

BCGS Intensive Week — Day 4: Morse Theory & Bott Periodicity (M. Zirnbauer)

PART 0: Some explanation of the words in the title. Motivation. Background.

(i) Morse Theory (developed in 1930–1965; Morse, Bott, Smale, Milnor ...)

One outcome of Morse Theory are the Morse inequalities:

for a compact manifold M with Morse function f , the number $m_q(f)$ of critical points of f with index q is no less than the Betti number $b_q(M)$.

- Fruitful new proof by E. Witten (1982) in "Supersymmetry and Morse Theory".

Witten's IDEA: de Rham complex Ω^*M = supersymmetric quantum mechanics
& deformation to a harmonic-oscillator problem.

See, e.g., <http://www.thp.uni-koeln.de/zirn> ~ "Topology for Physicists" (pp. 46–64).

(ii) Bott Periodicity Theorem

"Modern" proofs available (cf. Day 5); Bott's original approach (1958) uses Morse theory.

Complex case.

$$\mathbb{Z} = \pi_1(U_n) = \pi_2(U_{2n}/U_n \times U_n) = \pi_3(U_{2n}) = \pi_4(U_{4n}/U_{2n} \times U_{2n}) = \dots$$

$$0 = \pi_1(U_{2n}/U_n \times U_n) = \pi_2(U_{2n}) = \pi_3(U_{4n}/U_{2n} \times U_{2n}) = \pi_4(U_{4n}) = \dots$$

Real case.

$$\begin{aligned} \pi_0(0) \stackrel{!}{=} \mathbb{Z} &= \pi_1(0/0 \times 0) = \pi_2(U/0) = \pi_3(Sp/U) = \pi_4(Sp) = \\ &= \pi_5(Sp/Sp \times Sp) = \pi_6(U/Sp) = \pi_7(0/U) = \pi_8(0) = \pi_9(0/0 \times 0) \end{aligned}$$

$$\begin{aligned} \pi_0(0/0 \times 0) \stackrel{?!}{=} \mathbb{Z} &= \pi_1(U/0) = \pi_2(Sp/U) = \pi_3(Sp) = \pi_4(Sp/Sp \times Sp) = \\ &= \pi_5(U/Sp) = \pi_6(0/U) = \pi_7(0) = \pi_8(0/0 \times 0) = \pi_9(U/0) \end{aligned}$$

and 6 more such chains of identities (8-fold periodicity).

≥ 2008: enter the physics of topological insulators and superconductors

(classification of gapped free-fermion ground states with symmetries).

Periodic Table of topological insulators/superconductors

from Hasan & Kane, Rev. Mod. Phys. (2011):

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Quantum Hall Effect

He-3 (B phase)

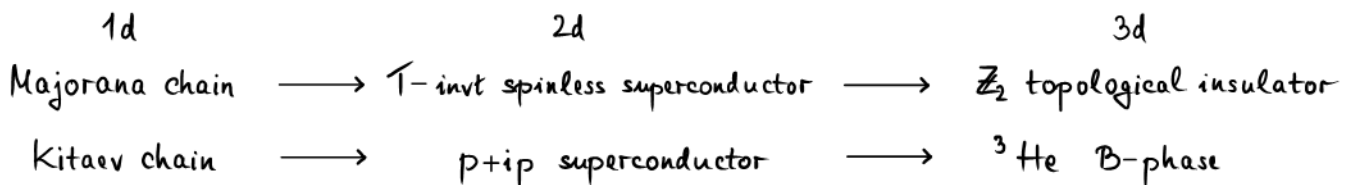
QSHI: HgTe

Majorana

Bi_2Se_3

TABLE I Periodic table of topological insulators and superconductors. The 10 symmetry classes are labeled using the notation of Altland and Zirnbauer (1997)

Diagonal map:



TODAY (modest goal): main ideas behind complex Bott periodicity.

CBP follows from the combination of two sub-results:

1. $\pi_d(U) = \pi_{d+1}(U/U \times U)$,
2. $\pi_{d+1}(U/U \times U) = \pi_{d+2}(U)$.

Both can be derived by using Morse theory, following Bott (1958). Roughly speaking, one shows that the unitary group furnishes a good approximation ^(→ homotopy type) to the loop space of a Grassmannian: $\Omega(U/U \times U) \approx U$, and vice versa.

