=2$

How to obtain the explicit form of P_2?

$\longrightarrow$ Use Casimir eigenvalues: $C_j = \vec{S}^2\big|_{\text{spin } j} = j(j+1)$

$$\overset{\vec{S}_1}{1} \otimes \overset{\vec{S}_2}{1} = \overbrace{0 \oplus 1 \oplus 2}^{\vec{S} = \vec{S}_1 + \vec{S}_2} \quad \Rightarrow \quad P_2 = \frac{(C-C_0)(C-C_1)}{(C_2-C_0)(C_2-C_1)}$$

$$C_j: \quad 0 \quad 2 \quad 6$$

Simplification using $C = (\vec{S}_1 + \vec{S}_2)^2 = \vec{S}_1^{\,2} + \vec{S}_2^{\,2} + 2\,\vec{S}_1 \cdot \vec{S}_2 = 4 + 2\,\vec{S}_1 \cdot \vec{S}_2$

$$P_2 = \frac{(4 + 2\,\vec{S}_1 \cdot \vec{S}_2)(2 + 2\,\vec{S}_1 \cdot \vec{S}_2)}{6 \cdot 4} = \frac{8}{24} + \frac{4+8}{24}\,\vec{S}_1 \cdot \vec{S}_2 + \frac{1}{6}(\vec{S}_1 \cdot \vec{S}_2)^2$$

Generalization to many sites:

$$H = \sum_k \left[\frac{1}{2}\,\vec{S}_k \cdot \vec{S}_{k+1} + \frac{1}{6}(\vec{S}_k \cdot \vec{S}_{k+1})^2 + \frac{1}{3} \right]$$

Properties:
- Each term non-negative $\Rightarrow$ H non-negative
- but: not mutually commuting (not as for Majorana case)
- H is gapped (see original paper)

A state $|\psi\rangle$ with $H|\psi\rangle = 0$ has to be annihilated by each single term $\Longrightarrow$ $|\psi_\alpha{}^\beta\rangle$ are the unique groundstates (OBC)

PBC: Unique groundstate $|\psi\rangle = \sum_\alpha |\psi_\alpha{}^\alpha\rangle$

II.3 Matrix product states (MPS)

One can write $|\psi_\alpha{}^\beta\rangle$ as an MPS:

$$|\psi_\alpha{}^\beta\rangle = \sum_{\{i_k\}} \left(A_{i_1}\cdots A_{i_L}\right)_\alpha{}^\beta \, |i_1\cdots i_L\rangle$$

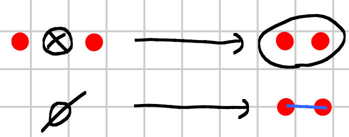
$\hookrightarrow$ 2×2 matrices

$$|\psi\rangle = \sum_{\{i_k\}} \mathrm{tr}\left(A_{i_1}\cdots A_{i_L}\right) |i_1\cdots i_L\rangle$$

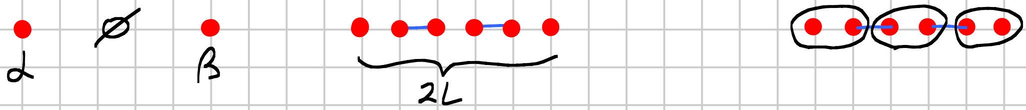
Abstract construction of valence bonds:

Ingredients: Intertwiner $A: L_{\frac{1}{2}} \otimes L_{\frac{1}{2}}^* \to L_1$

Intertwiner $I: L_0 \to L_{\frac{1}{2}} \otimes L_{\frac{1}{2}}^*$



Edge spaces $\xrightarrow{I^{\otimes(L-1)}}$ Auxiliary space $L_{\frac{1}{2}}^{\otimes 2L}$ $\xrightarrow{A^{\otimes L}}$ Physical space $L_1^{\otimes L}$



Explicitly: $I: 1 \longmapsto \sum_\alpha \langle\alpha| \otimes |\alpha\rangle$

$$A: |\alpha\rangle \otimes \langle\beta| \longmapsto \sum_i (A^i)_\alpha{}^\beta \, |i\rangle$$

