

Renormalization Group Methods: Landau-Fermi Liquid and BCS Superconductor

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This is the second part of the notes to the course on quantum theory of large systems of non-relativistic matter taught by J. Fröhlich at the 1994 Les Houches summer school. It is devoted to a sketchy exposition of some of the beautiful and important, recent results of J.Feldman and E.Trubowitz, and J. Feldman, H. Knörrer, D. Lehmann, J. Magnen, V. Rivasseau and E. Trubowitz. Their results are about a mathematical analysis of non-relativistic many-body theory, in particular of the Landau-Fermi liquid and BCS superconductivity, using Wilson's renormalization group methods and the techniques of the $\frac{1}{N}$ -expansion. While their work is ultimately aimed at a complete mathematical control (beyond perturbative expansions) of systems of weakly coupled electron gases at positive density and small or zero temperature, we can only illustrate some of their ideas within the context of perturbative solutions of Wilson-type renormalization group flow equations (we calculate leading-order terms in a $\frac{1}{N}$ -expansion, where N is an energy scale) and of one-loop effective potential calculations of the BCS superconducting ground state. We therefore urge the reader to consult the following original articles :

- [1] J. Feldman and E. Trubowitz, *Helv. Phys. Acta* **63**, 156 (1990)
G. Benfatto and G. Gallavotti, *J. Stat. Phys.* **59**, 541 (1991)
- [2] J. Feldman and E. Trubowitz, *Helv. Phys. Acta* **64**, 214 (1991)
- [3] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, *Helv. Phys. Acta* **65**, 679 (1992)

- [4] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, *Europhysics Letters* **24**, 437 (1993)
 - [5] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, *Europhysics Letters* **24**, 521 (1993)
 - [6] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, *Fermionic Many-Body Models*, in *Mathematical Quantum Theory I: Field Theory and Many-Body Theory*, J. Feldman, R. Froese and L. Rosen eds, CRM Proceedings and Lecture Notes, 1994.
 - [7] J. Feldman, D. Lehmann, H. Knörrer and E. Trubowitz, *Fermi Liquids in Two Space Dimensions*, in “Constructive Physics”, V. Rivasseau (ed.), *Lecture Notes in Physics* vol. 446, Berlin, Heidelberg, New York: Springer-Verlag, 1995.
- In Chapter 1, we review some of the standard material on thermodynamics, quantum statistical mechanics and functional integration. Sources for this material can be found in
- [8] D. Ruelle, *Statistical Mechanics (Rigorous results)*, London and Amsterdam: Benjamin, 1969.
 - [9] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, Berlin, Heidelberg and New York: Springer-Verlag, 1987 (I, second edition), and 1981 (II).
 - [10] J. W. Negele and H. Orland, *Quantum Many-Particle Systems*, *Frontiers in Physics* vol 68, New York: Addison-Wesley, 1987.

Refs. 6 and 7 above.

In Chapter 2, we consider weakly coupled electron gases. We recall the notion of scaling limit (large distance scales and low frequencies). We show that, in the scaling limit, systems of free electrons at positive density and small temperature in $d = 1, 2, 3, \dots$ space dimensions can be mapped onto systems of multi-flavour, free chiral Dirac fermions in $1 + 1$ space-time dimensions, the flavour index corresponding to a direction on the Fermi sphere. This is the first manifestation of a “principle of dimensional reduction” which says that, in the scaling limit (infrared domain), electron gases at positive density and very small temperatures have properties analogous to those of certain $1 + 1$ dimensional (relativistic) models of Dirac fermions with four-fermion interactions.

We then introduce the basic ideas underlying a renormalization group analysis of weakly coupled electron gases and show that there is a parameter, related to a (dimensionless) ratio of energy scales, that plays a rôle analogous to the number, N , of fermion flavours in $1 + 1$ dimensional models of Dirac fermions with four-fermion interactions, such as the Gross-Neveu model. This enables us to study the renormalization flow in weakly coupled electron gases by using $\frac{1}{N}$ -techniques. We follow ideas first presented in refs. [1], [4] and [5].

Useful additional references to Chapter 2 are:

- [11] K. G. Wilson and J. B. Kogut, Phys. Reports **12**, 7 (1974)
- [12] J. Fröhlich, R. Götschmann and P. A. Marchetti, “The Effective Gauge Field Action of a System of Non-Relativistic Electrons”, to appear in Commun. Math. Phys. 1995

(See also Chapter 5 of Part I.)

In Chapter 3, we derive the renormalization group flow equations, to leading order in a $\frac{1}{N}$ -expansion, for systems of non-relativistic electrons at positive density and zero temperature interacting through weak two-body potentials of short range (“screening”), following ideas in refs. [1] - [5]. Special attention is given to understanding the striking similarities with $1 + 1$ dimensional Gross-Neveu type models and to an analysis of the flow of BCS couplings (“BCS instability”). Besides refs. [1] - [7] and [11], the following references will prove useful:

- [13] A. L. Fetter and J.D. Walecka, Quantum Theory of Many-Particle Systems, New York: McGraw-Hill, 1971.
- [14] J. Solyom, Adv. Phys. **28**, 201 (1979); (RG methods for one-dimensional systems).
- [15] J. Polchinski, TASI Lectures 1992; (effective actions).
- [16] R. Shankar, Rev. Mod. Phys. **66**, 129 (1994); (RG methods for $(d \geq 2)$ -dimensional, non-relativistic many-particle systems).
- [17] D. J. Gross and A. Neveu, Phys. Rev. D **10**, 3235 (1974); ($\frac{1}{N}$ -expansion in the Mitter-Weisz-Gross-Neveu model).
- [18] K. Gawędzki and A. Kupiainen, Commun. Math. Phys. **102**, 1 (1985); (rigorous renormalization group analysis of the Gross-Neveu model).
- [19] G. Gallavotti, Rev. Mod. Phys. **57**, 471 (1985); (RG methods in scalar field theories).
- [20] J. Polchinski, Nucl. Phys. B **231**, 269 (1984)

In Chapter 4, we study the BCS superconducting ground state, the spontaneous breaking of the U(1) global gauge symmetry (the particle number symmetry) in the superconducting ground state and the emergence of a massless Goldstone boson associated with the broken symmetry. We make use of the Nambu-Gorkov formalism and large- N techniques, (refs. [2] - [5]). We emphasize the striking analogies between weakly coupled BCS s-wave superconductors and the $1 + 1$ dimensional, large- N chiral Gross-Neveu model. Our discussion of symmetry breaking is based on the loop expansion of the effective action of a charged, scalar field describing Cooper pairs. In the mean-field approximation, it reduces to the calculation of an effective potential that proceeds along the lines of the calculations in ref. [17]. In order to understand the dependence of symmetry breaking on dimension and temperature and the dynamics of Goldstone bosons, we calculate leading corrections to mean-field theory.

Useful additional references for Chapter 4 are:

- [21] P. -G. de Gennes, Superconductivity of Metals and Alloys, New York: Benjamin, 1966.
- [22] J. R. Schrieffer, Theory of Superconductivity, Menlo Park: Benjamin-Cummings, 1964.
- [23] A. A. Abrikosov, L. P. Gorkov and I. E. Dzyaloshinsky, Methods of Quantum Field Theory in Statistical Physics, New York: Dover, 1975.
- [24] S. Coleman and E. Weinberg, Phys. Rev. D **7**, 1888 (1973); and ref. [17].
- [25] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961)
- [26] J. Goldstone, Nuovo Cimento **19**, 154 (1961);
J. Goldstone, A. Salam and S. Weinberg, Phys. Rev. **127**, 965 (1961)
- [27] D. Mermin and H. Wagner, Phys. Rev. Letters **17**, 1133 (1966);
D. Mermin, J. Math. Phys. **8**, 1061 (1967)

For rigorous results on continuous symmetry breaking, see also

- [28] J. Fröhlich, B. Simon and T. Spencer, Commun. Math. Phys. **50**, 79 (1976);
J. Fröhlich and T. Spencer, in “Scaling and Self-Similarity in Physics”,
Progress in Physics, Basel, Boston: Birkhäuser, 1983.

