Braiding fluxes in Pauli Hamiltonians

Anyons for anyone

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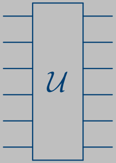
Outline

1. Motivation
   - Non abelian anyons
   - Aharonov Casher
2. Braiding fluxes
   - Zero modes
   - Adiabatically Moving fluxes
   - Metric and connection
   - Magic

Outline

Gates

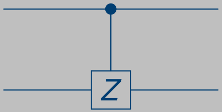
- Unitary: $|\psi\rangle \mapsto \mathcal{U}|\psi\rangle$



- $n - qubits \implies \dim(\mathcal{H}) = 2^n$
- Universal single qubit gates:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \text{———}\boxed{H}\text{———}, \quad \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \text{———}\boxed{e^{i\pi/4}}\text{—}$$

- Universal two qubits:



Anyons and quantum computing

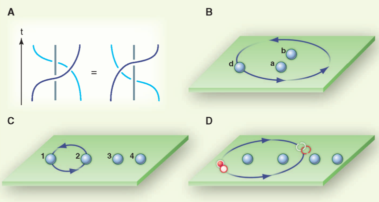
Desiderata

- Fault tolerance

Gap

$\mathcal{H} =$ Protected subspace

- Topological quantum computing—non-abelain anyons



Lindner & Stern, Science

Non abelian anyons

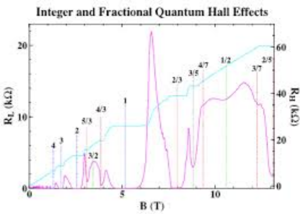
Theory and experiment

Theory

Localized modes of interacting) fermions or spins

Theoretical realization

- Anyons in FQHE
- Majoranas: $electron/\sqrt{2}$



Experiment

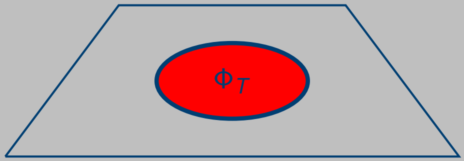
Fractional charges in FQHE, Evidence for Majorana

Outline

Aharonov Casher

Topological Zero modes

- Geometric setting:



- Pauli equation: spin $1/2$, $g = 2$

$$\left((-i\nabla - \mathbf{A})\cdot\sigma\right)^2 \geq 0, \quad \Phi_T = \frac{1}{2\pi}\int B\,dx\wedge dy$$

- Zero modes:

Aharnonov Casher

holomorphy

- Decoupling in 2-D:

$$(-i\nabla - \mathbf{A})\cdot\sigma = -2i\begin{pmatrix} 0 & \partial_z - iA_z \\ \bar\partial_z - i\bar A_z & 0 \end{pmatrix}$$

- Zero modes:

$$\big((-i\nabla - \mathbf{A})\cdot\sigma\big)(\psi,0)^t = 0, \Longrightarrow \underbrace{(\bar\partial_z - i\bar A)\psi = 0}_{1-st\ order\ pde}$$

- Holomorphy:

$$\psi(z,\bar z) \in Ker(\bar\partial_z - i\bar A) \ni \underbrace{P(z)}_{holomorphic}\ \psi(z,\bar z)$$

Aharonov and Casher

Index

- Poissons' equation–source B

$$\underbrace{\partial_z \bar\partial_z}_{\Delta} \log \psi_0 = \underbrace{i\partial_z \bar A}_{B}$$

- Polynomial decay:

$$\psi_0 = \exp(\Delta^{-1} B) \underset{z\to\infty}{\longrightarrow} |z|^{-\Phi_T}, \quad \Delta^{-1} = \frac{1}{2\pi} \log z$$

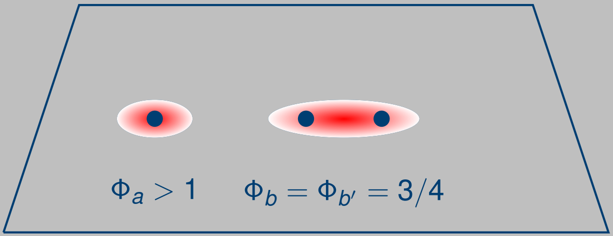
Aharonov-Casher Index theorem: Number of zero modes

