Critical spin chains from modular invariance

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Mathematics meet Physics

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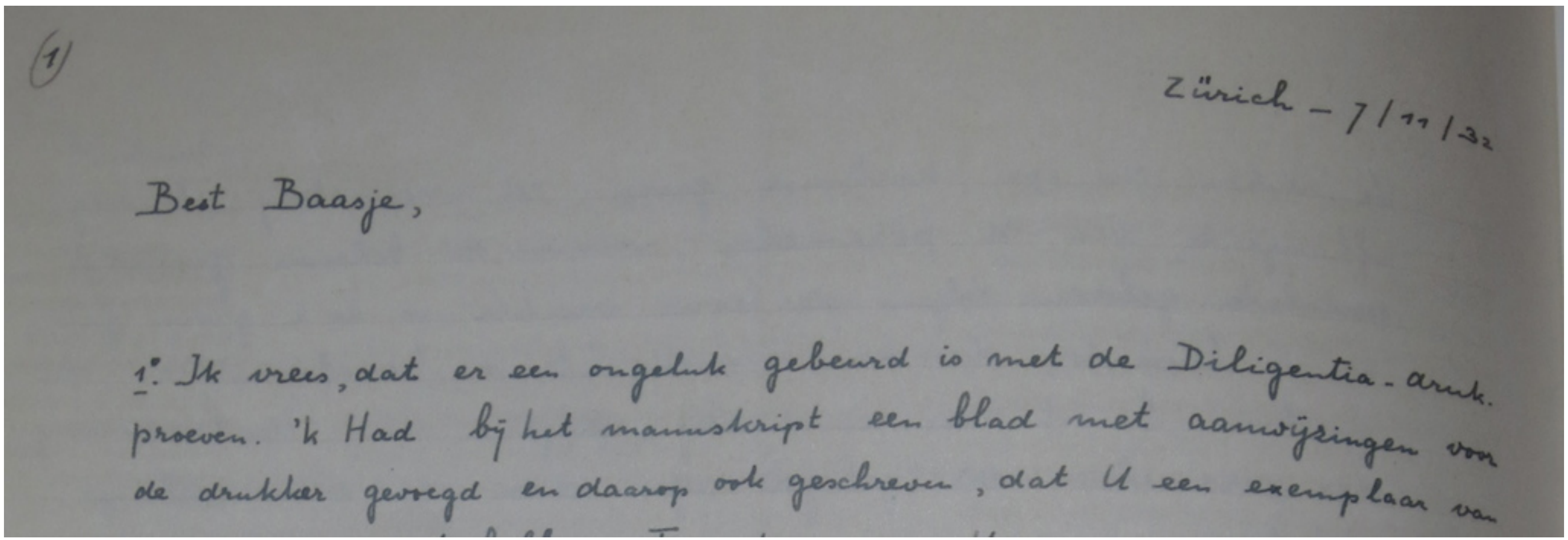

(1)

Zürich – 7/11/32

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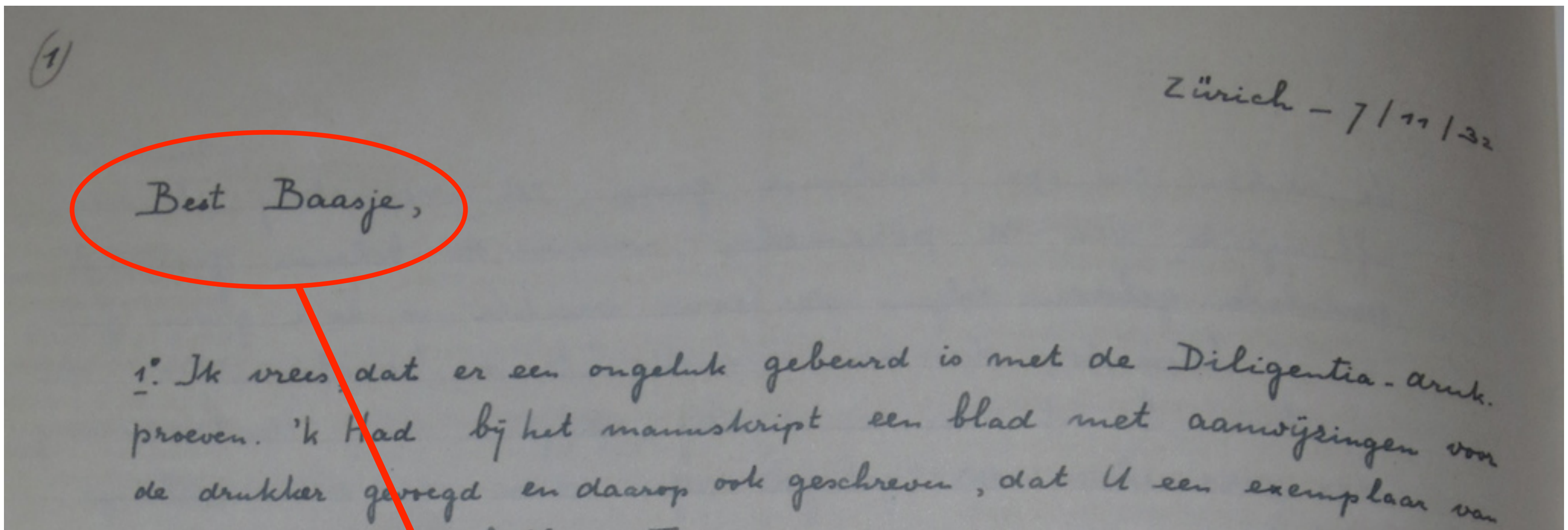

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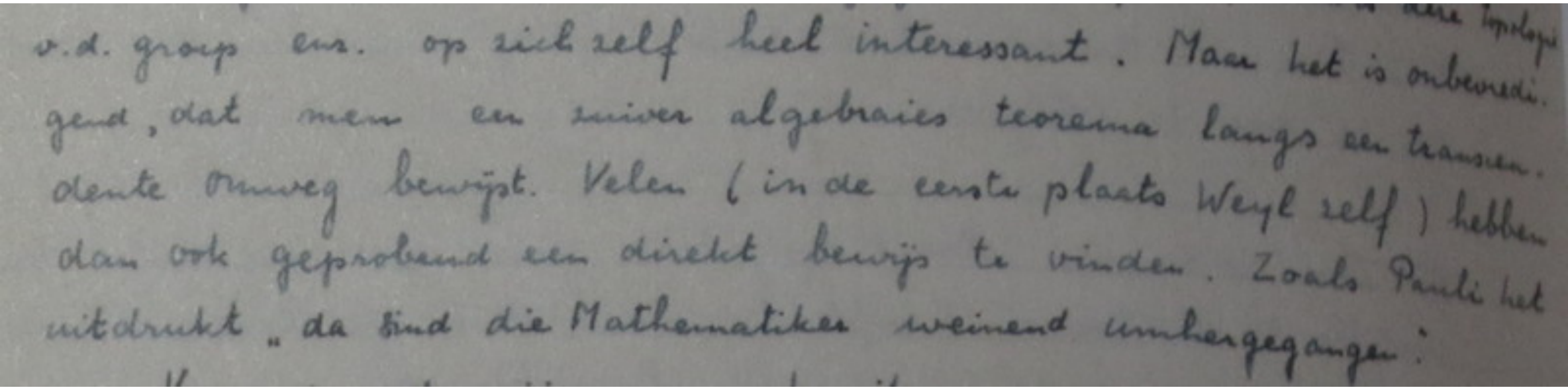

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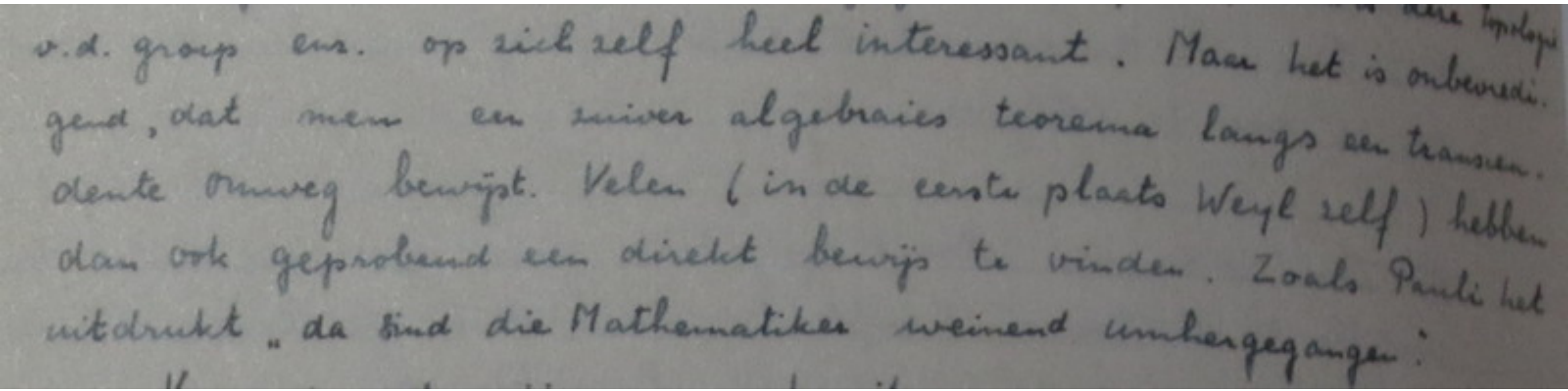
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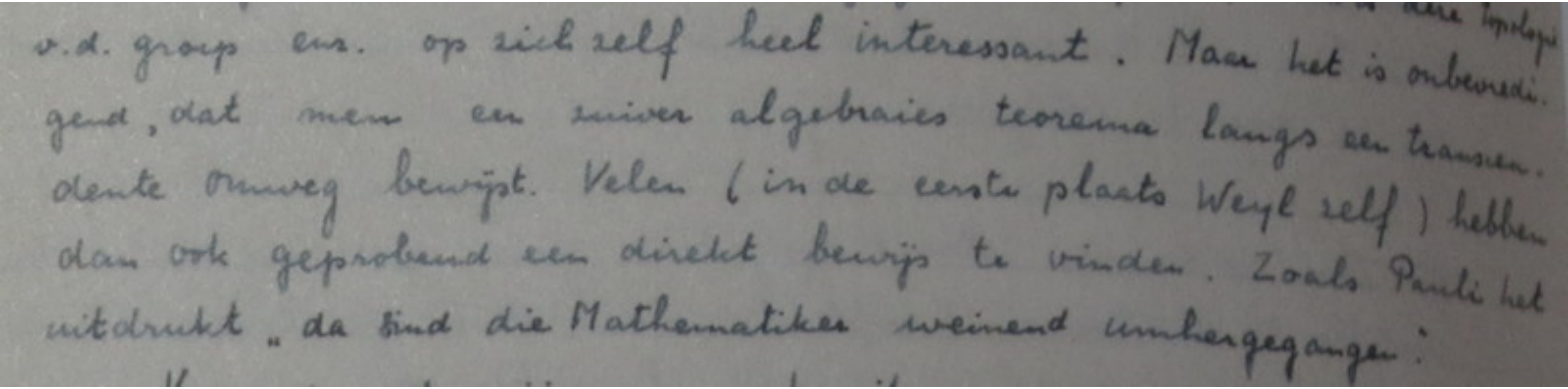