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1 Background material

In this chapter we recall some basic definitions of quantum statistical mechanics, introduce ensembles of identical particles, fermions and bosons, and express Euclidean correlation functions in terms of functional integrals.

1.1 Thermodynamics and quantum statistical mechanics

We first recall the definition of *statistical ensemble*. We assume that a quantum mechanical system is confined to a box $\Lambda \subset \mathbb{E}^3$ in 3-dimensional Euclidean space. Important characteristics of the system are its volume $|\Lambda|$ and the particle number $N = n|\Lambda|$, where n is the particle density.

The quantum-mechanical state space of an N -particle system in the box Λ is a separable Hilbert space $\mathcal{H}_\Lambda^{(N)}$, its dynamics is given by a selfadjoint Hamilton operator $H_\Lambda^{(N)}$ acting on $\mathcal{H}_\Lambda^{(N)}$, whose energy spectrum is discrete and bounded from below.

In statistical mechanics, one considers two standard ensembles: One either fixes the number of particles in the system (*canonical ensemble*), or one permits the number of particles to vary and keeps fixed only its mean value $\langle N \rangle$ (*grand canonical ensemble*). Here, N denotes the particle number operator, and $\langle (\cdot) \rangle$ is the expectation in some state of the system.

Quantities of interest in thermodynamics are *thermodynamic potentials*. One of the most important such potential is the free energy F :

$$\beta F(\beta, \Lambda, N) = -\log Z(\beta, \Lambda, N) \quad (1.1)$$

where the canonical partition function Z is defined by

$$Z(\beta, \Lambda, N) = \text{Tr}_{\mathcal{H}_\Lambda^{(N)}} \left(e^{-\beta H_\Lambda^{(N)}} \right) \quad (1.2)$$

The free energy refers to the canonical ensemble, since the number of particles is fixed. One also introduces a free energy per unit volume (per particle) by setting

$$f(\beta, \Lambda, N) = \frac{1}{|\Lambda|} F(\beta, \Lambda, N) \quad (= \frac{1}{N} F(\beta, \Lambda, N) , \text{ resp. }) \quad (1.3)$$

To an ordered sequence, $0 < \tau_n < \dots < \tau_2 < \tau_1 < \beta$, of *imaginary times*, and an n -tuple of observables (bounded operators on $\mathcal{H}_\Lambda^{(N)}$), $a_1, a_2 \dots a_n$, one associates a *temperature-ordered Green (correlation) function*

$$\frac{1}{Z} \text{Tr} \left(e^{-\beta H} e^{\tau_1 H} a_1 e^{-(\tau_1 - \tau_2) H} a_2 \dots a_n e^{-\tau_n H} \right) \quad (1.4)$$

(where we have omitted the subscripts Λ and superscripts N).

The grand canonical partition function is defined by

$$\Xi(\beta, \Lambda, \mu) = \text{Tr}_{\mathcal{H}_\Lambda} \left(e^{-\beta(H - \mu N)} \right) = e^{\beta \Omega(\beta, \Lambda, \mu)} \quad (1.5)$$

The coefficient μ is called *chemical potential*. The thermodynamic potential Ω is the so called *grand potential*. We denote by $\mathcal{H}_\Lambda = \oplus_N \mathcal{H}_\Lambda^{(N)}$ the direct sum of Hilbert spaces corresponding to different numbers of identical particles (bosons or fermions). The Hamiltonian H is then a direct sum of N -particle Hamiltonians. In each N -particle subspace, H coincides with $H^{(N)}$ (we again omit the subscript Λ). In nonrelativistic quantum theory, we assume that

the particle number operator, N , and the Hamiltonian commute; in relativistic theories this is usually not the case because of particle creation-annihilation processes.

The laws of thermodynamics are described by differential *thermodynamic relations*. In the grand canonical ensemble the most important such relation is

$$d\Omega(\beta, \Lambda, \mu) = \langle N \rangle d\mu + S_\Lambda dT \quad (1.6)$$

The differential, d , is taken with respect to $\beta = \frac{1}{k_B T}$ (where k_B is Boltzmann's constant) and μ ; the box Λ is kept fixed. The coefficient in front of $d\mu$ is the mean value (in an equilibrium state at inverse temperature β and chemical potential μ) of the particle number operator already encountered in the definition of the grand canonical ensemble. The coefficient of dT is the *entropy* of the system.

In the study of macroscopically large systems it is useful to consider the *thermodynamic limit*, $\Lambda \nearrow \mathbb{E}^3$, and to pass from *extensive* quantities (F, Ω, N, \dots) to *intensive* ones (f, p, n, \dots). For example, in the grand canonical ensemble one defines the *pressure*, p , by setting

$$p(\beta, \mu) = \lim_{\Lambda \nearrow \mathbb{E}^3} \frac{1}{|\Lambda|} \Omega(\beta, \Lambda, \mu) \quad (1.7)$$

(where $\Lambda \nearrow \mathbb{E}^3$ in the sense of van Hove or Fisher). The particle density and the specific entropy are then given by

$$n = \frac{\partial p}{\partial \mu}, \quad s = \frac{\partial p}{\partial T} \quad (1.8)$$

Grand canonical correlation functions are defined for systems in a box Λ in the same way as for the canonical ensemble. They satisfy the *KMS (Kubo-Martin-Schwinger) condition* which we now describe: Given a \ast -algebra \mathcal{A} of quantum-mechanical observables, we define a *state* on \mathcal{A} as a normalized ($\rho(\mathbb{1}) = 1$), positive ($\rho(a^\ast a) \geq 0, \forall a \in \mathcal{A}$), linear functional ($\rho(\alpha a + \beta b) = \alpha \rho(a) + \beta \rho(b), \forall a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$). An example of a state is an equilibrium state at some inverse temperature β defined by

$$\rho_\beta(a) = \frac{1}{\Xi} \text{Tr} (e^{-\beta(H-\mu N)} a) \quad (1.9)$$

It is commonly called *Gibbs state*. Introducing time-dependent observables in the Heisenberg picture by setting

$$a(t) = e^{it(H-\mu N)} a e^{-it(H-\mu N)}, \quad a \in \mathcal{A} \quad (1.10)$$

we can formulate the KMS condition as follows: For arbitrary elements a and b of \mathcal{A} , there exists a function $F_{ab}^\beta(z)$ analytic in the strip $-\beta < \text{Im } z < 0$ and continuous on its closure such that

$$\begin{aligned} F_{ab}^\beta(t) &= \rho_\beta(a(t) b) \\ F_{ab}^\beta(t - i\beta) &= \rho_\beta(b a(t)) \end{aligned} \quad (1.11)$$

Using the cyclic property of the trace

$$\mathrm{Tr}(ab) = \mathrm{Tr}(ba) \quad (1.12)$$

one can check that the KMS condition is satisfied for the Gibbs state (1.9). In the formulation (1.11), the KMS condition can be transferred to the thermodynamic limit. Actually, one can view the KMS condition as an equation for equilibrium states of the system. Any state ρ_β satisfying (1.11) is called an *equilibrium state* at inverse temperature β .

1.2 Systems of identical particles

In this section we consider a gas of identical quantum-mechanical particles, bosons or fermions. The *second quantization* of a system of identical particles is conventionally described by starting with the *Fock space*

$$\mathcal{H} = \oplus_{N=0}^{\infty} \mathcal{H}^{(N)} \quad (1.13)$$

where $\mathcal{H}^{(N)}$ is the N -particle Hilbert space, *i.e.*,

$$\begin{aligned} \mathcal{H}^{(N)} &= (\mathcal{H}^{(1)})^{\otimes_s N}, \text{ for bosons} \\ \mathcal{H}^{(N)} &= (\mathcal{H}^{(1)})^{\otimes_a N}, \text{ for fermions} \end{aligned} \quad (1.14)$$

and $\mathcal{H}^{(1)}$ is the Hilbert space of pure states of a single particle. The subscripts s and a indicate completely symmetric and antisymmetric tensor products, respectively, according to whether the particles are bosons or fermions. Furthermore, $(\mathcal{H}^{(1)})^{\otimes_s 0} \cong (\mathcal{H}^{(1)})^{\otimes_a 0} \cong \mathbb{C}|0\rangle$, where $|0\rangle$ is the vacuum vector in \mathcal{H} .