Comment. The first identity, say in the form of $\pi_d(U_n) = \pi_{d+1}(U_n/U_{n-n} \times U_n)$, can also be obtained from the long exact homotopy sequence of a fiber bundle,

$$U_n \hookrightarrow U_n/U_{n-n} \times \bullet \rightarrow U_n/U_{n-n} \times U_n,$$

whose total space (Stiefel manifold) is weakly contractible.

HERE: "tools" ($\sim$ focus on aspects that are computationally powerful).

PART 1: A basic tale of CW complexes

Cell decomposition.

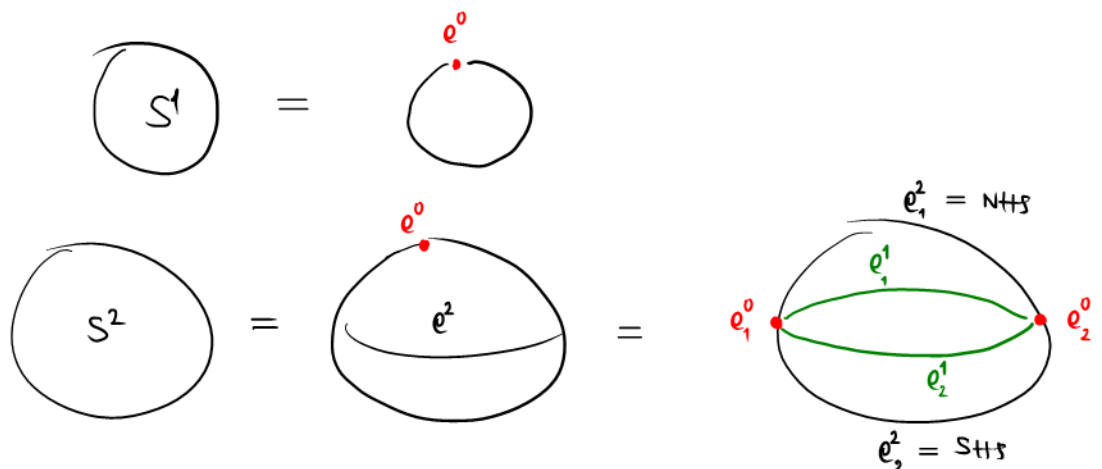
(closed) n -disk $D^n := \{x \in \mathbb{R}^n \mid |x| \leq 1\}$; $\partial D^n = S^{n-1}$ ($(n-1)$ -sphere)

n -cell is a space homeomorphic to $\text{int}(D^n)$ (open n -disk)

Defn. A **cell decomposition** of X (top. space) is a collection $\{e_\alpha\}_{\alpha \in I}$ of cells $e_\alpha \subset X$ such that $X = \bigsqcup_{\alpha \in I} e_\alpha$ (disjoint union).

n -skeleton $X^n := \bigsqcup_{\dim(e_\alpha) \leq n} e_\alpha$

Examples.



$$\mathbb{RP}^2 \cong S^2 / \mathbb{Z}_2 = e_1^0 \sqcup e_1^1 \sqcup e_2^2$$

Definition. Hausdorff space X with finite cell decomposition $X = \bigsqcup_{\alpha \in I} e_\alpha$

is called a **CW complex** iff for each n -cell e there exists $\varphi_e: D^n \rightarrow X$

such that 1. the restriction $\varphi_e: \text{int}(D^n) \rightarrow e$ is a homeomorphism,

2. the image of $\varphi_e: \partial D^n \rightarrow X$ is contained in X^{n-1} .

If there are infinitely many cells, one poses the additional requirements of

- "closure finiteness": for each e the closure $\bar{e}$ intersects only finitely many other cells.
- "weak topology": a subset $A \subset X$ is closed iff $A \cap \bar{e}$ is closed for every cell e .

Note: $\bar{e} = \varphi_e(D^n)$ (needs Hausdorff).

X^n ($n = 0, 1, 2, \dots$) is always closed (in fact, X^n is a CW complex).

Exercise: find a CW complex for S^3 , T^2 , $\mathbb{RP}^n$, Klein bottle.

PART 1 (continued)

initially just a set

Topological space X . Surjective map $p: X \rightarrow Y$ induces

"identification" topology on Y : $C \subset Y$ open $\iff p^{-1}(C) \subset X$ open

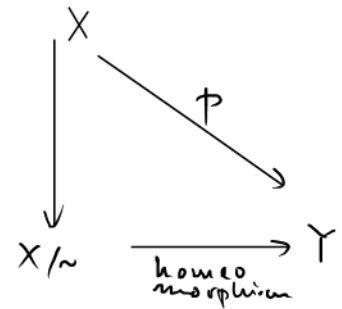
Example: principal bundle $p: X \rightarrow X/G$ (a.k.a. quotient topology)

e.g., $X = \mathbb{R}^2 \setminus \{0\}$, $G = SO_2$, $X/G = \mathbb{R}_+$.