$$D = \lceil \Phi_T \rceil - 1$$

Confined and free zero modes

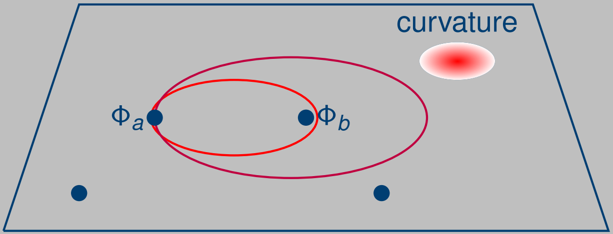
$\Phi_a > 1$ vs $\Phi_a < 1$

Two types of Charge-Flux composite



Braiding fluxes

Gates from braiding fluxes

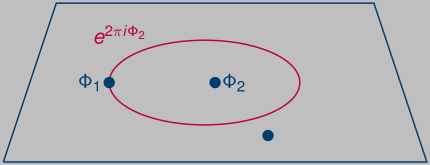


- What gates can you make by braiding fluxons?
- Catch 22: Holonomy without curvature!
- $\Phi_a \in \mathbb{R}$; Think of $1/2 < \Phi_a < 1$
- No gap protection

Adiabatic evolution

AB-Anyons

- Adiabatic evolution for moving fluxes
  - Gapless
  - Gauge issues
  - Defrosting
- Confined zero modes
  - Super Critical fluxons; $\Phi_a > 1$
  - Aharonov-Bohm abelian phases

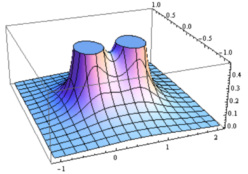


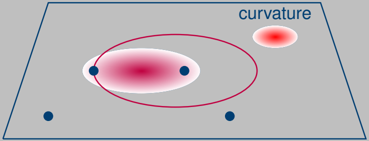
Localized zero modes

Deconfined modes

Anyons

- Holonomy–Abelian & non-abelian
- curvature & topological





- Topological if: $D = N - 1$
- Identical fluxes $1 - \frac{1}{N} < \Phi < 1$
- Burau rep of braid group : $\begin{pmatrix} 1-\nu & \nu \\ 1 & 0 \end{pmatrix}, \quad \nu = e^{-2\pi i \Phi}$

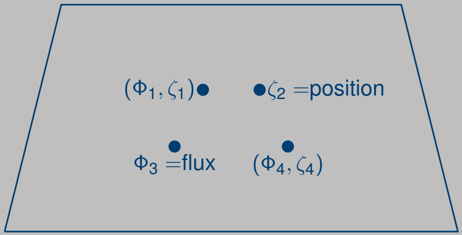
Outline

Aharonov and Casher

Fluxons

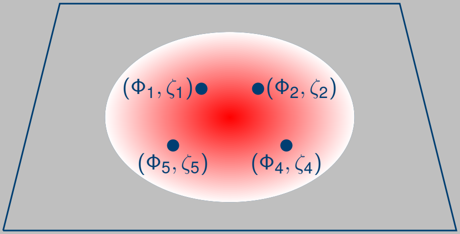
- Log-Superposition:

$$\bar{\partial}_z \log \psi = i\bar{A} \Longrightarrow (A_1 + A_2, \psi_1 \psi_2)$$



Weak individuals, $\Phi_a < 1$, strong community, $\Phi_T > 1$

Point fluxes



- Zero modes; $0 < \Phi_a < 1$

$$\psi(z;\zeta) = \underbrace{P(z)}_{polynom} \prod_a (z - \zeta_a)^{-\Phi_a}, \quad \deg(P) < \Phi_T - 1$$

Outline

Bad defrosting

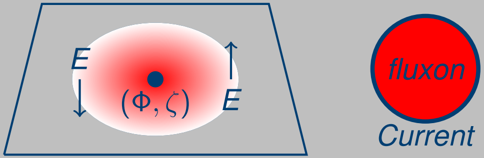
Dead frozen

- Defrosted Hamiltonian

$$\underbrace{\zeta \mapsto \zeta(t)}_{\textit{control}} \quad H(A_\zeta) \mapsto H\big(A_{\zeta(t)}\big)$$

- Wrong sources

$$\Box A = J$$

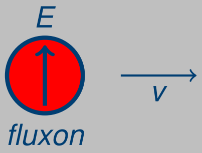


Gauge fields of moving flux

Defrosting and Gauge freedom

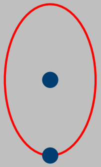
- Motion generates weak electric fields

$$\underbrace{\mathbf{E} = -\mathbf{v} \times \mathbf{B}}_{localized\ on\ fluxon}$$