Let ξ denote the position and spin of a particle. Creation- and annihilation-operators, $\Psi^*(\xi)$ and $\Psi(\xi)$, are defined as operator-valued distributions on \mathcal{H} . They satisfy the commutation relations

$$\begin{aligned} [\Psi^\#(\xi), \Psi^\#(\eta)]_\pm &= 0 \\ [\Psi(\xi), \Psi^*(\eta)]_\pm &= \delta(\xi - \eta) \end{aligned} \quad (1.15)$$

where the commutator $[A, B]_- := AB - BA$ is chosen for bosons (integer spin particles), and the anti-commutator $[A, B]_+ := AB + BA$ is chosen for fermions (particles with half-integer spin). Moreover, $\Psi(\xi)|0\rangle = 0$.

The operators

$$\begin{aligned} \Psi(f) &:= \int d\xi \Psi(\xi) \bar{f}(\xi) \\ \Psi^*(f) &:= \int d\xi f(\xi) \Psi^*(\xi) \end{aligned} \quad (1.16)$$

$f \in \mathcal{H}^{(1)}$, then generate a $*$ -algebra $\mathcal{A}_{\mathcal{H}^{(1)}}$, with $(\Psi(f))^* = \Psi^*(f)$, and

$$\begin{aligned} [\Psi^\#(f), \Psi^\#(g)]_\pm &= 0 \\ [\Psi(f), \Psi^*(g)]_\pm &= \langle f, g \rangle \mathbb{I} \end{aligned} \quad (1.17)$$

where $\langle f, g \rangle = \int d\xi \bar{f}(\xi)g(\xi)$ denotes the scalar product on $\mathcal{H}^{(1)}$. For fermions, $\Psi^*(f)$ and $\Psi(f)$ are bounded operators: $\|\Psi(f)\| = \|\Psi^*(f)\| = \sqrt{\langle f, f \rangle}$.

Rather than starting from the Hilbert space \mathcal{H} and a concrete realization of (1.15) by operators acting on \mathcal{H} , one can start from the abstract $*$ -algebra $\mathcal{A}_{\mathcal{H}^{(1)}}$ generated by the identity and creation- and annihilation operators (1.16), with relations (1.17). We obtain representations of $\mathcal{A}_{\mathcal{H}^{(1)}}$ from states via the so-called GNS (Gelfand-Naimark-Segal) construction, as follows: Consider a state ρ on a $*$ -algebra \mathcal{A} , i.e., ρ is a normalized ($\rho(\mathbb{I}) = 1$), positive ($\rho(a^*a) \geq 0, \forall a \in \mathcal{A}$), linear functional ($\rho(\alpha a + \beta b) = \alpha\rho(a) + \beta\rho(b), \forall a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$) on \mathcal{A} . Divide \mathcal{A} by the left-ideal

$$\mathcal{K} := \{a \in \mathcal{A} | \rho(ba) = 0, \forall b \in \mathcal{A}\} \quad (1.18)$$

Define a Hilbert space \mathcal{H} as the norm-closure of $V := \mathcal{A}/\mathcal{K}$ (where $\langle [a], [b] \rangle := \rho(a^*b)$, i.e., $\|[a]\|^2 := \rho(a^*a)$) and a representation Π_ρ of \mathcal{A} via $\langle [a], \Pi_\rho(c)[b] \rangle := \rho(a^*cb)$. For the *cyclic vector* $\Omega_\rho := [\mathbb{I}] \in V$, one has that $\rho(c) = \langle \Omega_\rho, \Pi_\rho(c)\Omega_\rho \rangle, \forall c \in \mathcal{A}$.

The usual Fock space (cf. (1.13)) is obtained by choosing $\rho = \rho_0$, where ρ_0 is uniquely determined by the equations $\rho_0(a\Psi(f)) = 0 = \rho_0(\Psi^*(f)b), \forall a, b \in \mathcal{A}_{\mathcal{H}^{(1)}}$. They imply *Wick's theorem*,

$$\rho_0\left(\prod_{i=1}^m \Psi(f_i) \prod_{j=n}^1 \Psi^*(g_j)\right) = \begin{cases} 0 & : m \neq n \\ \sum_{\text{permutations } P} \epsilon_P \prod_{i=1}^n \langle f_i, g_{P(i)} \rangle & : m = n \end{cases} \quad (1.19)$$

where

$$\epsilon_P = \begin{cases} 1 & , \text{ for bosons} \\ \text{sign } P & , \text{ for fermions} \end{cases} \quad (1.20)$$

In this example, the Hilbert space \mathcal{H} obtained from the GNS construction is the Fock space, and the cyclic vector Ω_{ρ_0} is the vacuum vector $|0\rangle$.

A useful operation is

$$a = \prod_i \Psi^\#(f_i) \rightarrow : a : \quad (1.21)$$

the so-called *Wick-ordering*. Up to a sign, $: a :$ is the same monomial as a , but with all Ψ^* 's written to the left of all Ψ 's. As usual, the sign is equal to 1, for bosons, and to $(-1)^t$, for fermions, where t denotes the number of transpositions necessary to move all Ψ^* 's to the left of all Ψ 's.

Remark. In order to become familiar with the GNS-construction, one may consider, as a trivial example, the $*$ -algebra $\mathcal{A} := M_n(\mathbb{C})$ of complex $n \times n$ matrices: Choosing $\rho(a) := \langle x, ax \rangle$, where $a \in M_n(\mathbb{C})$, $\langle \cdot, \cdot \rangle$ denotes the usual scalar product on \mathbb{C}^n , and x is an

arbitrary unit vector in \mathbb{C}^n , one easily recovers the representation of $M_n(\mathbb{C})$ on \mathbb{C}^n . In this example, $\mathcal{K} = \{a \mid ax = 0\}$ (consisting of all matrices having their n -th column identically equal to zero if x is the n -th basis-vector of \mathbb{C}^n), and $\mathcal{A}/\mathcal{K} \cong \mathbb{C}^n$ has complex dimension n . Another example is obtained by considering a density matrix (e.g. a Gibbs state) on $M_n(\mathbb{C})$. It leads to a reducible representation of $M_n(\mathbb{C})$.

Returning to $\mathcal{A}_{\mathcal{H}^{(1)}}$, it is often convenient to choose an orthonormal basis $\{h_n\}_{n=1}^\infty$ in $\mathcal{H}^{(1)}$, and to expand annihilation- and creation operators according to

$$\begin{aligned}\Psi(\xi) &= \sum_1^\infty \Psi_n h_n(\xi) \\ \Psi^*(\xi) &= \sum_1^\infty \Psi_n^* \overline{h_n}(\xi)\end{aligned}\tag{1.22}$$

with

$$\begin{aligned}[\Psi_n^\#, \Psi_m^\#]_\pm &= 0 \\ [\Psi_n, \Psi_m^*]_\pm &= \delta_{mn}\end{aligned}\tag{1.23}$$

We assume that every basis element h_n , $n = 1, 2, 3, \dots$, is a C^∞ -vector for the one-particle Hamiltonian $H_\Lambda^{(1)}$. A typical second-quantized Hamiltonian has the form

$$\begin{aligned}H &= \int d\xi \Psi^*(\xi) \left(-\frac{\hbar^2 \Delta_\Lambda}{2m} \Psi \right) (\xi) \\ &\quad + g \int \Psi^*(\xi_1) \Psi^*(\xi_2) V(\xi_1, \xi_2) \Psi(\xi_2) \Psi(\xi_1) d\xi_1 d\xi_2\end{aligned}\tag{1.24}$$

where Δ_Λ is the Laplacian with, e.g., Dirichlet boundary conditions at $\partial\Lambda$, and V is a two-body potential. One can rewrite H as

$$H = T(\Psi^*, \Psi) + V(\Psi^*, \Psi)\tag{1.25}$$

where $T(\Psi^*, \Psi) \equiv \sum_{m,n} \Psi_m^* A_{mn} \Psi_n$ is a positive-definite quadratic form in $\Psi = (\Psi_m)$ and $\Psi^* = (\Psi_m^*)$ corresponding to the kinetic energy of particles (first term on the right side of (1.24)), and V involves terms of higher order in the operators Ψ and Ψ^* and is even in Ψ, Ψ^* and gauge-invariant.