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Casimir & B.L. v.d. Waerden give an algebraic proof, using a Casimir operator

Outline

⋆ Low-energy description of 2-d topological phases: anyon models

⋆ Topological phase transitions in 2-d:
- condensation
- modular invariance

⋆ Analogue on the level of spin chains: Ising examples

⋆ Beyond condensation: parafermions

A little about anyon models

Moore, Seiberg,....

An anyon model consist of a set of particles $\mathcal{C} = \{\mathbf{1}, a, b, c, \ldots, n\}$

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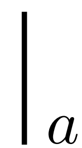

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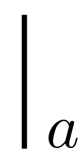

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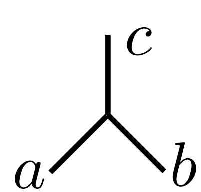

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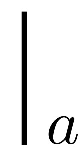

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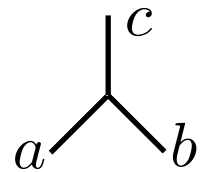
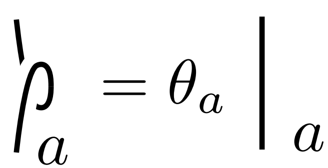


Twisting of a particle



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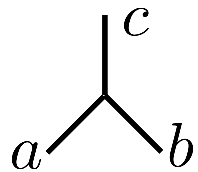

A boson has
$\theta_b = 1 \quad (h_b \in \mathbb{Z})$

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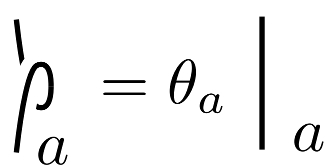

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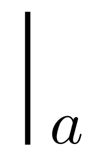
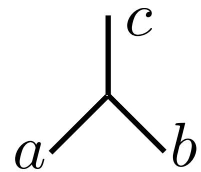
Anyons are represented by their 'worldlines' 

Fusion is represented as 

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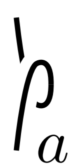
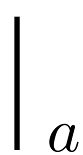
Twisting of a particle 

$= \theta_a$ 

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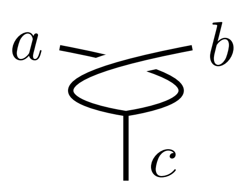
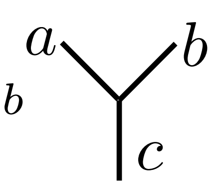
Braiding of particles 

$= R_c^{a,b}$ 

$\qquad R_c^{a,b} = \pm e^{\pi i (h_c - h_a - h_b)}$

Condensation in anyon models

Bais, Slingerland, 2009

Condensation amounts to identifying a boson with the vacuum

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$$a \times a = \mathbf{1} + b + \cdots \rightsquigarrow a \times a = \mathbf{1} + \mathbf{1} + \cdots$$

vacuum twice

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In CFT language, one condenses a boson by adding it to the chiral algebra, and in the end, one has constructed a new modular invariant partition function

Modular invariant partition functions

A conformal field theory splits in two pieces, a chiral and anti-chiral part.

To each chiral sector (primary field), one associates a ‘character’, describing the number of states in this sector

$$\chi_\phi(q) = q^{h_\phi - c/24}\left(a_0 q^0 + a_1 q^1 + a_2 q^2 + \cdots\right) \qquad q = e^{2\pi i \tau}$$

The constants a_j are non-negative integers, and τ is the modular parameter, describing the shape of the torus (next slide).

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The full partition function is obtained by combining the chiral halves, and summing over the primary fields:

$$Z_{\mathrm{cft}} = \sum_j |\chi_{\phi_j}|^2$$

Modular invariant partition functions

One should be able to put the cft on the torus: partition function should be invariant under re-parametrization of the torus!

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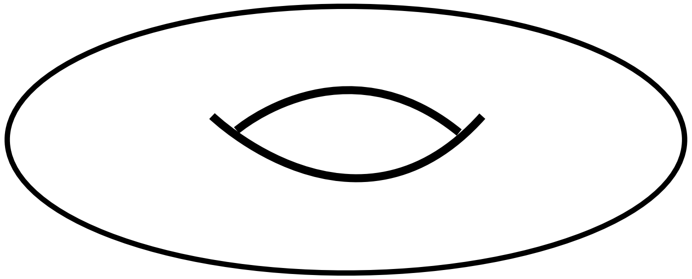

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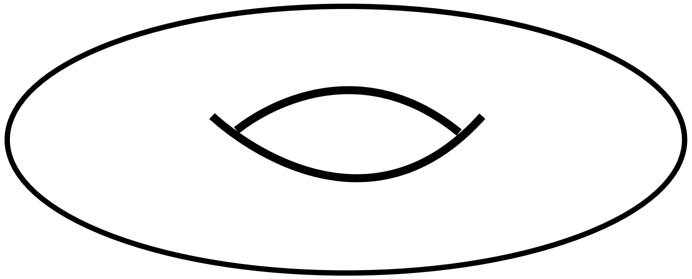

Shape of the torus is encoded by the modular parameter τ.
The transformations S and T do not change the shape of the torus.

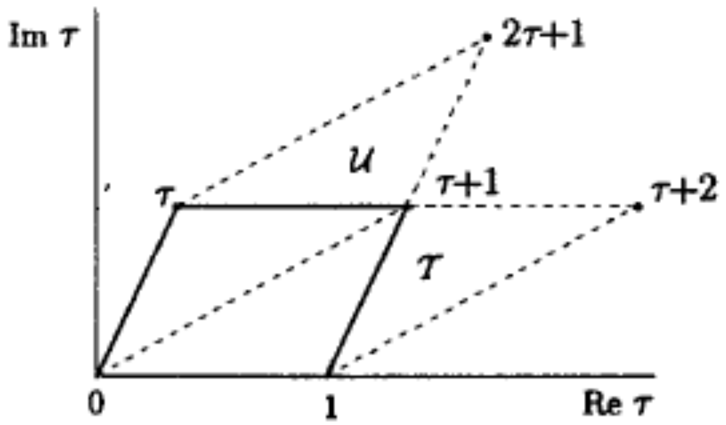

image: BYB

$$T : \tau \to \tau + 1$$
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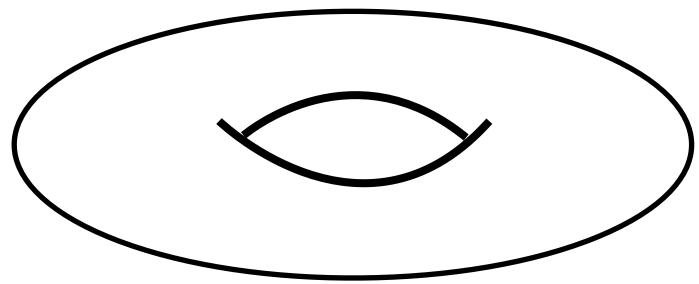

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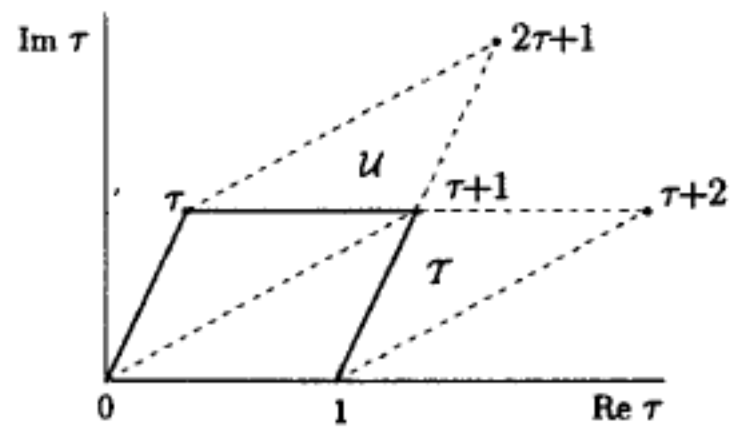

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The most general way to combine the chiral halves:

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Finding all invariants is, in general, a hard task, but progress has been made (minimal models, $su(2)_k$, $su(3)_k$, parafermions...)

Cappelli et al., Gepner et al., ...

Example: Ising^2 theory

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Apart from the diagonal invariant, one also finds a block diagonal invariant:

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One sees identification of sectors: $(\mathbf{1},\mathbf{1}) \sim (\psi,\psi) \qquad (\mathbf{1},\psi) \sim (\psi,\mathbf{1})$

Confined sectors: $(\mathbf{1},\sigma)\,, (\psi,\sigma)\,, (\sigma,\mathbf{1})\,, (\sigma,\psi)$

Split sector: (σ,σ)

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One sees identification of sectors: $(\mathbf{1},\mathbf{1}) \sim (\psi,\psi) \qquad (\mathbf{1},\psi) \sim (\psi,\mathbf{1})$

Confined sectors: $(\mathbf{1},\sigma)\,,(\psi,\sigma)\,,(\sigma,\mathbf{1})\,,(\sigma,\psi)$

Split sector: (σ,σ)

The resulting invariant describes the u(1)$_4$ cft, and the construction amounts to the orbifold construction.