Defrosted potentials

$$\mathbf{A} = \mathbf{A}(z - \zeta(t)), \quad \underbrace{A_0 = -\mathbf{v} \cdot \mathbf{A}(z - \zeta(t))}_{Inertial\ frame}$$



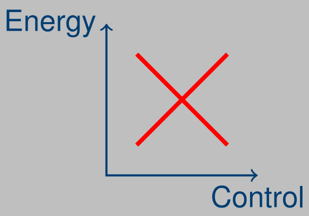
- Closed path in control $\zeta_a \Longrightarrow$ closed path in $(A_0, \mathbf{A})$

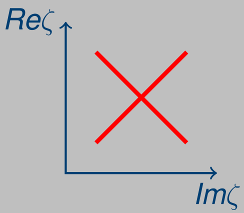
Topology in Gappless Adiabatic evolution

What is the time scale?

- Gapless. Distance between fluxon defines time scale:

$$\underbrace{\textit{time scale} = \frac{m}{h}(\textit{distance})^2,}_{\textit{dim analysis}} \quad \underbrace{\textit{distance} = |\zeta_a - \zeta_b|}_{\textit{length scale}}$$





Parallel transport

Connection

- Zero modes:

$$\underbrace{\mathcal{P}_D}_{projection} : \underbrace{Span\{z^j\psi_0 | j = 0, \dots, D-1\}}_{zero\ modes}, \quad \underbrace{\langle z|\psi_0\rangle = \prod_a (z - \zeta_a(t))^{-\Phi_a}}_{\zeta_a=\zeta_b=\dots\Rightarrow|\psi_0\rangle=\infty}$$

- Evolution within $\mathcal{P}_D$

$$\psi(z,t) = \underbrace{P(z,t)}_{polynom}\psi_0, \quad P(z,t) = \sum_0^D p_j(t) z^j,$$

Connection

$$\underbrace{\mathcal{P}_D D_t \psi = 0}_{No\ motion}, \quad \underbrace{D_t = \partial_t - iA_0}_{covariant\ derivative}$$

Outline

The connection

Metric

- Geometric–independent of time schedule:

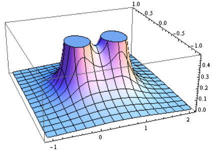
$$i\mathcal{P}_{\mathcal{D}} d|\psi\rangle = \mathcal{P}_D \left(\sum_a \underbrace{d\mathbf{x}_a}_{flux\ displace} \cdot \underbrace{\mathbf{A}_a|\psi\rangle}_{i\partial_a|\psi\rangle} \right)$$

- A (non-orthogonal) basis

$$z^j|\psi\rangle_0, \quad j = 0, \ldots, D-1$$

- Hilbert space metric

$$(\mathbf{g})_{jk}\underbrace{(\zeta, \bar{\zeta})}_{control} = \langle \psi_0 | \bar{z}^j z^k | \psi_0 \rangle$$



- Diverges when fluxons collide:

$$(\mathbf{g})_{jk}(\zeta_a = \zeta_b = \ldots) = \infty$$

Beauty parlor

Connection

$$P(z,t)=\sum_0^D p_j(t)z^j \Longrightarrow p(t)=(p_0,\dots,p_{D-1})$$

$$0=(d+\mathcal{A})\,p,\quad \mathcal{A}=\underbrace{\mathbf{g}^{-1}(\partial_\zeta \mathbf{g})}_{semi\ pure\ gauge}$$

- Semi-pure

$$\mathcal{A}=\underbrace{\mathbf{g}^{-1}(\partial \mathbf{g})}_{semi\ pure\ gauge}\neq\underbrace{\mathbf{g}^{-1}d\mathbf{g}}_{pure\ gauge},\quad d=\partial+\bar\partial$$

Factorization

holomorphic × anti-holomorphic

- Heuristics

$$(\mathbf{g})_{jk}(\zeta,\bar\zeta) = \underbrace{\langle\psi_0(\zeta)|}_{anti-holomorphic}\ \bar z^j z^k\ \underbrace{|\psi_0(\zeta)\rangle}_{holomorphic},$$