The time-evolution of $a \in \mathcal{A}_{\mathcal{H}^{(1)}}$ in the Heisenberg picture is given by

$$\dot{a} = i [H - \mu N, a]\tag{1.26}$$

(the term proportional to μ drops out for gauge-invariant observables) implying that

$$i \dot{\Psi}_n = A_{nm} \Psi_m + \frac{\partial^L V}{\partial \Psi_n^*} - \mu \Psi_n\tag{1.27}$$

and

$$-i \dot{\Psi}_n^* = \Psi_m^* A_{mn} + \frac{\partial^R V}{\partial \Psi_n} - \mu \Psi_n^*\tag{1.28}$$

where repeated indices are to be summed over, and $\frac{\partial^L V}{\partial \Psi_n^*}$ is obtained from V by (anti-) commuting any factor of Ψ_n^* in V to the very left and then omitting it and $\frac{\partial^R V}{\partial \Psi_n}$ is obtained by (anti-) commuting any factor of Ψ_n in V to the very right and then omitting it.

1.3 Functional integrals: Bosons

The goal of this section is to rewrite the correlation functions (see Eq.(1.4)) in terms of functional integrals. For this purpose we replace the field operators by classical (Grassmann) integration variables for bosons (fermions) and integrate the exponential of the action functional over all configurations of these variables. To obtain well-defined quantities, a cutoff has to be introduced (the same for bosons and fermions). Referring to the second quantized Hamiltonian introduced previously, we define $H^{[\kappa]}$ by replacing A and V by

$$A_{nm}^{[\kappa]} = \begin{cases} A_{nm} & \text{if } n, m \leq \kappa \\ 0 & \text{, otherwise} \end{cases} \quad (1.29)$$

$$V^{[\kappa]}(\Psi^*, \Psi) = \begin{cases} V & \text{all indices } \leq \kappa \\ 0 & \text{, otherwise} \end{cases} \quad (1.30)$$

respectively. From now on, bosons and fermions are treated separately. For an arbitrary operator a on \mathcal{H} we define

$$a(\tau) := e^{\tau(H^{[\kappa]} - \mu N)} a e^{-\tau(H^{[\kappa]} - \mu N)}$$

where τ denotes *imaginary time*. Thus, we obtain τ -dependent fields $\Psi_n(\tau), \Psi_n^*(\tau)$. We replace the bosonic field operators by complex variables

$$\Psi_n(\tau), \Psi_n^*(\tau) \rightarrow \psi_n(\tau), \psi_n^*(\tau) \quad (1.31)$$

satisfying $\psi_n^*(\tau) = \overline{\psi_n(\tau)}$, with $\tau \in [0, \beta]$. In terms of the complete orthonormal system $\{h_n\}_{n=1}^\infty$ in $\mathcal{H}_\Lambda^{(1)}$, we understand the ψ_n 's as modes in the decomposition

$$\psi(\tau, \xi) = \sum \psi_n(\tau) h_n(\xi) \quad (1.32)$$

Furthermore, we assume that $A^{[\kappa]} \geq 0$ and $V^{[\kappa]}[\psi^*, \psi] \geq 0$, $\forall \psi^*, \psi$.

We define the Euclidean, i.e., imaginary-time, bosonic action by

$$S^{[\kappa]}[\psi^*, \psi] \equiv \int_0^\beta d\tau \left[\sum_{n \leq \kappa} (\psi_n^* \frac{\partial}{\partial \tau} \psi_n)(\tau) + H^{[\kappa]}(\psi^*(\tau), \psi(\tau)) - \mu N(\tau) \right] \quad (1.33)$$

The system is always assumed to be confined to a compact box Λ .

As the variable τ takes values in the interval $[0, \beta]$, we must specify boundary conditions for the variables $\psi(\tau), \psi^*(\tau)$ at $\tau = 0, \beta$. They are determined by the KMS condition for correlation functions.

Let us consider the correlation function of two bosonic operators $\Psi^*(\tau)$ and $\Psi(0)$. By analyticity in the complex time variable, the KMS condition implies that

$$\rho_\beta(\Psi^*(\tau)\Psi(0)) = \rho_\beta(\Psi(0)\Psi^*(\tau - \beta)) \quad (1.34)$$

The imaginary-time correlation function of operators a_1, \dots, a_n is defined as

$$\langle a_1(\tau_1) \cdots a_n(\tau_n) \rangle_{\beta, \mu} := \frac{1}{\Xi} \text{Tr} (e^{-\beta(H^{[\kappa]} - \mu N)} T a_1(\tau_1) \cdots a_n(\tau_n)) \quad (1.35)$$

The imaginary-time ordering operator “T” rearranges the operators in decreasing time order, and the imaginary times τ_1, \dots, τ_n are assumed to be non-negative, pairwise different and smaller than β . Using the KMS condition we obtain the identities

$$\begin{aligned} \langle \Psi^*(\tau) \Psi(0) \rangle_{\beta, \mu} &= \rho_\beta(\Psi^*(\tau) \Psi(0)) = \rho_\beta(\Psi(0) \Psi^*(\tau - \beta)) \\ &= \langle \Psi(0) \Psi^*(\tau - \beta) \rangle_{\beta, \mu} = \langle \Psi^*(\tau - \beta) \Psi(0) \rangle_{\beta, \mu} \end{aligned} \quad (1.36)$$

This simple consideration implies periodic boundary conditions for bosonic variables:

$$\psi^*(\tau + \beta) = \psi^*(\tau) \ , \ \psi(\tau + \beta) = \psi(\tau) \quad (1.37)$$

Introducing integration measures

$$\mathcal{D}\psi = \prod_{m \leq \kappa} d\psi_m \quad , \quad \mathcal{D}\psi^* = \prod_{m \leq \kappa} d\psi_m^* \quad (1.38)$$

we have the following result.

Lemma : *The cutoff correlation functions can be written in terms of functional integrals as follows:*

$$\begin{aligned} \langle \prod_i \Psi_{n_i}(\tau_i) \prod_j \Psi_{m_j}^*(\sigma_j) \rangle_{\beta, \mu} &\equiv \frac{1}{\Xi} \text{Tr} \left(e^{-\beta(H^{[\kappa]} - \mu N)} T \prod_i \Psi_{n_i}(\tau_i) \prod_j \Psi_{m_j}^*(\sigma_j) \right) \\ &= \frac{1}{\Xi} \int \mathcal{D}\psi^* \mathcal{D}\psi e^{-S^{[\kappa]}[\psi^*, \psi]} \prod_i \psi_{n_i}(\tau_i) \prod_j \psi_{m_j}^*(\sigma_j) \end{aligned} \quad (1.39)$$

where $n_i, m_j \leq \kappa$. The integral on the r.h.s. is supplied with periodic boundary conditions, Eq. (1.37). For $\mu < 0$, both sides of the equality are well-defined in the limit $\kappa \rightarrow \infty$, and the equality continues to hold in this limit.

It is easy to verify this lemma at the level of formal power series, or by appealing to Schwinger-Dyson equations. A rigorous proof is somewhat more complicated, but the result is standard.

1.4 Functional integrals: Fermions

Our discussion for fermions follows the procedure outlined for bosons, but some room will be given to understanding the properties of integration over anticommuting variables.