Permutation invariants

In some cases, one can permute some of the labels of the primary fields (anyons), without changing the fusion rules. Let π be such a permutation:

$$n_{\pi(a),\pi(b),\pi(c)} = n_{a,b,c} \qquad \mathcal{S}_{\pi(a),\pi(b)} = \mathcal{S}_{a,b}$$

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Example in the Ising^2 theory, starting from the diagonal invariant:

$$Z_\pi = |\chi_{(\mathbf{1},\mathbf{1})}|^2 + |\chi_{(\sigma,\sigma)}|^2 + |\chi_{(\psi,\psi)}|^2 + \chi_{(\mathbf{1},\sigma)}\chi^*_{(\sigma,\mathbf{1})} + \chi_{(\sigma,\mathbf{1})}\chi^*_{(\mathbf{1},\sigma)} + \chi_{(\mathbf{1},\psi)}\chi^*_{(\psi,\mathbf{1})} + \chi_{(\psi,\mathbf{1})}\chi^*_{(\mathbf{1},\psi)} + \chi_{(\psi,\sigma)}\chi^*_{(\sigma,\psi)} + \chi_{(\sigma,\psi)}\chi^*_{(\psi,\sigma)}$$

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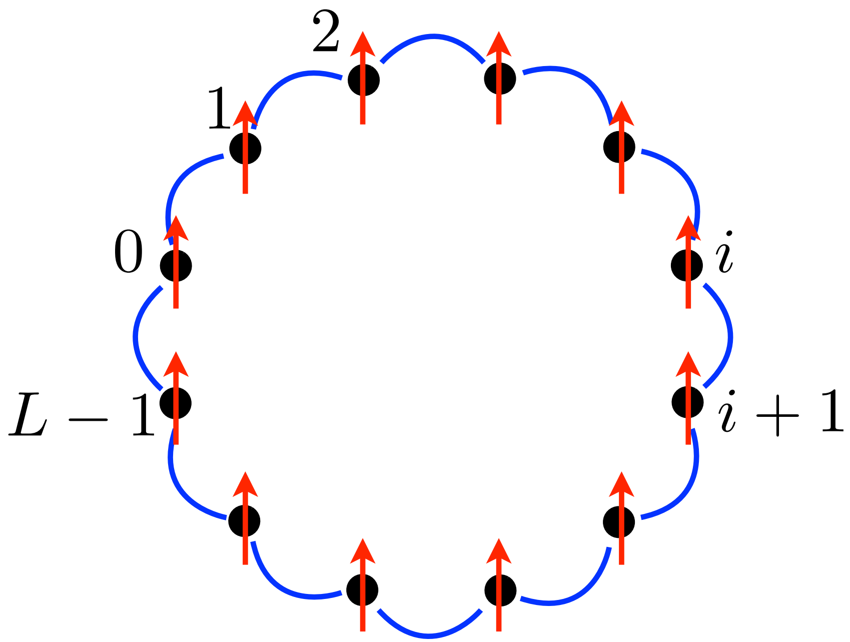
In this case, we have $Z_\pi = Z$, but that's not generic.

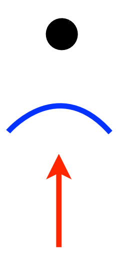
Transverse field Ising model (critical)



$$H_{\mathrm{TFI}} = \sum_{i=0}^{L-1} \sigma_i^z + \sigma_i^x \sigma_{i+1}^x$$

Note: we use periodic boundary conditions: $\sigma_{j+L}^{\alpha} \equiv \sigma_j^{\alpha}$ crucial for our purposes!

Symmetry: $\mathcal{P} = \prod_i \sigma_i^z$



spin-1/2

$\sigma_i^x \sigma_{i+1}^x$

σ_j^z

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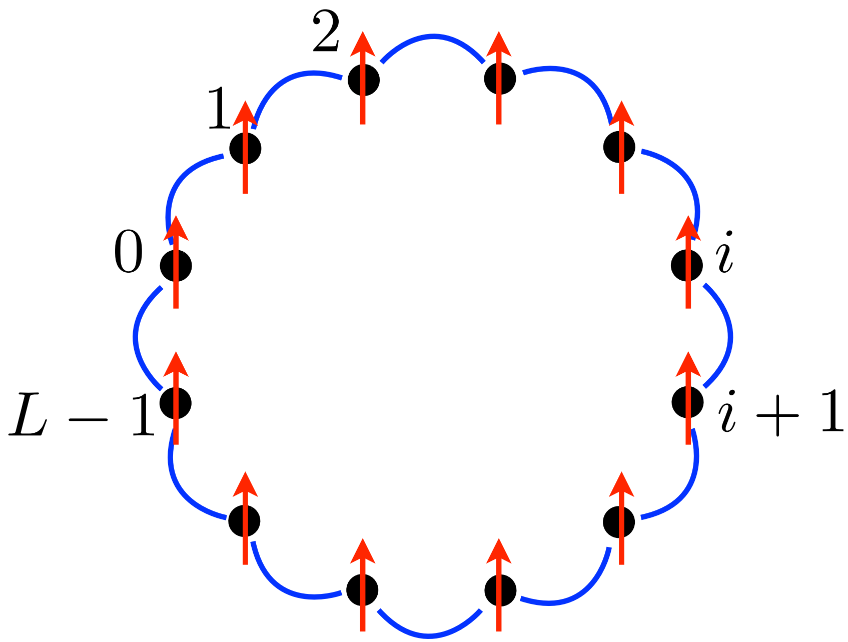

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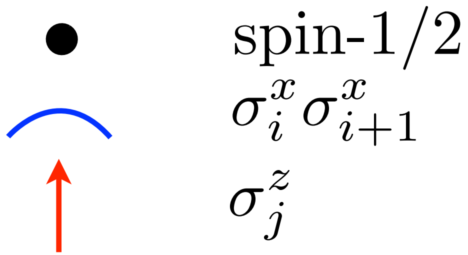


Define fermionic levels (Jordan-Wigner):

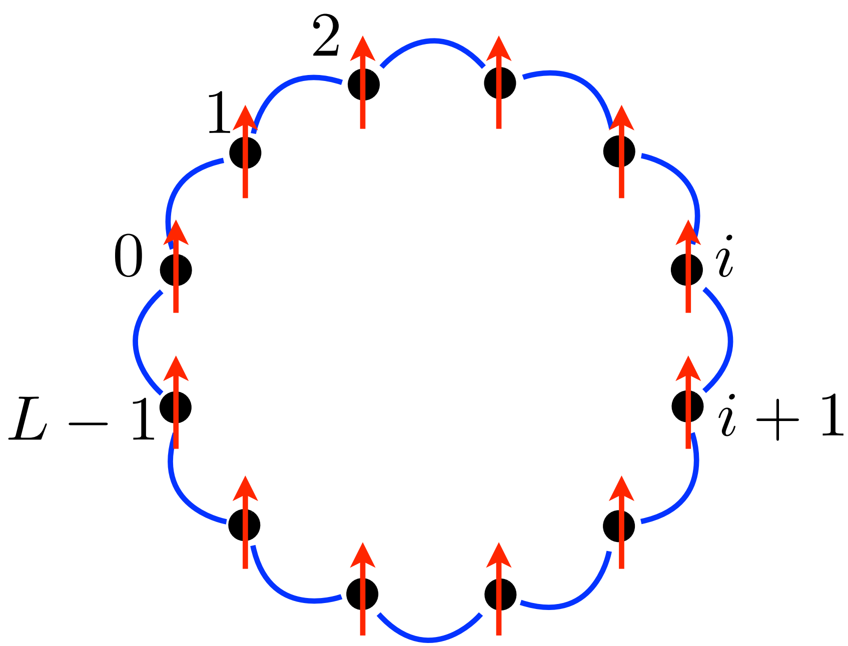
$$|\uparrow\rangle = |0\rangle \qquad |\downarrow\rangle = |1\rangle$$

$$\sigma_i^z = 1 - 2c_i^\dagger c_i \qquad \prod_i \sigma_i^z = (-1)^F$$

$$c_i = \Big(\prod_{j<i} \sigma_i^z\Big)\sigma_i^+ \qquad c_i^\dagger = \Big(\prod_{j<i} \sigma_i^z\Big)\sigma_i^-$$

Lieb, Schultz, Mattis, (1961)
Pfeuty, (1970)

Transverse field Ising model



$$H_{\mathrm{TFI}} = \sum_{i=0}^{L-1} \sigma_i^z + \sigma_i^x \sigma_{i+1}^x$$

$$= \sum_{j=0}^{L-1} (2c_j^\dagger c_j - 1) +$$

$$\sum_{j=0}^{L-2} (c_j - c_j^\dagger)(c_{j+1} + c_{j+1}^\dagger) +$$

$$- (-1)^F (c_{L-1} - c_{L-1}^\dagger)(c_0 + c_0^\dagger)$$

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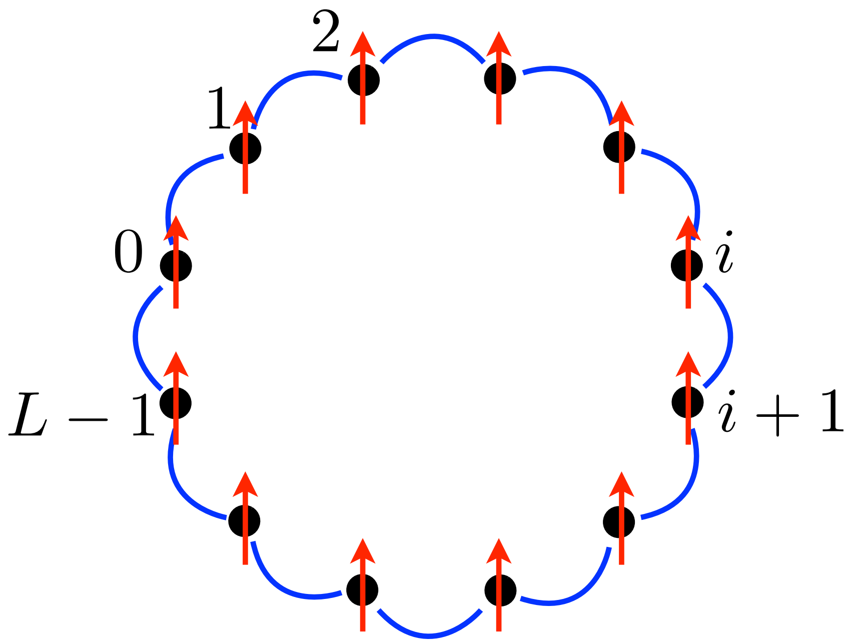