Factorization of metric

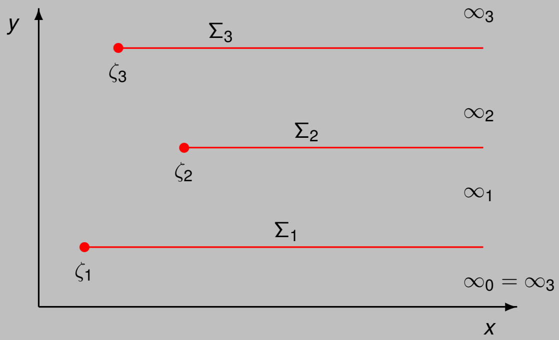
$$\underbrace{\mathbf{g}(\zeta,\bar\zeta,\Phi)}_{D\times D} = \underbrace{\Psi^*(\zeta;\Phi)}_{D\times(N-1)}\ \underbrace{\mathbf{G}(\Phi)}_{(N-1)\times(N-1)}\ \underbrace{\Psi(\zeta;\Phi)}_{(N-1)\times D}$$

Branch structure Ψ

Fluxons and cuts

- The matrix Ψ

$$\Psi_{ak}(\zeta) = \int_{\xi_N}^{\zeta_a} dz\, z^k \psi_0(z;\zeta), \quad a \in 1, \dots, N-1, \quad k \in 0, \dots, D-1$$



Outline

The magic when $D = N - 1$

Conservation laws

$$0 = \big(d + \underbrace{\mathbf{g}^{-1}(\partial \mathbf{g})}_{connection}\big)\, p \Longrightarrow 0 = \big(\mathbf{g} d + (\partial \mathbf{g})\big)\, p$$

$$= \underbrace{\big(\Psi^* G \Psi\, d + \partial(\Psi^* G \Psi)\big)\, p}_{factorization} = \underbrace{\Psi^* G\big(\Psi\, d + (\partial \Psi)\big)\, p}_{holomorphy}$$

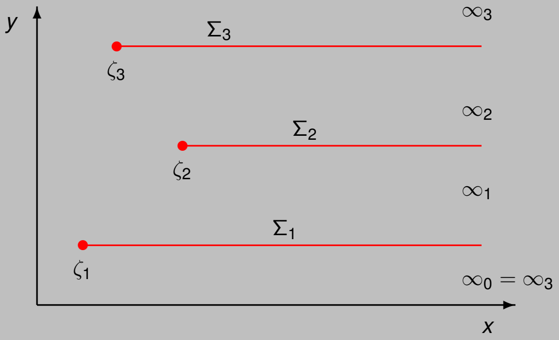
$$= \Psi^* G\big(\Psi\, d + \underbrace{(d\Psi)}_{holomorphy}\big)\, p = \underbrace{\Psi^* G}_{D\times(N-1)}\; d(\Psi\, p)$$

$D = N - 1$

- $\Psi^* G = \square$ a square matrix
- Invertible (since $g > 0$)
- $d(\Psi\, p) = 0 \Longrightarrow p(\zeta)$ a function on control space
- Curvature localized at branch points ζ_a

Monodromy

Branched surface

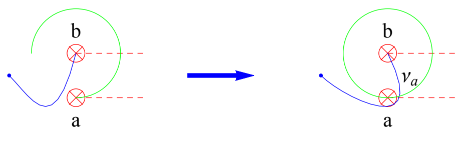


$d(\Psi\, p) = 0$

- Ψ a function on branched control space
- Monodromy of p induced from Ψ
- Holonomy is topological

Monodromy

- $\Psi_{aj}(\zeta) = \int_{\zeta_N}^{\zeta_a} d\xi \, \xi^j \prod_{b=1}^N \underbrace{(\xi - \zeta_b)^{-\Phi_b}}_{\textit{branched}}$
- What happens to Ψ_b as fluxon a goes around it:



- The monodromy matrix, non-abelian

$$\mathbf{M}(\nu_a, \nu_b) = \begin{pmatrix} 1 - \nu_a + \nu_a\nu_b & \nu_a(1-\nu_b) \\ 1 - \nu_a & \nu_a \end{pmatrix}, \quad \det \mathbf{M} = \nu_a\nu_b$$

- $\textit{Eigenvalues}(\mathbf{M}) = \{1, \nu_a\nu_b\}$

Summary

Pauli Anyons

- Point-like fluxes are non-abelian anyons
- When $\Phi = N - 1$ braiding of fluxes is topological

- Outlook
  - Spin connection
  - Conic Anyons (Kenneth)

Further Reading I

- J. Preskill, Lecture Notes
- O. Kenneth and J. Avron, ArXiv & Ann. Phys. 2014.
- Y. Aharonov and R. Casher, Phys. Rev A, 1979

Integrals: $dz \wedge d\bar{z}$

- $D = N - 1$: p a function on branched control space.
- p has the monodromy of Ψ
- Stokes
- Cuts and more