Just as in the previous section, we replace the operators Ψ, Ψ^* by symbols ψ, ψ^*

$$\Psi_m, \Psi_n^* \rightarrow \psi_m, \psi_n^* \quad (1.40)$$

where ψ_n and ψ_m^* are independent *Grassmann variables*:

$$\{\psi_n, \psi_m\} = \{\psi_n^*, \psi_m^*\} = \{\psi_n, \psi_m^*\} = 0 \quad (1.41)$$

To integrate over Grassmann variables, we define the *Berezin integral* by requiring that

$$\{d\psi_n^\#, \psi_m^\#\} = \{d\psi_n^\#, d\psi_m^\#\} = 0 \quad (1.42)$$

and

$$\int d\psi_n \psi_n = \int d\psi_n^* \psi_n^* = 1 \quad (1.43)$$

$$\int d\psi_n 1 = \int d\psi_n^* 1 = 0 \quad (1.44)$$

We recall some properties of the Berezin integral that follow easily from the definition and standard combinatorics: Introducing $\mathcal{D}^{[n]}\psi = d\psi_n \dots d\psi_1$ and $\mathcal{D}^{[n]}\psi^* = d\psi_1^* \dots d\psi_n^*$, we have that

$$1 = \prod_{i=1}^n \int d\psi_i \psi_i \prod_{j=1}^n \int d\psi_j^* \psi_j^* = \int \mathcal{D}^{[n]}\psi^* \mathcal{D}^{[n]}\psi \psi_1 \dots \psi_n \psi_n^* \dots \psi_1^* \quad (1.45)$$

It is easy to see that, for $M^{[n]}[\psi^*, \psi]$ a monomial in $\psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*$,

$$\int \mathcal{D}^{[n]}\psi^* \mathcal{D}^{[n]}\psi M^{[n]}[\psi^*, \psi] = 0 \quad (1.46)$$

unless $M^{[n]}[\psi^*, \psi] = \lambda \psi_1 \dots \psi_n \psi_n^* \dots \psi_1^*$, $\lambda \in \mathbb{C}$. From Eqs. (1.45) and (1.46) and from the standard definition of determinants we derive that

$$\int \mathcal{D}^{[n]}\psi^* \mathcal{D}^{[n]}\psi e^{-(\psi^*, K^{[n]}\psi)} = \det K^{[n]} \quad (1.47)$$

where $K^{[n]}$ is a regular $n \times n$ matrix. In fact, this determinant formula may be used as a basis for a definition of integration over Grassmann variables.

It is convenient to introduce the left derivative $\frac{\delta}{\delta\psi_i}$ by setting

$$\begin{aligned} \frac{\delta}{\delta\psi_{i_k}} \psi_{i_1} \dots \psi_{i_k} \dots \psi_{i_n} &= (-1)^{(k+1)} \left(\frac{\delta}{\delta\psi_{i_k}} \psi_{i_k} \right) \psi_{i_1} \dots \psi_{i_{k-1}} \psi_{i_{k+1}} \dots \psi_{i_n} \\ &= (-1)^{(k+1)} \psi_{i_1} \dots \psi_{i_{k-1}} \psi_{i_{k+1}} \dots \psi_{i_n} \end{aligned} \quad (1.48)$$

where we assume that i_1, \dots, i_n are pairwise different. The derivative $\frac{\delta}{\delta\psi_i^*}$ is defined similarly. All anticommutators involving derivatives vanish, except for

$$\{\psi_i, \frac{\delta}{\delta\psi_j}\} = \{\psi_i^*, \frac{\delta}{\delta\psi_j^*}\} = \delta_{ij} \quad (1.49)$$

These rules yield an integration by parts formula for Berezin integrals. Let $F^{[n]}[\psi^*, \psi]$ and $G^{[n]}[\psi^*, \psi]$ be two monomials in $\psi_1, \dots, \psi_n, \psi_1^*, \dots, \psi_n^*$ of degrees f and g . Then

$$\begin{aligned} \int \mathcal{D}^{[n]}\psi^* \mathcal{D}^{[n]}\psi F[\psi, \psi^*] \frac{\delta G[\psi, \psi^*]}{\delta\psi_m} &= \\ (-1)^{f+1} \int \mathcal{D}^{[n]}\psi^* \mathcal{D}^{[n]}\psi \frac{\delta F[\psi, \psi^*]}{\delta\psi_m} G[\psi, \psi^*] & \end{aligned} \quad (1.50)$$

A similar formula holds when $\frac{\delta}{\delta\psi_m}$ is replaced by $\frac{\delta}{\delta\psi_m^*}$. In particular, choosing G to be the exponential $\exp\{-(\psi^*, K^{[n]}\psi)\}$, we arrive at the following identity

$$\int \mathcal{D}^{[n]}\psi^* \mathcal{D}^{[n]}\psi \psi_n e^{-(\psi^*, K^{[n]}\psi)} F[\psi, \psi^*] = (K^{[n]-1})_{nm} \int \mathcal{D}^{[n]}\psi^* \mathcal{D}^{[n]}\psi e^{-(\psi^*, K^{[n]}\psi)} \frac{\delta F[\psi, \psi^*]}{\delta\psi_m^*} \quad (1.51)$$

We define the Euclidean action for fermions by the same formula as in Eq. (1.33). We find the boundary conditions at $\tau = 0, \beta$ for Grassmann variables ψ^*, ψ by using the KMS condition. The calculation proceeds along the same lines as for commuting variables, but we get an extra minus sign in the last equality of (1.36), due to fermionic imaginary-time ordering. Thus, we conclude that the boundary conditions for fermionic variables are *anti-periodic*:

$$\psi^*(\tau + \beta) = -\psi^*(\tau), \psi(\tau + \beta) = -\psi(\tau) \quad (1.52)$$

With these boundary conditions, formula (1.39) holds true for fermions. The proof is much easier than the one for bosons, since, after imposing a cutoff, it reduces to pure multi-linear algebra involving (1.50) and (1.51).

In the following chapters we shall not explicitly refer to the finite-volume (infrared) cutoff Λ and the ultraviolet cutoff ($\kappa < \infty$) anymore, and we shall work with functional integrals in a formal way. In most instances it is, however, straightforward to justify our manipulations. For simplicity, we shall study systems at zero temperature; but our method can be used for positive temperatures as well.

We shall assume that the reader is familiar with basic notions and results in the quantum theory of systems of non-relativistic, non-interacting fermions at positive density, such as the Fermi sphere and -surface, the Fermi momentum k_F and the Fermi velocity $v_F = \frac{k_F}{m}$. We shall use units such that $\hbar = 1$ (and thus shall not distinguish between “wave vector” and “momentum”).

2 Weakly coupled electron gases

On a *microscopic scale* ($\approx 10^{-10}\text{m}$), many systems of condensed matter physics can be described approximately in terms of non-relativistic electrons — which are fermions — with two-body interactions, moving in a static background. We are interested in studying such systems in thermal equilibrium at some temperature T and chemical potential μ . The Heisenberg equations of motion and the equations for equilibrium states (KMS condition) of the microscopic system are, i.g., not exactly solvable. Our main interest lies, however, in predicting transport quantities like conductivity, which only depend on physical properties of the system at *mesoscopic length scales* ($\approx 10^{-6}\text{m}$) characterized by a dimensionless parameter $\lambda \gg 1$ (to be thought of as the ratio of meso- to microscopic scale). Such quantities are therefore calculable from processes involving momenta of order k_F/λ around the Fermi

surface, i.e., from properties of the scaling limit (large λ , low frequencies) of the system. One can try to extract information on the scaling limit of the system without explicitly solving the microscopic equations. Techniques that sometimes allow one to do this involve a principle of *dimensional reduction* — i.e., the observation that, in the scaling limit, systems of non-relativistic (free) electrons in d spatial dimensions behave like systems of multi-flavour Dirac fermions in 1+1 dimensions — as well as *renormalization-group-improved* perturbation theory around the non-interacting electron gas. These techniques will be explained in the following chapters.