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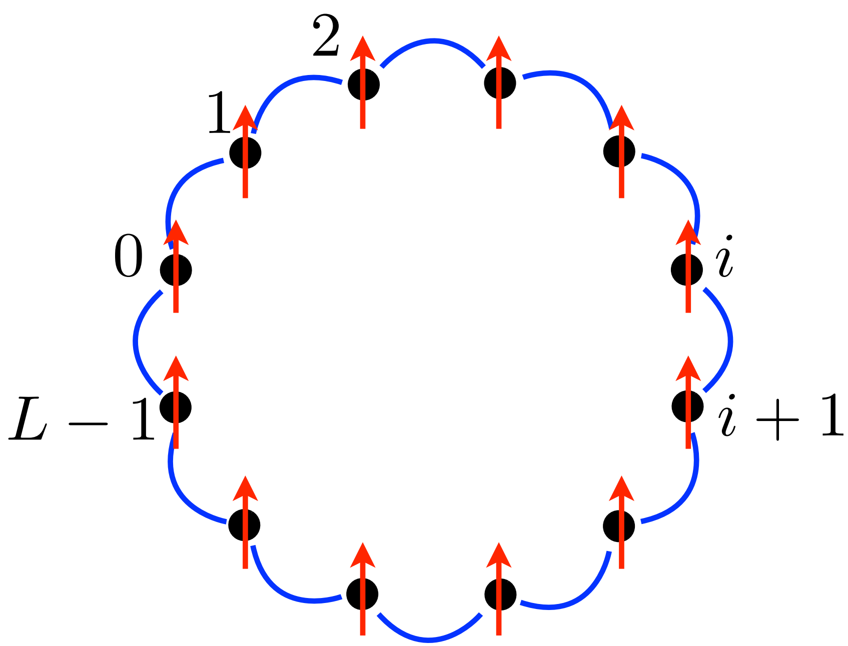
$$- (-1)^F (c_{L-1} - c_{L-1}^\dagger)(c_0 + c_0^\dagger)$$

The parity of the number of fermions is conserved!
The fermion boundary conditions depend on the symmetry sector:

For F even: *anti*-periodic boundary conditions
For F odd: periodic boundary conditions

Transverse field Ising model



Solution: go to k-space, and perform a diagonalize a 2x2 matrix (or, in general 2 L x 2 L if couplings are disordered).

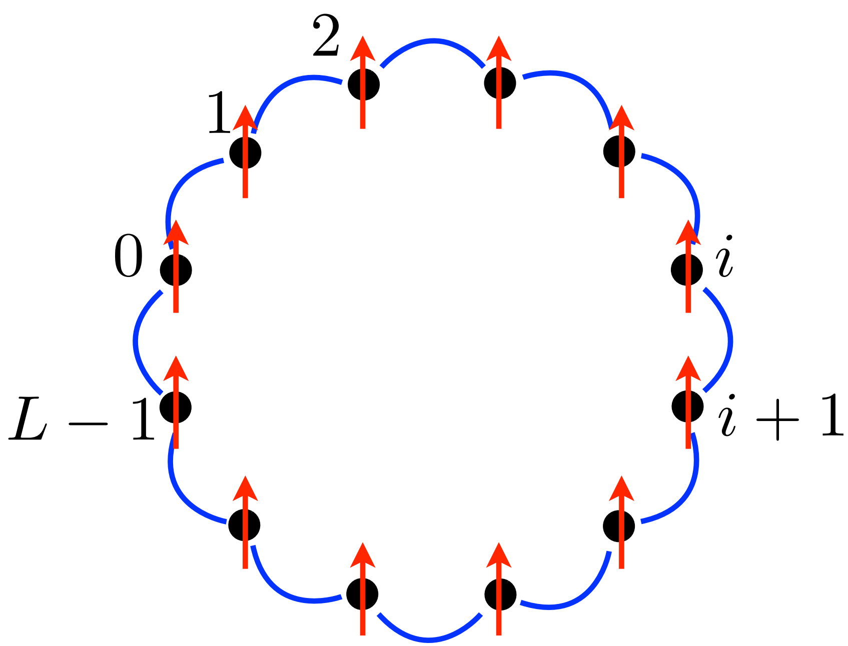
$$H_{\mathrm{TFI}} = \sum_{i=0}^{L-1} \sigma_i^z + \sigma_i^x \sigma_{i+1}^x$$

$$= \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - 1/2\right)$$

$$\epsilon_k = 2\sqrt{2 - 2\cos\left(\frac{2\pi k}{L}\right)}$$

Momenta k: half integer for even F
integer for odd F

Transverse field Ising model



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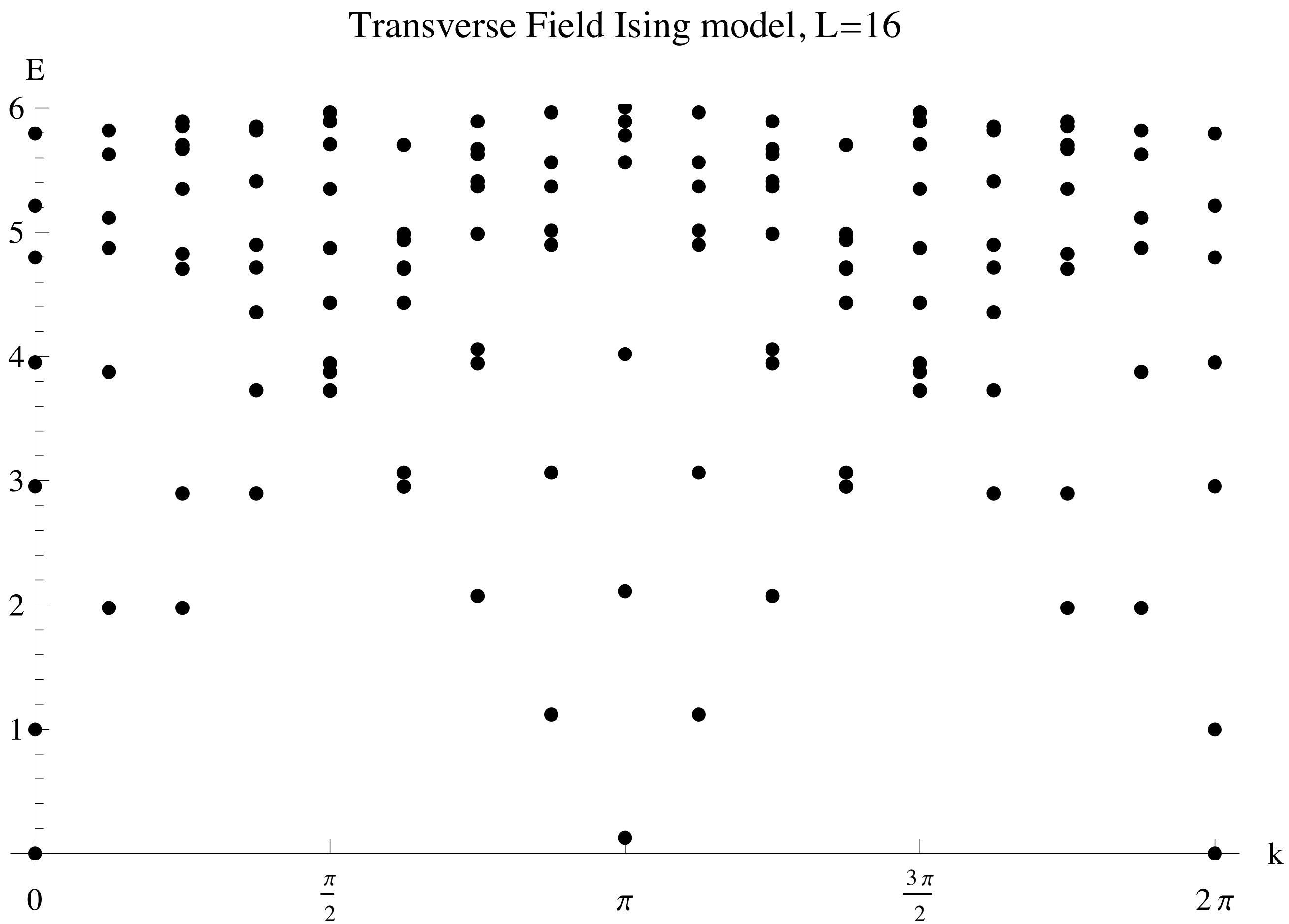
Conformal field theory: spectrum is described in the following way:

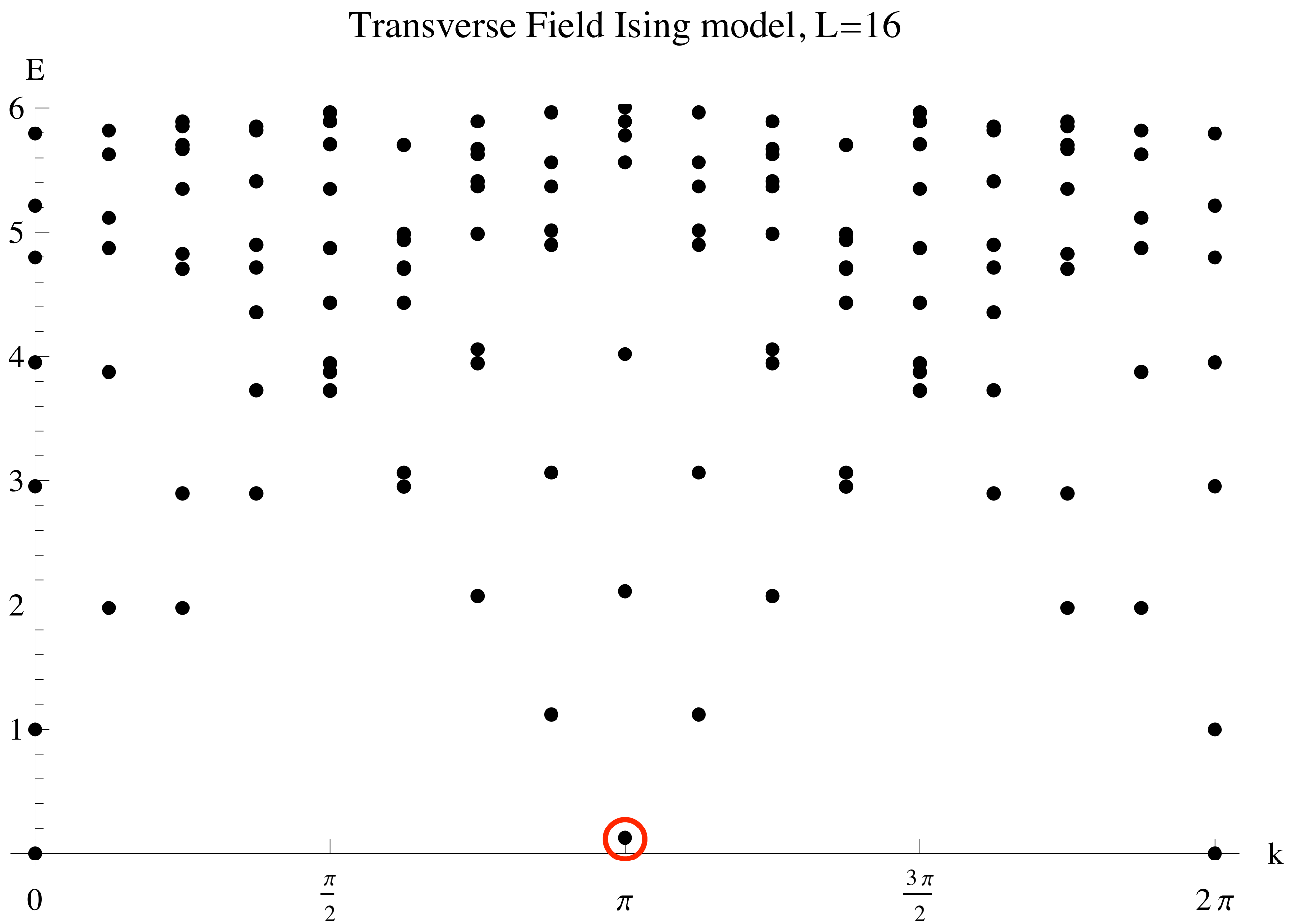
$$\epsilon_i = E_0 L + \frac{2\pi v}{L}\left(-\frac{c}{12} + h_l + h_r + n_l + n_r\right)$$

Transverse field Ising model



Conformal field theory: ‘towers’ of states with: $\Delta E = 1$ $\Delta k = \frac{2\pi}{L}$

Transverse field Ising model



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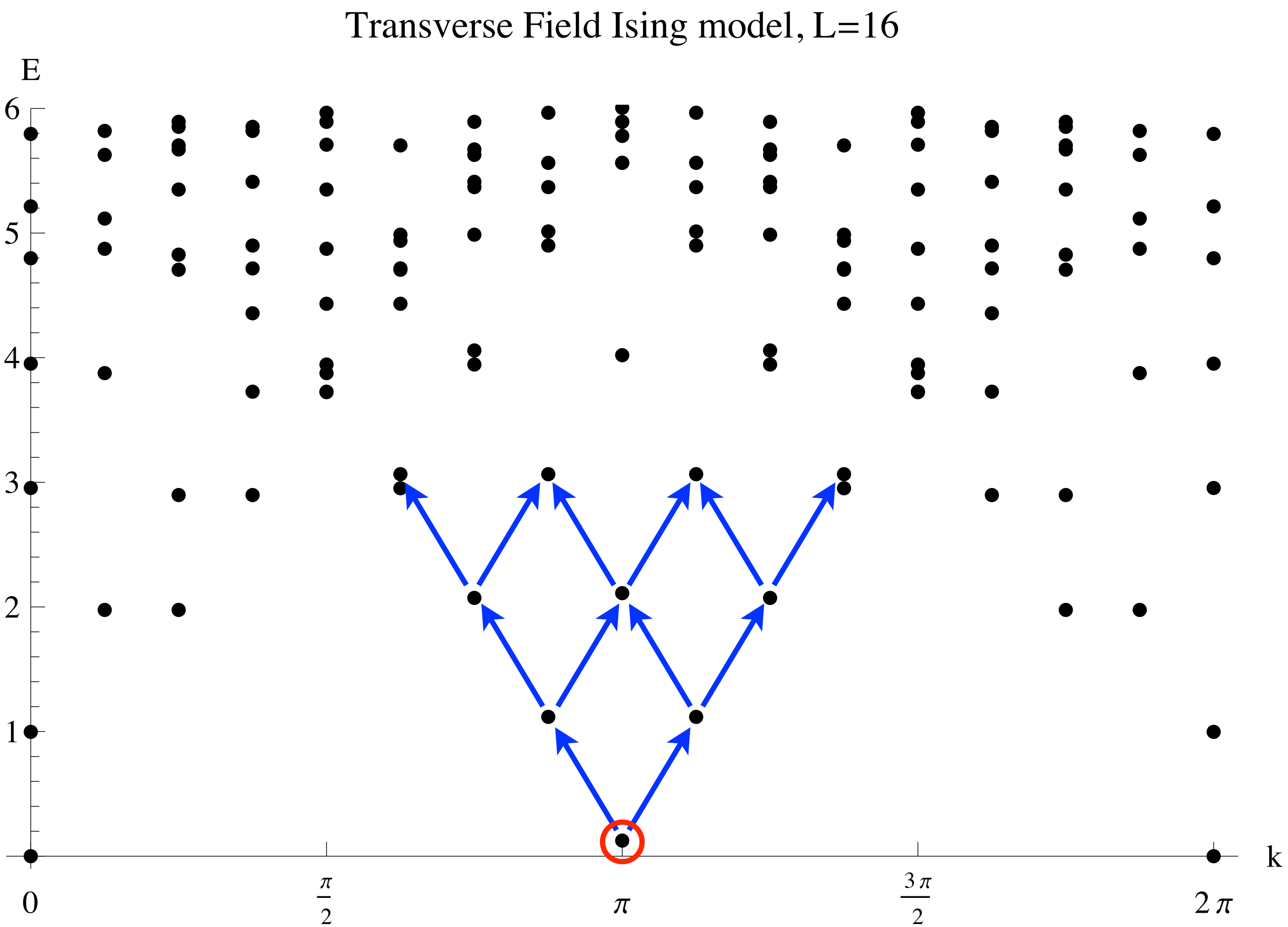

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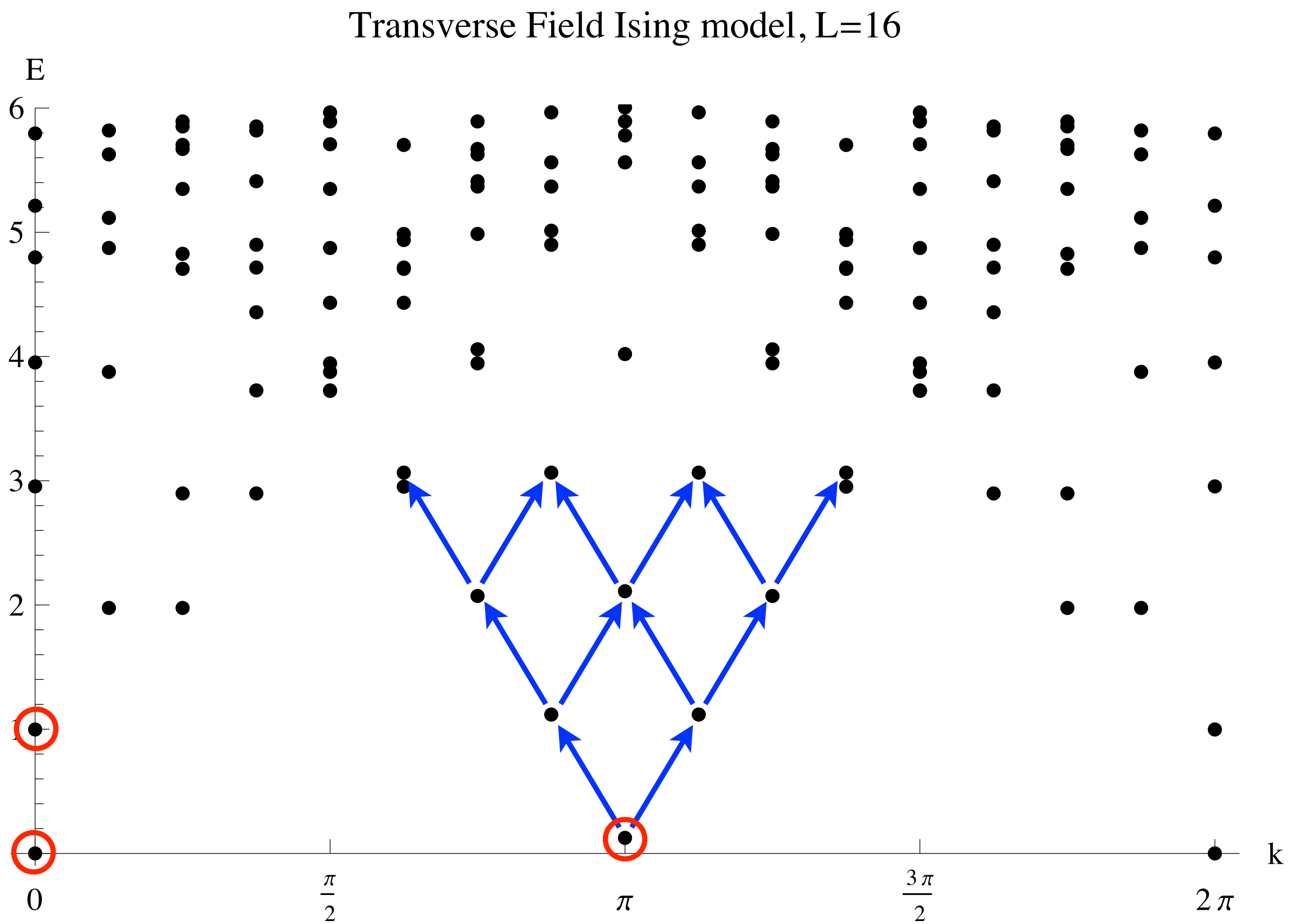