Note: identification map $p: X \rightarrow Y$

is the same as an equivalence relation

$$x_1 \sim x_2 \iff p(x_1) = p(x_2)$$



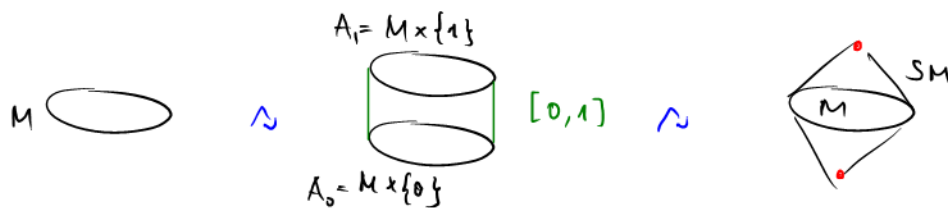
Collapsing a subspace. subspace $A \subset X$. Then

$X/A := X/\sim$ where equiv. classes are A and all $\{x\}$ for $x \in X \setminus A$.

Example. Suspension $SM := M \times [0,1] / (M \times \{0\}) \cup (M \cup \{1\})$

More precisely: $X = M \times [0,1]$, $A_0 = M \times \{0\}$, $A_1 = M \times \{1\}$

equivalence classes: A_0, A_1 , all points outside of $A_0 \cup A_1$



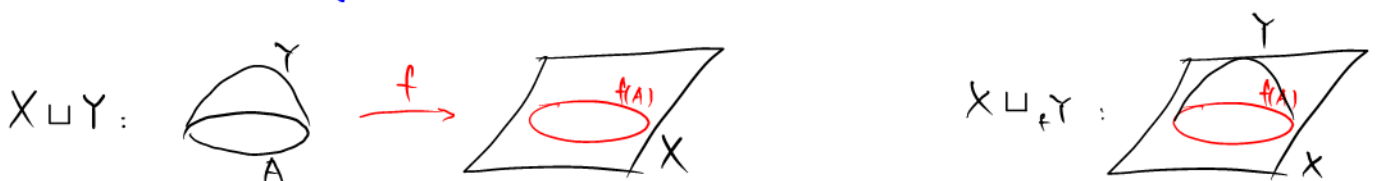
Attaching one space to another by map.

Spaces X, Y . Closed subspace $A \subset Y$. Map $f: A \rightarrow X$.

Disjoint union $X \sqcup Y$. Define $X \sqcup_f Y := (X \sqcup Y) / \sim$

equiv. classes $\{y\}$ for $y \in Y \setminus A$, $\{x\}$ for $x \in X \setminus f(A)$, $\{y, f(y)\}$ for $y \in A$.

Example: attaching a 2-cell. $Y = D^2$, $A = \partial D^2 = S^1$.



Note. For a CW complex X with n -skeleton X^n ($n = 0, 1, 2, \dots$)
 X^n is obtained from X^{n-1} by attaching the n -cells.

Exercise.

- 1) $\pi_d(S^n) = 0$ for $1 \leq d < n$. Hint: $S^n = \{p\} \cup_f D^n$
 with $f(\partial D^n) = p$.
- 2) $\pi_d(X) = \pi_d(X \cup_f D^n)$ ($f: \partial D^n \rightarrow X$) for $d < n-1$
- 3) Cellular homology ... (cf. tomorrow)

PART 2. Morse Theory – fundamental theorems

Manifold M , function $f: M \rightarrow \mathbb{R}$, $M^a := \{x \in M \mid f(x) \leq a\}$.

Assume f smooth and proper.

Thm 1. If f has **no** critical values in $[a, b]$, then M^a is homotopy equivalent to M^b (in fact, a deformation retract of M^b).

Thm 2. Let $x \in M$ be non-degenerate critical point of f of index n .

If x is the only critical point in $f^{-1}[f(x) - \varepsilon, f(x) + \varepsilon]$, then $M^{f(x)+\varepsilon}$ is homotopy equivalent to $M^{f(x)-\varepsilon}$ with an n -cell attached.