Hence: $$|\alpha\rangle\otimes\langle\beta| \xmapsto{I} \sum_{\{\sigma_\ell\}} |\alpha\rangle\otimes\langle\sigma_1|\otimes|\sigma_1\rangle\otimes\cdots\otimes\langle\sigma_{L-1}|\otimes|\sigma_{L-1}\rangle\otimes\langle\beta|$$

$$\xmapsto{A} \sum_{\{\sigma_\ell\}}\sum_{\{i_\ell\}} (A_{i_1})_{\alpha}^{\ \sigma_1}(A_{i_2})_{\sigma_1}^{\ \sigma_2}\cdots(A_{i_L})_{\sigma_{L-1}}^{\ \beta}\,|i_1\rangle\otimes\cdots\otimes|i_L\rangle$$

$$= \sum_{\{i_\ell\}} (A_{i_1}\cdots A_{i_L})_{\alpha}^{\ \beta}\,|i_1\cdots i_L\rangle$$

In the standard basis for $L_{\frac{1}{2}}$ $(\pm\frac{1}{2})$ and L_1 $(\pm 1, 0)$ one has

$$\{A_i\} = \{\sigma^z, \pm\sqrt{2}\,\sigma^\pm\}$$

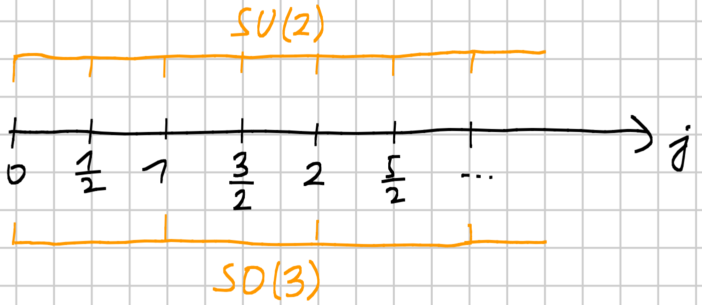
II.4 Symmetry fractionalization

Obviously $\frac{1}{2}$ (boundary spin) $= \frac{1}{2}\cdot 1$ (physical spin)

But: How would that compare to $1 = \frac{1}{2}\cdot 2$?

Def: Symmetry fractionalization (in 1D systems)
$\Longleftrightarrow$ Distinct symmetry of physical and emergent degrees of freedom

Reps of su(2):



What's the difference?

$\underbrace{SU(2) = \mathrm{Spin}(3)}_{\text{simply-connected}}$ is the double cover of $\underbrace{SO(3)}_{\text{not...}}$ $(= SU(2)/\mathbb{Z}_2)$

(all loops contractible) (path in $SU(2)$ from $+1$ to -1 closed in $SO(3)$ but not in $SU(2)$)

This follows from the identification

$$\vec{x} \in \mathbb{R}^3 \longleftrightarrow \vec{x}\cdot\vec{\sigma} \in H_2 = \{\text{traceless, hermitean } 2\times 2 \text{ matrices}\}$$

Action of $SU(2)$ on H_2: $M \longmapsto UMU^+$, $U \in SU(2)$

$\Rightarrow$ Induces action $R(U)$ of $SU(2)$ on $\mathbb{R}^3$

But: $R(U) = R(-U) \rightarrow$ only $SU(2)/\mathbb{Z}_2$ acts faithfully

This corresponds to the action of $SO(3)$

Actually, H_2 even admits a natural action of $U(2)$ since $UMU^+ = (e^{i\alpha}U)M(e^{i\alpha}U)^+$ and hence $SO(3) = U(2)/U(1)$.

<u>One says:</u> – $SU(2)$ is a <u>central extension</u> (by $\mathbb{Z}_2$) of $SO(3)$

– $U(2)$ is a <u>central extension</u> (by $U(1)$) of $SO(3)$

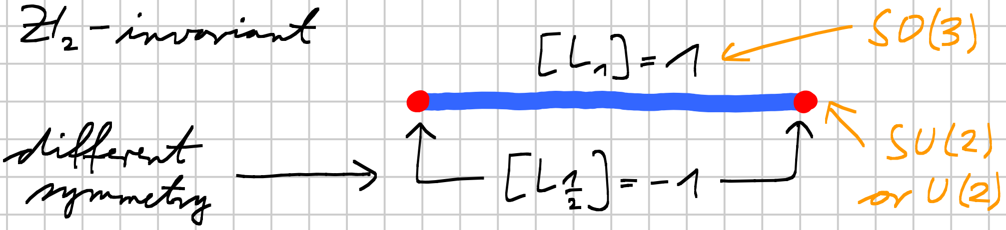
<u>Irreducible $SU(2)$ reps fall into classes:</u>