2.1 Free electrons and dimensional reduction

In this section, we show how a system of non-relativistic free electrons in d -dimensional space can be approximated by multi-flavour relativistic fermions in 1+1 dimensional space-time. We start from the action

$$S_0(\psi^*, \psi) = \sum_{\sigma} \int d^{d+1}x \psi_{\sigma}^*(x) (\partial_0 - \frac{1}{2m} \Delta - \mu) \psi_{\sigma}(x) \quad (2.1)$$

with a prescribed chemical potential μ (related to the Fermi momentum k_F by $\mu = \frac{k_F^2}{2m}$, and to the particle density by $n = 2\tau_d k_F^d / (2\pi)^d$, where τ_d is the volume of the d -dimensional unit ball, and the factor 2 accounts for the spin orientations). For simplicity, we work at zero temperature.

The Euclidean free fermion Green's function is given by ($x = (t, \vec{x})$, $y = (s, \vec{y})$, $t > s$, with t and s now denoting *imaginary time*)

$$\begin{aligned} G_{\sigma\sigma'}^0(x - y) &= \langle e^{tH} \Psi_{\sigma}(\vec{x}) e^{-(t-s)H} \Psi_{\sigma'}^*(\vec{y}) e^{-sH} \rangle_{\mu} \\ &= -\delta_{\sigma\sigma'} \int \tilde{d}^{d+1}k \frac{e^{-ik_0(t-s) + i\vec{k}(\vec{x}-\vec{y})}}{ik_0 - (\frac{k^2}{2m} - \mu)} \end{aligned} \quad (2.2)$$

$$= \delta_{\sigma\sigma'} \int \tilde{d}^d k \Theta\left(\frac{k^2}{2m} - \mu\right) e^{-(t-s)(\frac{k^2}{2m} - \mu)} e^{i\vec{k}(\vec{x}-\vec{y})} \quad (2.3)$$

where

$$\tilde{d}k := \frac{dk}{2\pi} \quad (2.4)$$

The equality (2.3) follows with the help of the residue theorem. In the following, we are interested in physics at a mesoscopic length scale, or, more precisely, in the scaling limit (very low momenta and frequencies). Then it suffices to determine the leading contribution to $G_{\sigma\sigma'}^0(x - y)$ at arguments x and y with $v_F|t - s| + |\vec{x} - \vec{y}| \approx \lambda/k_F$ (where $v_F = \frac{k_F}{m}$ is the Fermi velocity). This contribution comes from modes with momenta in a shell $S_F^{(\lambda)}$ of thickness k_F/λ around the Fermi sphere S_F :

$$\begin{aligned} S_F &= \{\vec{k} \in \mathbb{R}^d \mid \vec{k}^2 = k_F^2\} \\ S_F^{(\lambda)} &= \left\{ \vec{k} \in \mathbb{R}^d \mid \text{dist}(\vec{k}, S_F) \leq \frac{k_F}{2\lambda} \right\} \end{aligned}$$

States of low energy describe electron-hole pairs with momenta in the vicinity of the Fermi surface of the system.

Let us rewrite (2.2) in terms of new variables $\vec{\omega}$, p_{\parallel} and p_0 , with $k_F \vec{\omega} \in S_F$, $p_0 = k_0$, and $\vec{k} = (k_F + p_{\parallel})\vec{\omega}$. If $\vec{k} \in S_F^{(\lambda)}$ then $|p_{\parallel}| \ll k_F$, and we can approximate the integrand of (2.2) by dropping the p_{\parallel}^2 term in the denominator. For large λ and small frequencies,

$$G_{\sigma\sigma'}^0(x - y) \approx \delta_{\sigma\sigma'} \int \frac{d\sigma(\vec{\omega})}{(2\pi)^{d-1}} k_F^{d-1} e^{ik_F \vec{\omega}(\vec{x} - \vec{y})} G_c(t - s, \vec{\omega}(\vec{x} - \vec{y})) \quad (2.5)$$

where $d\sigma(\vec{\omega})$ is the uniform measure on the unit sphere, and G_c is the two-dimensional, chiral propagator

$$G_c(\tau, \xi) = - \int \tilde{d}p_0 \tilde{d}p_{\parallel} \frac{e^{-ip_0\tau + ip_{\parallel}\xi}}{ip_0 - v_F p_{\parallel}} \quad (2.6)$$

We will set $v_F = 1$ in the remainder of this section. Introducing the complex variable $z = i\tau + \xi$ and its complex conjugate \bar{z} , it is easy to verify that $G_c(z)$ satisfies

$$-2i\bar{\partial}G_c(z) = \delta^{(2)}(z)$$

In other words, G_c is the Green's function of a *chiral Dirac fermion* in 1+1 dimensions. Explicitly,

$$G_c(z) = \frac{i}{\pi z} \quad (2.7)$$

One can further approximate the $\vec{\omega}$ -integration in Eq. (2.5) by a summation over discrete directions $\vec{\omega}_j$ by dividing $S_F^{(\lambda)}$ into N small boxes $B_{\vec{\omega}_j}$, $j = 1, \dots, N$, of roughly cubical shape: $B_{\vec{\omega}_j}$ is centered at $\vec{\omega}_j \in S_F$ and has approximate side length k_F/λ ; note that $N \approx \text{const } \lambda^{d-1}$. Thus

$$G_{\sigma\sigma'}^0(x - y) \approx -\delta_{\sigma\sigma'} \sum_{\vec{\omega}_j} e^{i\vec{\omega}_j(\vec{x} - \vec{y})} \int \tilde{d}p_0 \tilde{d}p_{\parallel} \tilde{d}p_{\perp} \frac{e^{-ip_0(t-s) + i\vec{p}(\vec{x} - \vec{y})}}{ip_0 - p_{\parallel}} \quad (2.8)$$

where $\vec{p} = p_{\parallel}\vec{\omega} + \vec{p}_{\perp}$ is a vector in $B_{\vec{\omega}_j} - k_F\vec{\omega}_j$, and $p_0 \in \mathbb{R}$.

We have shown that, in the scaling limit, the behaviour of a d -dimensional, non-relativistic, non-interacting electron gas is described by N flavours of free, chiral Dirac fermions in a 1+1 dimensional space-time. The statement is to be understood in an appropriate sense, since the propagator $G_c(t-s, \vec{\omega}(\vec{x} - \vec{y}))$ actually depends on the “flavour index” $\vec{\omega}$. But the energy of an electron or a hole with momentum \vec{k} only depends on p_{\parallel} , where $p_{\parallel} = \vec{k}\vec{\omega} - k_F$, and $\vec{\omega} = \frac{\vec{k}}{|\vec{k}|}$; it is proportional to $|p_{\parallel}|$, just as for relativistic fermions in 1 + 1 dimensions.

2.2 Weakly coupled electrons and the renormalization group (RG)

We have presented a somewhat unusual description of the free fermion system, because expressions such as (2.8) provide a convenient starting point for treating interacting fermions by a perturbation expansion. In particular, one may hope that interesting physical quantities have an expansion in powers of $1/\lambda$ analogous to the $1/N$ expansion in the Gross-Neveu

model. Later we will see that this is the case for weakly coupled systems, and, as in the Gross-Neveu model, the large number of flavours N encountered in the mesoscopic regime will play an important rôle in the actual calculations. We will, however, also see that perturbation expansions usually cannot be applied “naively”, but have to be improved by applying renormalization group (RG) methods.