Proof uses **Morse lemma**: in some neighborhood U of a non-degenerate critical point x with index n there exists a chart $\{x_1, \dots, x_m\}$ such that

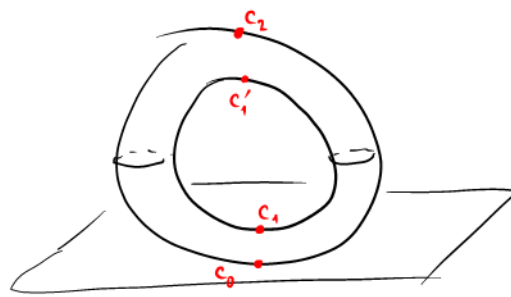
$$f = f(x) - x_1^2 - \dots - x_n^2 + x_{n+1}^2 + \dots + x_m^2 \quad \text{holds on } U.$$

Fact. For any M , there exist functions with no degenerate critical pts (Morse fcts) and with no two critical values the same.

~ Corollary. every manifold is a CW complex with one n -cell for each critical point (of a Morse fctn) of index n .

Example: 2-torus

$f = \text{height fctn}$



Critical values

$$c_0 < c_1 < c_1' < c_2$$

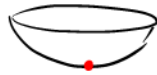
M_a

$$a < 0$$



empty

$$c_0 < a < c_1$$



disk

$$c_1 < a < c_1'$$



cylinder

$$c_1' < a < c_2$$



genus-1 surface
with boundary

$$c_2 < a$$

2-torus

homotopy
equivalent
to:

nothing

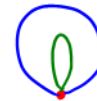
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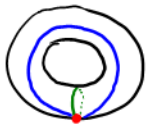
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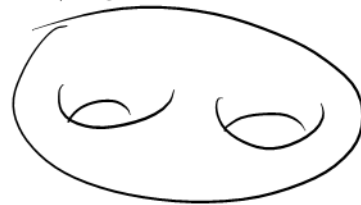


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Exercises. CW complex for Riemann surface of genus 2

... for the suspension of M
(given that of M)



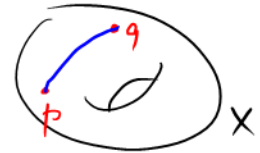
PART 3. Bott Periodicity Theorem (sketch of complex case).

(said to be one of the most surprising theorems in topology)

Setting. Riemannian manifold X (compact, connected).

$v = (p, q; h)$ homotopy class h of curves in X joining p to q

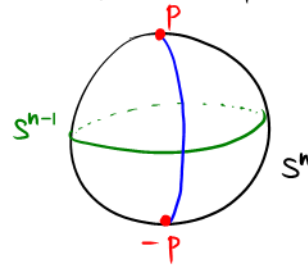
$X^v =$ space of minimal geodesics $\quad \text{---} \parallel \text{---}$



Thm (Bott). If X is a symmetric space, then so is X^v .

Remark. The most interesting situation occurs when q is the antipode of p .

Example 1. $X = S^n$, $v = (p, -p; \cdot) : X^v = S^{n-1}$



Example 2.

Let $C_0(n) := \{ J \in \text{End}(\mathbb{C}^{2n}) \mid J^\dagger = -J, J^2 = -1 \}$.

Subspace $X = \{ J \in C_0(n) \mid \dim E_{+i}(J) = n \} \simeq G_n(\mathbb{C}^{2n}) \simeq U_{2n}/U_n \times U_n$.

Let $v = (J_1, -J_1; \cdot)$ with, say, $J_1 = i\sigma_3 \otimes 1_n$.

$$\uparrow J = g(i\sigma_3 \otimes 1_n)g^\dagger$$

CLAIM: $X^v \simeq U_n$. Proof of $U_n \hookrightarrow X^v$:

Let $C_1(n) = \{ J \in C_0(n) \mid J J_1 + J_1 J = 0 \}$.

For $J \in C_1(n)$ consider $\gamma(t) = e^{(\pi t/2) J_1 J} J_1 e^{-(\pi t/2) J_1 J} = e^{\pi t J_1 J} J_1$.

Properties of γ : (i) $\gamma(0) = J_1$ and $\gamma(1) = -J_1$ | from $(J_1 J)^2 = -1$

(ii) $\gamma(\frac{1}{2}) = -J$ | from $e^{(\pi/2) J_1 J} = J_1 J$

(iii) γ is minimal geodesic in X .

$J \in C_1(n)$ is a pair of linear transformations $E_{+i}(J_1) \xrightleftharpoons[v]{u} E_{-i}(J_1)$.