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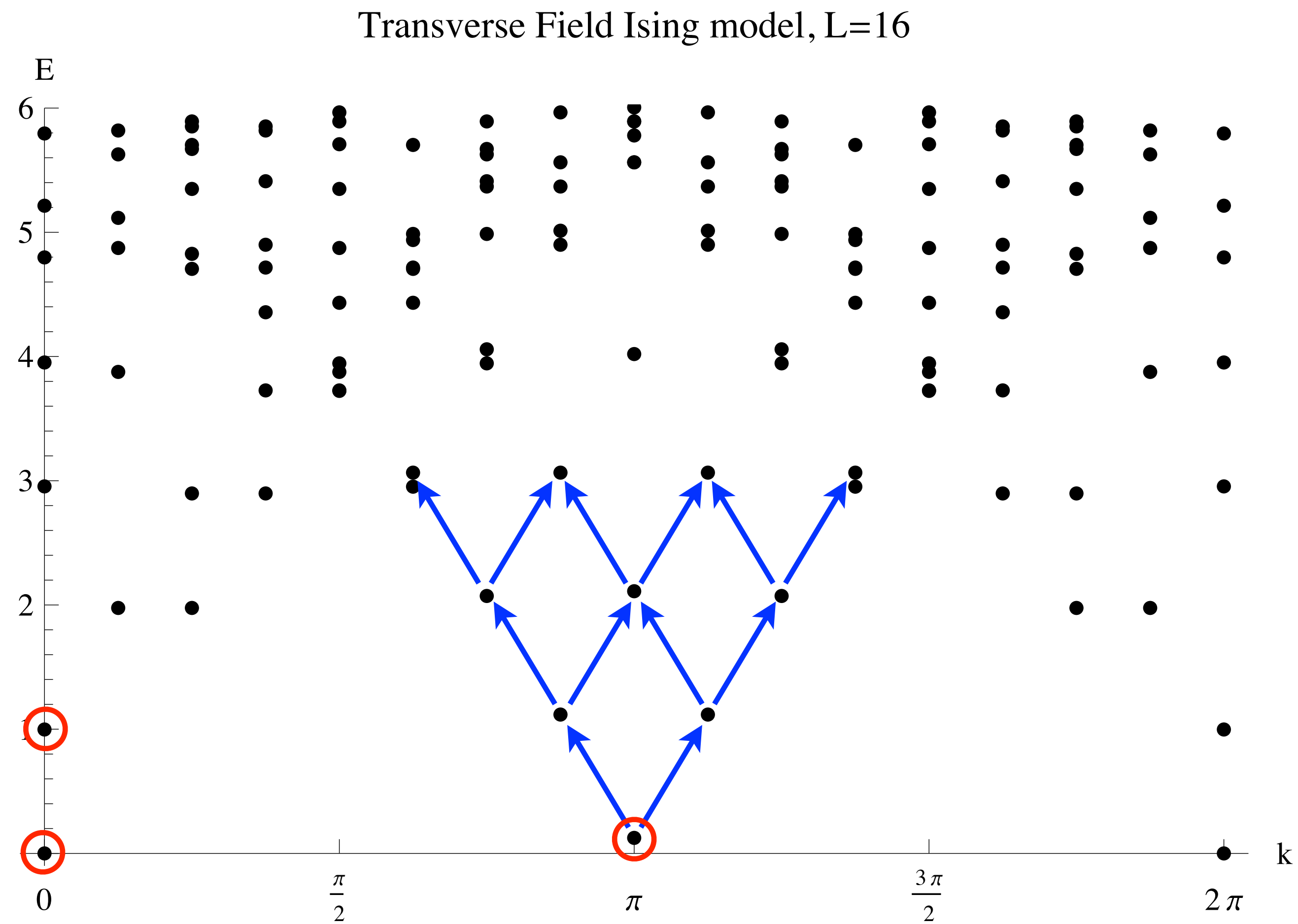

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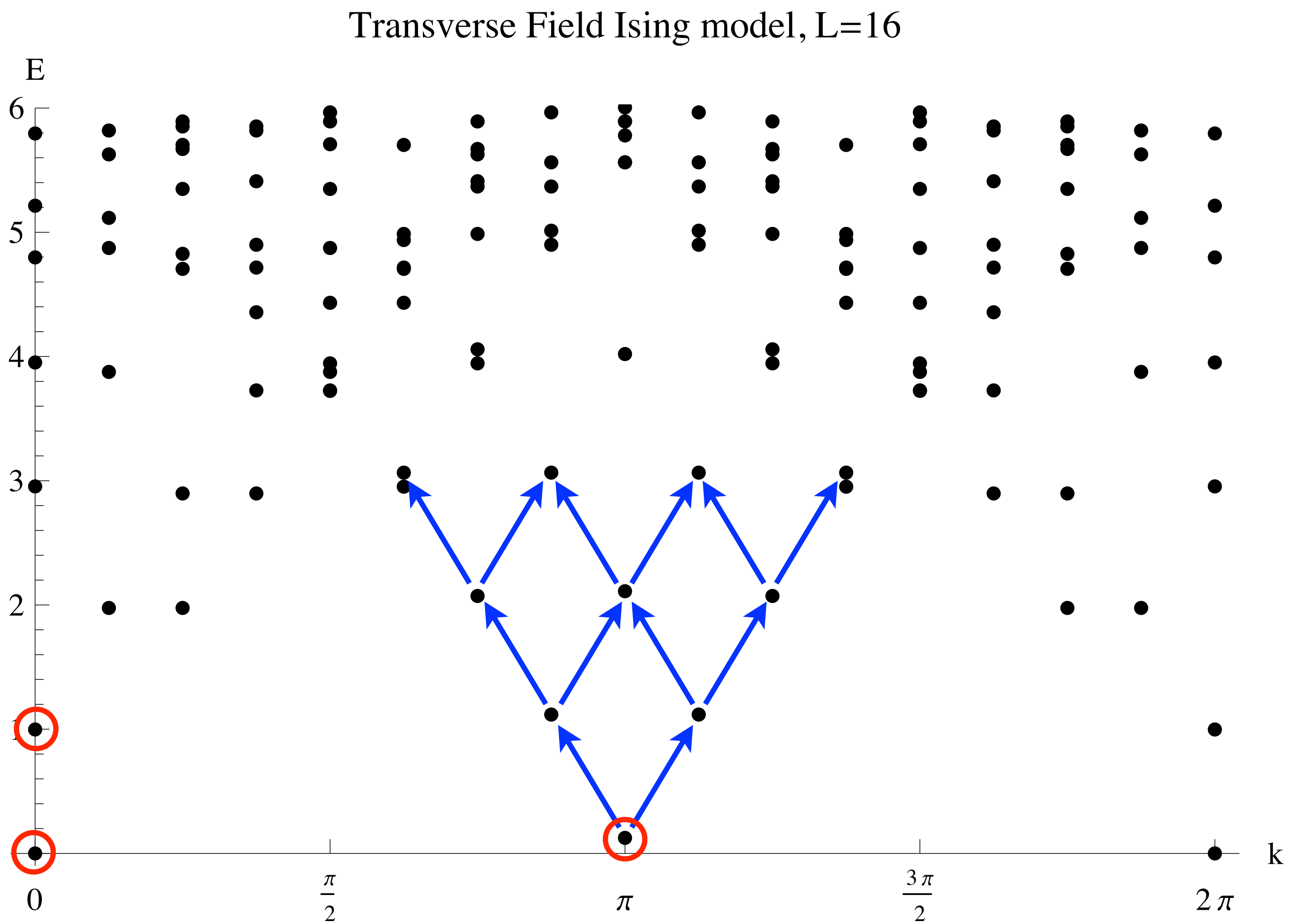

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Ising conformal field theory: $h_1 = 0 \quad h_\sigma = \frac{1}{16} \quad h_\psi = \frac{1}{2}$