We consider systems with a Euclidean action of the form

$$S(\psi^*, \psi) = S_0(\psi^*, \psi) + S_I(\psi^*, \psi)$$

where $S_0(\psi^*, \psi)$ is the quadratic term (2.1), and $S_I(\psi^*, \psi) = g_0 P(\psi^*, \psi)$ is some higher order polynomial interaction to be specified below; we assume that the dimensionless coupling constant g_0 is small. Since correlation functions of the fermion fields $\Psi^\#(x)$ do not have good scaling behaviour, we first split off the oscillatory factors associated with the “direction index” $\vec{\omega}$ (see, e.g., Eq. (2.8)) by expanding the fermions

$$\Psi(x) = \sum_{\vec{\omega}} e^{i\vec{\omega}\vec{x}} \Psi_{\vec{\omega}}(x) \quad (2.9)$$

into quasi-particle operators $\Psi_{\vec{\omega}}(x)$. We are interested in calculating *connected correlators* (temperature-ordered Green functions) of the form

$$(\lambda^\alpha)^{2n} \langle \Psi_{\vec{\omega}_1}(\lambda x_1) \cdots \Psi_{\vec{\omega}_n}(\lambda x_n) \Psi_{\vec{\omega}'_1}^*(\lambda x'_1) \cdots \Psi_{\vec{\omega}'_n}^*(\lambda x'_n) \rangle_\mu^c \quad (2.10)$$

at large distance scales λ ; the factor in front of the bracket, involving the scaling dimension α of the quasi-particle fields, accounts for the scaling behaviour of the quasi-particle operators. Strictly speaking, instead of considering the expectation values above, one ought to “smear out” the fields $\Psi_{\vec{\omega}}^\#$ with some test function (an approximate δ -function) $h(\vec{\xi})$:

$$\Psi_{\vec{\omega},\lambda}^\#(x) = (\lambda)^{-(d+1)} \int d^{d+1}y h((\vec{x} - \vec{y})/\lambda) \Psi_{\vec{\omega}}^\#(y)$$

Because we are interested in the large-scale behaviour, we can work with test functions h of compact support in momentum space, e.g. with $\text{supp } \hat{h} = \{k \in \mathbb{R}^{d+1} | \vec{k}^2 \leq k_F^2\}$. After Fourier transformation, the connected correlation functions of smeared quasi-particle fields are given by

$$(\lambda^{\alpha-d-1})^{2n} \hat{h}(k_1) \cdots \hat{h}(k'_n) \widehat{G_{2n}^c}(k_1/\lambda, \dots, k'_n/\lambda)$$

with

$$\widehat{G_{2n}^c} = \langle \hat{\Psi}_{\vec{\omega}_1}(k_1) \cdots \hat{\Psi}_{\vec{\omega}_n}(k_n) \hat{\Psi}_{\vec{\omega}'_1}^*(k'_1) \cdots \hat{\Psi}_{\vec{\omega}'_n}^*(k'_n) \rangle_\mu^c$$

According to Chapter 1, we can express this correlator in terms of a functional integral

$$\widehat{G_{2n}^c} = \frac{1}{\Xi} \int \mathcal{D}\hat{\psi}^* \mathcal{D}\hat{\psi} e^{-S(\hat{\psi}^*, \hat{\psi})} \hat{\psi}_{\vec{\omega}_1}(k_1) \cdots \hat{\psi}_{\vec{\omega}'_n}^*(k'_n)$$

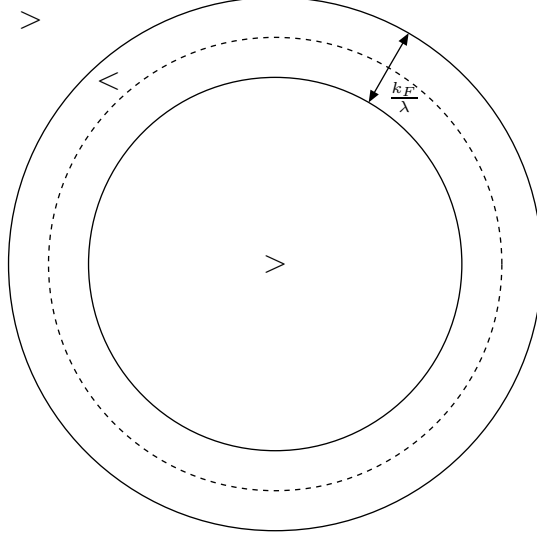


Figure 1: Integrating out degrees of freedom

where $\vec{k}_j \in B_{\vec{\omega}_j} - k_F \vec{\omega}_j$, etc. (because of the restriction on the support of the test functions). We now decompose the fermion fields into “slow” and “fast” modes by writing $\hat{\psi} = \hat{\psi}_< + \hat{\psi}_>$, with

$$\begin{aligned} \text{supp } \hat{\psi}_> &\subset \mathbb{R} \times (\mathbb{R}^d \setminus S_F^{(\lambda)}) && (\text{region } >) \\ \text{supp } \hat{\psi}_< &\subset \mathbb{R} \times S_F^{(\lambda)} && (\text{region } <) \end{aligned}$$

The large-scale behaviour of the system is described by the effective action $S_{\text{eff}}(\hat{\psi}_<^*, \hat{\psi}_<)$ given by

$$e^{-S_{\text{eff}}(\hat{\psi}_<^*, \hat{\psi}_<)} = \frac{1}{\Xi_>} \int \mathcal{D}\hat{\psi}_>^* \mathcal{D}\hat{\psi}_> e^{-S(\hat{\psi}_>^*, \hat{\psi}_<^*, \hat{\psi}_>, \hat{\psi}_<)} \quad (2.11)$$

where we perform the functional integral over fast modes only; the normalization factor $\Xi_>$ is chosen so as to ensure that $S_{\text{eff}}(0, 0) = 0$. Obviously, S_{eff} depends on our choice of the scale λ .

The quadratic part, S_0 , of the action splits into two pieces:

$$S_0(\hat{\psi}^*, \hat{\psi}) = S_{0,>}(\hat{\psi}_>^*, \hat{\psi}_>) + S_{0,<}(\hat{\psi}_<^*, \hat{\psi}_<)$$

The first term and the normalization factor can be absorbed into the definition of the integration measure

$$dP(\hat{\psi}_>^*, \hat{\psi}_>) = \frac{1}{\Xi_>} \mathcal{D}\hat{\psi}_>^* \mathcal{D}\hat{\psi}_> e^{-S_{0,>}(\hat{\psi}_>^*, \hat{\psi}_>)}$$

and we have to calculate

$$\begin{aligned} e^{-S_{\text{eff}}(\hat{\psi}_<^*, \hat{\psi}_<)} &\equiv e^{-S_{0,<}(\hat{\psi}_<^*, \hat{\psi}_<)} \int dP(\hat{\psi}_>^*, \hat{\psi}_>) e^{-S_I(\hat{\psi}_>^*, \hat{\psi}_<^*, \hat{\psi}_>, \hat{\psi}_<)} \\ &= \exp\{-S_{0,<} - \langle S_I \rangle_{G_>}^0 + \frac{1}{2} \langle S_I; S_I \rangle_{G_>} - \frac{1}{3!} \langle S_I; S_I; S_I \rangle_{G_>} - \dots\} \end{aligned} \quad (2.12)$$

The abbreviations

$$\begin{aligned}\langle A; B \rangle &:= \langle AB \rangle - \langle A \rangle \langle B \rangle \\ \langle A; B; C \rangle &:= \langle ABC \rangle - \langle A; B \rangle \langle C \rangle - \langle C; A \rangle \langle B \rangle - \langle B; C \rangle \langle A \rangle - \langle A \rangle \langle B \rangle \langle C \rangle\end{aligned}$$

etc., denote connected correlators; the subscript “ $G^0_{>}$ ” indicates that the expectations $\langle (\cdot) \rangle_{G^0_{>}}$ are calculated with the help of the infrared-cutoff free fermion propagator $G^0_{>}$, in accordance with the functional measure $dP(\hat{\psi}^*, \hat{\psi}_{>})$. The second equality in (2.12) states that the effective action is given by sums over connected diagrams. This is the so-called *linked cluster theorem*. Eq. (2.12) also shows that S_{eff} contains i.g. far more interaction terms than S_I ; for weakly coupled systems, however, the original couplings will remain the dominant ones. The proofs of equation (2.12) and of the linked cluster theorem are standard. We therefore omit them. *Feynman rules* for the computation of S_{eff} are easily derived (we do not present the details).