- The element $-1 \in SU(2)$ acts on L_j as $(-1)^{2j}$
- Can label each irrep L_j with an element of $\mathbb{Z}_2$:

$$[L_j] = (-1)^{2j} \in \mathbb{Z}_2$$

- Reps of $SO(3)$ need to satisfy $[L_j] = 1$ (integer spin)
- Reps with $[L_j] = -1$ are <u>projective reps</u> of $SO(3)$

<u>Remark:</u> For the AKLT chain $[L_{\frac{1}{2}}] = -1$ can be interpreted as a non-trivial topological $\mathbb{Z}_2$-invariant



II.7 Projective representations and central extensions

Def: A projective representation of a group G is a map $\rho: G \to \mathcal{U}(V)$ such that

$$\rho(g_1)\rho(g_2) = \omega(g_1, g_2)\rho(g_1 g_2)$$

with $\omega: G \times G \to \mathcal{U}(1)$

Associativity: $[\rho(g_1)\rho(g_2)]\rho(g_3) = \rho(g_1)[\rho(g_2)\rho(g_3)]$

$$\Longleftrightarrow \omega(g_1, g_2)\,\omega(g_1 g_2, g_3) = \omega(g_2, g_3)\,\omega(g_1, g_2 g_3) \qquad (*)$$

Solution ω: „cocycle"

Observations:

- Let $f: G \to \mathcal{U}(1)$ be an arbitrary function. Then

$$\alpha(g_1, g_2) = \frac{f(g_1) f(g_2)}{f(g_1 g_2)} \longrightarrow \text{„coboundary"}$$

is a solution of $(*)$ $\to$ should be regarded as „trivial"

- ω cocycle, α coboundary $\Rightarrow$ $\alpha \cdot \omega$ cocycle
- $\{\text{cocycles}\}$, $\{\text{coboundaries}\}$ are abelian groups

$$\Rightarrow H^2(G, \mathcal{U}(1)) = \frac{\{\text{cocycles}\}}{\{\text{coboundaries}\}} \text{ is an abelian group,}$$

$\downarrow$

an example of a cohomology group („group cohomology")

Def: Two projective representations belong to the same class if their cocycles agree up to a coboundary

<u>Def:</u> A group H is called <u>central extension</u> of another group G (by A) if

i) A is an abelian subgroup of H

ii) $G = H/A$

One writes: $1 \to A \longrightarrow H \longrightarrow G \longrightarrow 1$

<u>Theorem:</u> The cohomology group $H^2(G, U(1))$ classifies

i) Projective representations of G

ii) Central extensions of G

<u>Note:</u> Every projective representation of G lifts to an ordinary representation of a central extension of G

<u>Remarks:</u> – The cohomology groups $H^2(G, U(1))$ classify topological phases of bosonic 1D systems with on-site symmetry G

– There are higher cohomology groups $H^d(G, U(1))$ (partially) such phases in d dimensions

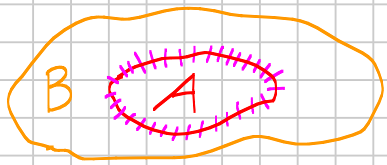
III. Aspects of entanglement

Many characteristics of a physical system can be inferred from

1. Entanglement entropy: – Gapped or gapless?
   – Existence of topological order
2. Entanglement spectra: – Low energy (boundary) theory
   – Projective representations, ...

Remarkable fact: All this is contained (to some extent) in a single groundstate wavefunction!

Gapped phases: „Area law"



Finite correlation length ⇒ Entanglement localized at interface

$$S_E = \alpha \, \mathrm{Area}(A) - \gamma + \dots$$

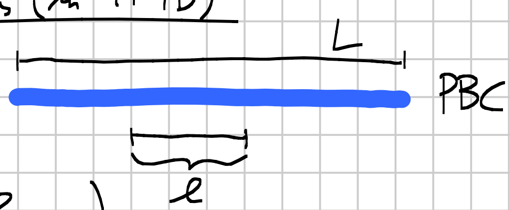
1D: $S_E = \ln n + \cdots$ (n = number of boundary modes)

2D: $S_E = \alpha \cdot L - \gamma + \cdots$ ($D = e^{\gamma}$ = total quantum dimension)

↳ non- ↳ universal [Kitaev, Preskill '06]

Entanglement scaling for critical systems (in 1+1D)

$$S_E = \frac{c}{3}\left[\frac{L}{\pi}\sin\frac{\pi}{L}\ell\right] + \mathrm{const.}$$



↳ universal (c = central charge)