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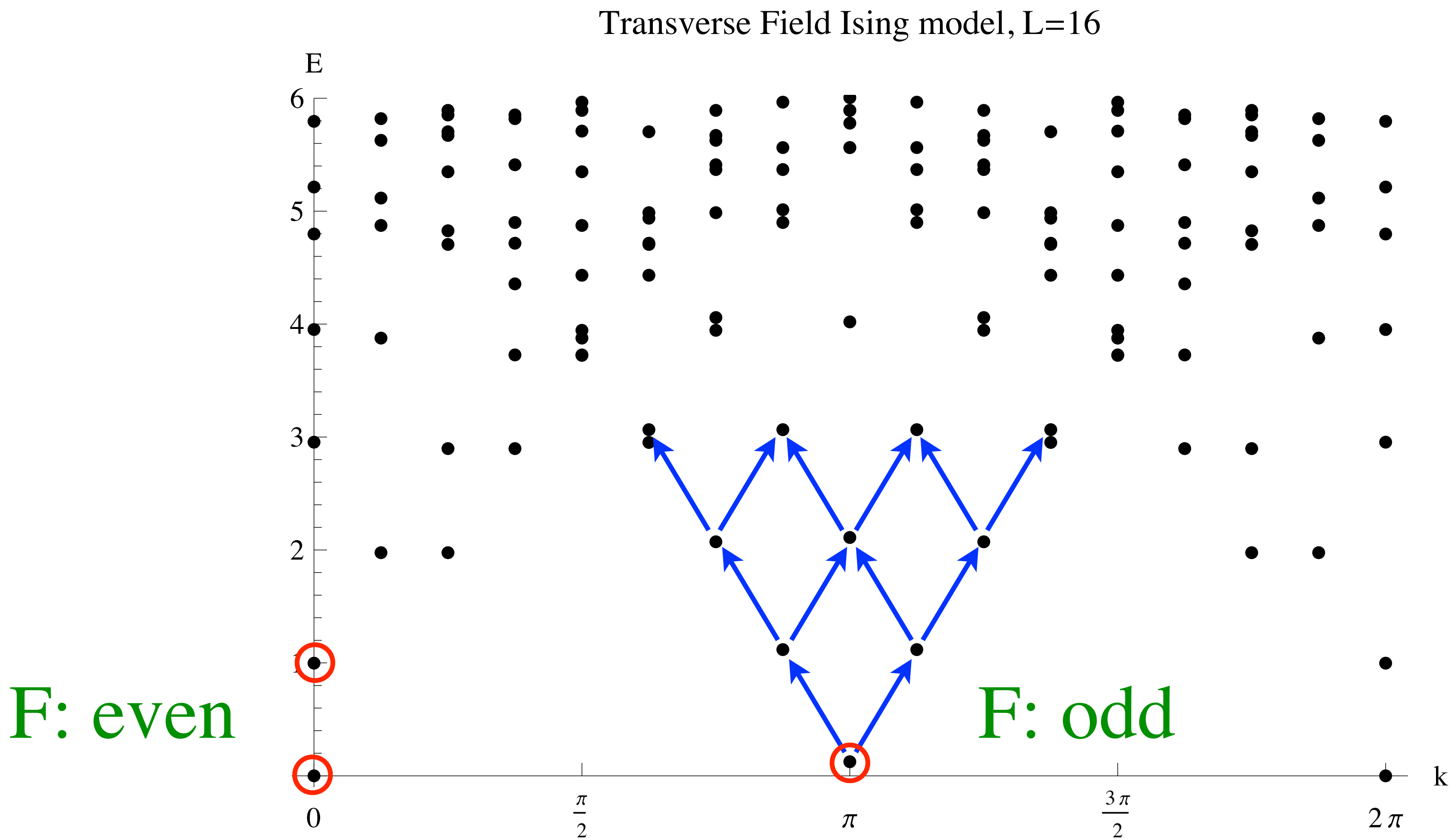

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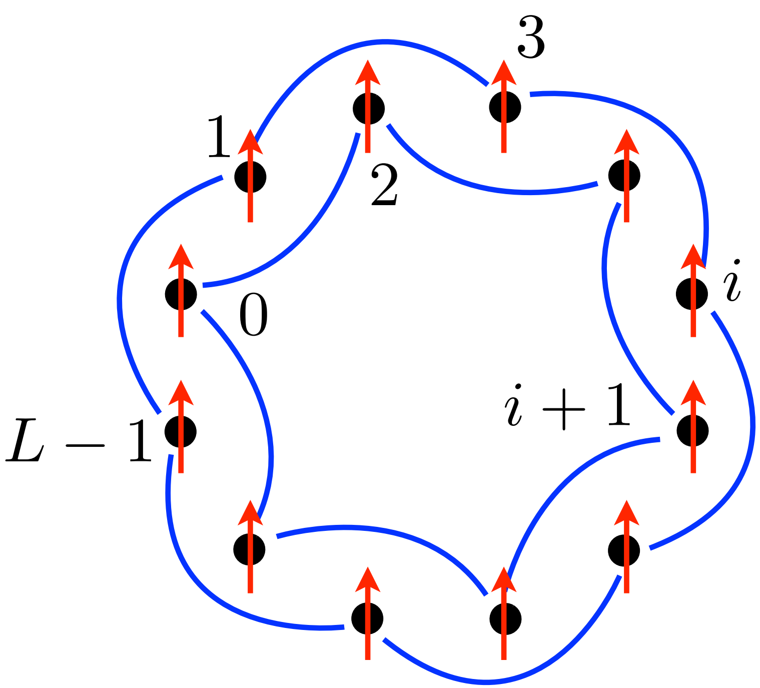
CFT sectors, Ising case

Relation between symmetry sector, boundary conditions for the fermions and cft sectors (primaries):

sym. sector $\mathcal{P} = (-1)^F$	boundary condition	fields
1	A	$\mathbf{1}$, ψ
-1	P	σ

TFI, next nearest neighbor interaction



$$H^{(2)}_{\mathrm{TFI}} = \sum_{i=0}^{L-1} \sigma^z_i + \sigma^x_i \sigma^x_{i+2}$$

Spectrum is the 'product' of two spectra of the TFI model with L/2

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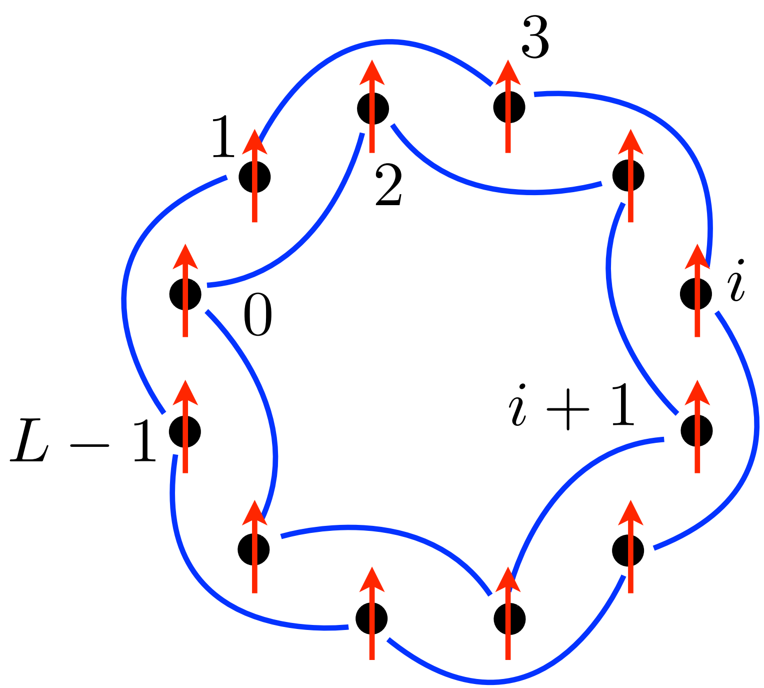

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Symmetries: $\mathcal{P}_e = \prod_{i,\mathrm{even}} \sigma_i^z = (-1)^{F_e}$ $\quad \mathcal{P}_o = \prod_{i,\mathrm{odd}} \sigma_i^z = (-1)^{F_o}$

Both the number of fermions on the even and the odd sites is conserved modulo two

CFT sectors

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H_{TFI}

sym. sector $\mathcal{P}$	boundary condition	fields
1	A	$\mathbf{1}, \psi$
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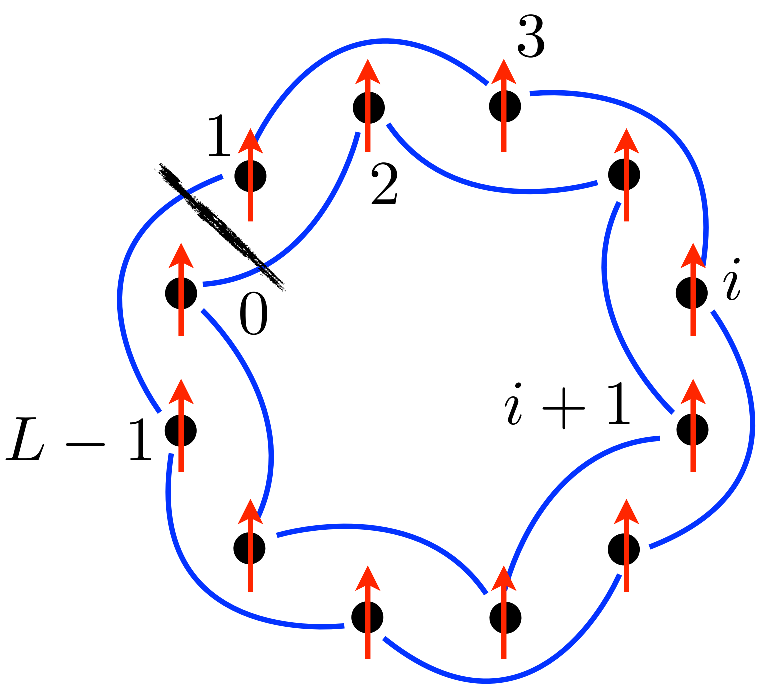
$H^{(2)}_{\mathrm{TFI}}$

$(\mathcal{P}_e, \mathcal{P}_o)$	$(\mathrm{BC_e}, \mathrm{BC_o})$	fields
$(1, 1)$	(A, A)	$(\mathbf{1}, \mathbf{1}),\ (\mathbf{1}, \psi),\ (\psi, \mathbf{1}),\ (\psi, \psi)$
$(1, -1)$	(A, P)	$(\mathbf{1}, \sigma),\ (\psi, \sigma)$
$(-1, 1)$	(P, A)	$(\sigma, \mathbf{1}),\ (\sigma, \psi)$
$(-1, -1)$	(P, P)	(σ, σ)

Adding a boundary term

We now change our model, by adding a 'boundary term', that changes the boundary condition of one chain, depending on the symmetry sector of the other.



$$H^{(2)}_{\mathrm{TFI}} = \sum_{i=0}^{L-1} \sigma^z_i + \sigma^x_i \sigma^x_{i+2}$$

$$H_{\mathrm{Boundary}} = \left(\mathcal{P}_o - \mathbf{1}\right)\sigma^x_{L-2}\sigma^x_0 + \left(\mathcal{P}_e - \mathbf{1}\right)\sigma^x_{L-1}\sigma^x_1$$

What is this new model $H^2_{\mathrm{TFI}} + H_{\mathrm{Boundary}}$?