$J^2 = -1 \sim v = -u^{-1}$; $J^\dagger = -J \sim u^{-1} = u^\dagger$.

Hence $C_1(n) \simeq U_n$ (as sets).

Exercise. For $X = C_1(n)$ and $v = (J_2, -J_2; \text{any})$ with, say, $J_2 = i\sigma_1 \otimes 1_n$, find X^v .

Hint. Extend the construction of before.

(sketch cont'd)

$\Omega_v X :=$ space of all (piecewise differentiable) curves $\text{---} \parallel \text{---}$.

Take the length fctn $l(\cdot)$ as a Morse function (actually, Morse-Bott function) for $\Omega_v X$.

By generalizing the procedure for constructing a CW complex from a Morse function,

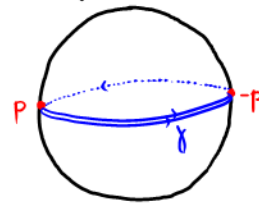
Bott shows that X^ν is a 'good' approximation to $\Omega_v X$:

as the index-0 critical manifold of l , the space X^ν of minimal geodesics captures the homotopy type of $\Omega_v X$ up to corrections (which take the form of cells attached to X^ν) due to positive-index critical manifolds (constituted by non-minimal geodesics):

$$\Omega_v X \simeq X^\nu \sqcup e^\mu \sqcup \dots$$

where $\dim e^\mu \geq |\nu| :=$ smallest non-zero index of a geodesic; the index of a geodesic turns out to be the (properly counted) number of p-conjugate points in its interior.

Example. $X = S^2$, $\nu = (p, -p)$, $X^\nu = S^1$



index(γ) = 2

Thm (1, Bott): $\pi_d(X^\nu) = \pi_d(\Omega_v X)$ for $0 < d < |\nu| - 1$.

Remark. $\pi_d(\Omega_v X) = \pi_{d+1}(X)$.

Application (condensed matter physics).

Recall: a topological insulator in symmetry class A (U(1) symmetry $\simeq$ charge is conserved) is modeled by $\left\{ \begin{array}{l} \text{a rank-}n \text{ complex vector (sub)bundle } V \xrightarrow{p} M, \quad p^{-1}(k) \equiv V_k \text{ space of valence states at momentum } k \\ \simeq \\ \text{a classifying map } M \rightarrow Gr_n(\mathbb{C}^N), \quad k \mapsto V_k. \end{array} \right.$

In d dimensions and for $M = S^d$, such objects are classified by $\pi_d(Gr_n(\mathbb{C}^N))$.

Pertinent consequences of Bott's results.

- $\pi_d(U_n) = \pi_{d+1}(U_{2n}/U_n \times U_n)$ for $d < [\text{insert}]$
- $\pi_d(U_{2n}/U_n \times U_n) = \pi_{d+1}(U_{2n})$ for $d < 2n+1$,
more generally, $\pi_d(U_N/U_n \times U_{N-n}) = \pi_{d+1}(U_N)$ for $[\text{insert}]$
- Periodicity Theorem: $\pi_d(U/U \times U) = \pi_{d+2}(U/U \times U)$; also $\pi_d(U) = \pi_{d+2}(U)$.
- $\pi_{\text{even}}(U/U \times U) = \mathbb{Z}$ (QHE) $\quad \pi_{\text{even}}(U) = 0$ (symmetry class AIII:
 $\pi_{\text{odd}}(U/U \times U) = 0$ (no QHE) $\quad \pi_{\text{odd}}(U) = \mathbb{Z}$ Su-Schrieffer-Heeger model)

Appendix

Literature.

R. Bott, The stable homotopy of the classical groups, Ann. Math. 70 (1959) 313-337.

J. Milnor, Morse theory (Princeton University Press, 1963).

R. Bott, The periodicity theorem for the classical groups and some applications

Adv. Math. 4 (1970) 353-411.

Symmetric space. On a Riemannian manifold M one has for every point $p \in M$ an operation σ_p of geodesic inversion (the Riemannian analog of the Euclidean-geometry operation of reflection at a point) in some neighborhood of p . M is called a **locally symmetric space** if for all p the map σ_p is an isometry (on its domain of definition). M is called a **globally symmetric space** if the locally defined isometry σ_p extends for all p to an isometry $\sigma_p: M \rightarrow M$.

The Bott Periodicity Theorem makes a statement about the homotopy groups of globally symmetric spaces of "classical type" — these are the spaces mentioned in PART 0.