CFT sectors

Relation between symmetry sector, boundary conditions for the fermions and cft sectors (primaries):

$$H_{\mathrm{TFI}}^{(2)}$$

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The new model has $u(1)_4$ critical behaviour, i.e., the other modular invariant in the Ising^2 theory. This is the critical behaviour of the XY chain!

TFI^2 v.s. XY chain

One can explicitly relate the TFI^2 to the XY chain:

$$H_{\text{TFI}}^{(2)} + H_{\text{Boundary}} = H_{\text{XY}}$$

$$\left(\sum_{i=0}^{L-1} \sigma_i^z + \sigma_i^x \sigma_{i+2}^x\right) + \left(\mathcal{P}_o - \mathbf{1}\right)\sigma_{L-2}^x \sigma_0^x + \left(\mathcal{P}_e - \mathbf{1}\right)\sigma_{L-1}^x \sigma_1^x =$$

$$\sum_i \tau_i^x \tau_{i+1}^x + \tau_i^y \tau_{i+1}^y$$

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One uses a modified version of the transformation for open chains (used, f.i., by D. Fisher, but dating back to the 70's:

$$\tau_{2j}^z = \sigma_{2j}^y \sigma_{2j+1}^y \qquad \tau_{2j+1}^z = \sigma_{2j}^x \sigma_{2j+1}^x$$

$$\tau_{2j}^x = \prod_{i \le 2j} \sigma_i^x \qquad \tau_{2j+1}^x = \prod_{i \ge 2j+1} \sigma_i^y$$

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So, by changing boundary conditions, one can change spin chains, such that one realizes CFT that is a different modular invariant of the original one!

Can we construct interesting chains?

Let's consider the product of n Ising models. Condensing the bosons gives the following modular invariant:

$$\mathrm{Ising}^n \longrightarrow \mathrm{so}(n)_1$$

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$$H_{\text{su}(2)_2} = \sum_i \tau_i^x \tau_{i+1}^x + \tau_i^y \tau_{i+1}^z \tau_{i+2}^y$$

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Generalization to arbitrary $so(n)_1$ critical chains is straightforward

Can we construct interesting chains?

$$H_{\mathrm{su}(2)_2} = \sum_i g_x \tau_i^x \tau_{i+1}^x + g_y \tau_i^y \tau_{i+1}^z \tau_{i+2}^y$$

This hamiltonian can be solved by Jordan-Wigner, in terms of a *single* fermion. The critical point $g_x = g_y$ is described by *su(2)*$_2$:

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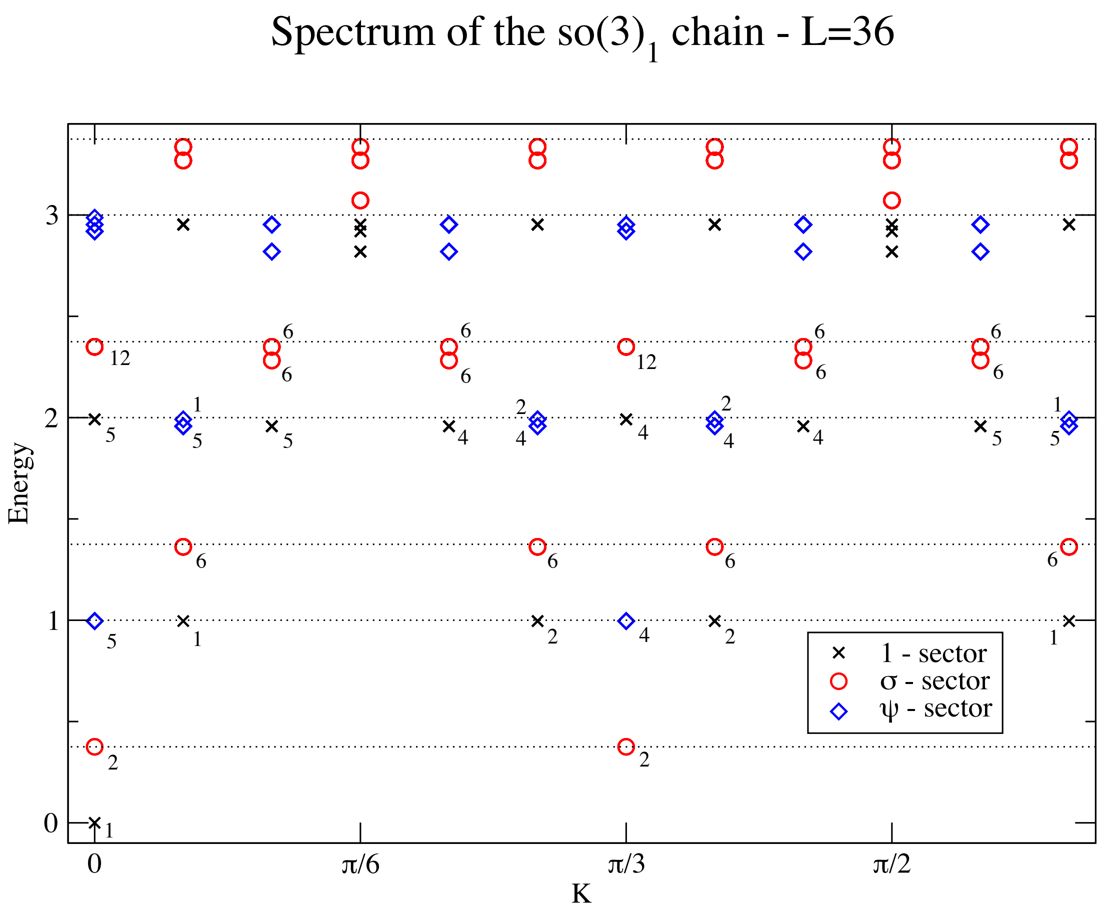

$$H_{\mathrm{su}(2)_2} = \sum_k \epsilon_k \left(\gamma_k^\dagger \gamma_k - 1/2\right)$$

$$\epsilon_k = 2\sqrt{2 + 2\cos\left(6k\pi/L\right)}$$

Going beyond condensation

To go beyond condensation transitions, we consider the 3-state Potts chain (compare: Fendley & Qi's talks).

$$H_{3-\text{Potts}} = -\sum_i X_i X_{i+1}^\dagger + Z_i + \text{h.c.}$$

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \qquad X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \omega = e^{2\pi i/3}$$

$$X^3 = \mathbf{1} \qquad Z^3 = \mathbf{1} \qquad XZ = \omega Z X$$

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The 3-state Potts chain at it's critical point: Z_3 parafermion CFT, with c=4/5.

Field content: $\{\mathbf{1}, \psi_1, \psi_2, \tau_0, \tau_1, \tau_2\}$

Scaling dimensions: $h_{\mathbf{1}} = 0$, $h_{\psi_{1,2}} = 2/3$, $h_{\tau_0} = 2/5$, $h_{\tau_{1,2}} = 1/15$

Going beyond condensation

The 3-state Potts chain at it's critical point: Z_3 parafermion CFT, with c=4/5.
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Z_{Diagonal}

$h_l + h_r$	'degenaracy'
0	1
$2/15$	4
$4/15$	4
$4/5$	2
$14/15$	4
$4/3$	4
$22/15$	8
$8/5$	1
$32/15$	4
$8/3$	4

$Z_{\text{Permutation}}$

$h_l + h_r$	'degenaracy'
0	1
$4/15$	4
$4/5$	2
$14/15$	4
$17/5$	8
$4/3$	4
$22/15$	8
$8/5$	1
$8/3$	4

Going beyond condensation

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$$\begin{aligned} H^{(2,\mathrm{perm})}_{3-\mathrm{Potts}} &= -\Big(\sum_i X_i X^\dagger_{i+2} + Z_i + \mathrm{h.c.}\Big) \\ &\quad - \big(\prod_{i,\mathrm{even}} Z^\dagger_i - \mathbf{1}\big) X^\dagger_{L-1} X_1 - \big(\prod_{i,\mathrm{odd}} Z_i - \mathbf{1}\big) X^\dagger_{L-2} X_0 + \mathrm{h.c.} \\ &= -\Big(\sum_i \tilde{X}_i \tilde{X}^\dagger_{i+1} + \tilde{Z}_i \tilde{Z}_{i+1} + \mathrm{h.c.}\Big) \end{aligned}$$

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This model is closely related to the 'quantum torus chain' Qin et al.

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By coupling three 3-state Potts chains, one can again condense a boson:

$$H^{(3,\mathrm{cond})}_{3-\mathrm{Potts}} = -\Big(\sum_i X_i X^\dagger_{i+1} + Z_i Z_{i+1} Z_{i+2} + \mathrm{h.c.}\Big)$$

Conclusions

We can construct interesting spin-chains in analogy with 2d topological condensation transitions as well as modular invariance

Construction works for Jordan-Wigner solvable models, ‘BA’ solvable models, and non-integrable models.

Latter category (non discussed here): S=1 Blume-Capel model, giving a N=1 susy cft, (A,E) exceptional modular invariant.

Study of the phase diagrams of the new models is underway

Open questions:

Can we do this without coupling several chains together (4-state Potts?)
How general is this method?
Can we learn something about the modular invariant partition functions?

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Thank you for your attention!