In order to analyze the perturbation series in (2.12) further, we must specify the interactions in the system: We assume that the leading term in S_I is a two-body interaction

$$S_I(\psi^*, \psi) = \frac{g_0}{2} k_F^{1-d} \sum_{\sigma, \sigma'} \int d^{d+1}x d^{d+1}y : \psi^*_{\sigma}(x) \psi_{\sigma}(x) v(\vec{x} - \vec{y}) \delta(x^0 - y^0) \psi^*_{\sigma'}(y) \psi_{\sigma'}(y) : \quad (2.13)$$

where $v(\vec{x} - \vec{y})$ is a smooth short-range potential (we choose units such that $\hbar = 1$; the factor k_F^{1-d} then ensures that g_0 is dimensionless). It is useful to estimate how close to S_F one can get with “naive” perturbation theory: The free propagator, $-(ik_0 - \vec{\omega} \vec{k})^{-1}$, is regular in region $>$, it is in fact of order λ/k_F . The size of the two-body interaction associated with the graph



(2.14)

is therefore roughly equal to $|g_0 \lambda^2|$ (which is dimensionless). The wavy line represents the interaction potential which is of order g_0 , the oriented solid lines represent electron “half-propagators” each of which is of order $\lambda^{\frac{1}{2}}$. As long as $\lambda \ll 1/\sqrt{g_0}$, there are no convergence problems with the perturbation series — except that the number of diagrams seemingly grows too fast; but their relative signs ensure appropriate cancellations (Pauli principle).

This crude estimate shows that we cannot pass to the limit $\lambda \rightarrow \infty$ without meeting infrared divergence problems in the perturbation series. The method to control the scaling limit is RG improved perturbation theory. It allows us to analyze the large-scale behaviour of the system by carefully keeping track of the (relative) growth of the various terms in the effective action (quadratic part as well as couplings) when the integration over fermion modes approaches those near the Fermi surface. In fact, for the interaction (2.13), the actual

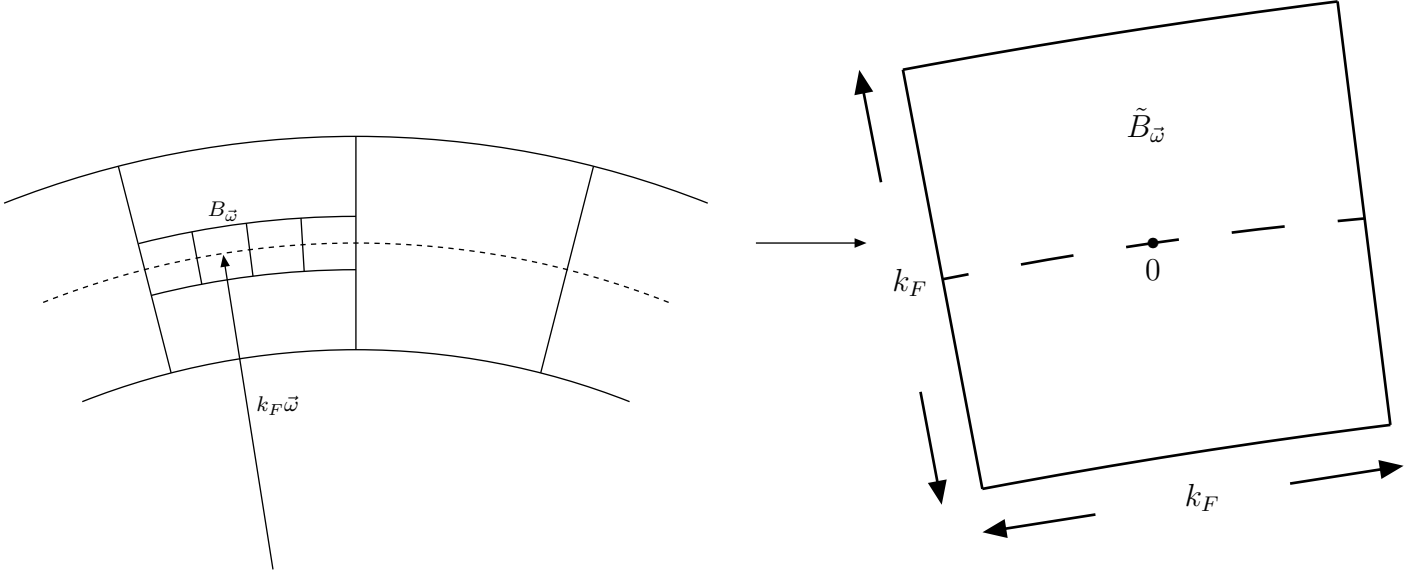


Figure 2: The renormalization group procedure

RG computations are not terribly involved, since S_{eff} has an expansion in powers of $1/\lambda$, and focusing on the leading order in $\frac{1}{\lambda}$ drastically reduces the number of diagrams that have to be calculated.

We shall apply an iterative RG scheme patterned on Wilson’s approach. In the “zeroth” step, we choose some large scale λ_0 such that $\lambda_0 \ll 1/\sqrt{g_0}$ and calculate the corresponding effective action $S_{\text{eff}}^{(0)}$ perturbatively, as in (2.12) — to leading order in $1/\lambda_0$. The action $S_{\text{eff}}^{(o)}$ depends on a collection of modes corresponding to wave vectors that are located in a shell of width $\frac{k_F}{\lambda_0}$ around the Fermi surface S_F . Although not essential for our method, we divide this shell into $N \approx \text{const } \lambda_0^{d-1}$ sectors (boxes), as in the previous section. Furthermore, we rescale all momenta so that, instead of belonging to $B_{\vec{\omega}_j}$, they are contained in boxes $\tilde{B}_{\vec{\omega}_j}$ of side length $\approx k_F$. Our RG procedure consists in iterating the following two steps:

Dec Choose some (fixed) integer $M > 1$ and integrate over the fermion modes corresponding to wave vectors in $S_F^{(\lambda)} \setminus S_F^{(M\lambda)}$ (where $\lambda = M^j \lambda_0$, for some j).

Resc Divide each sector in $S^{(M\lambda)}$ into M^{d-1} new sectors to restore the cubical shape of the sectors. Then rescale all momenta by $k \mapsto \tilde{k} = Mk$.

To determine the RG flow of the couplings, we have to find out how the various terms in the action transform under rescaling and which diagrams contribute, during the integration process, to leading order in $1/\lambda$.

3 The renormalization group flow

In this chapter we attempt to explicitly implement the renormalization group procedure described in the last chapter. Our aim is to understand the renormalization group flow to lowest order in $\frac{1}{\lambda_0}$ and for small values of g_0 .

3.1 Scaling of action and fields

We shall begin with the discussion of the second step and of the scaling dimensions of the different terms in the effective action. Since the decimation of degrees of freedom is most explicitly described in momentum-space, our renormalization group procedure is implemented in the momentum-space representation. Thus, we first calculate the action in terms of the sector fields discussed in the previous chapter.

Let $\lambda \gg 1$, and assume that the degrees of freedom not lying in the shell $S_F^{(\lambda)}$ have already been integrated out. We divide $S_F^{(\lambda)}$ into boxes of roughly cubical shape and approximate side length $\frac{k_F}{\lambda}$. The number of such boxes is $N \approx \omega_{d-1} \lambda^{d-1}$, where ω_{d-1} is the surface volume of the unit sphere in d spatial dimensions. Let $B_{\vec{\omega}}$, $|\vec{\omega}| = 1$, denote the box which is centered at $k_F \vec{\omega}$. The support of those modes

$$\hat{\psi}_\sigma(k) = \int d^{d+1}x e^{i(k_0 t - \vec{k} \vec{x})} \psi_\sigma(x) \quad (3.1)$$

that have not been integrated out, yet, is contained in $\mathbb{R} \times S_F^{(\lambda)}$. *Sector fields* are defined as

$$\psi_{\vec{\omega}, \sigma}(x) := \int_{\mathbb{R} \times B_{\vec{\omega}}} d^{d+1}k e^{-i(k_0 t - (\vec{k} - k_F \vec{\omega}) \vec{x})} \hat{\psi}_\sigma(k) \quad (3.2)$$

The Fourier transform of $\psi_{\vec{\omega}, \sigma}(x)$ has support in $\mathbb{R} \times (B_{\vec{\omega}} - k_F \vec{\omega})$, and

$$\psi_\sigma(x) = \sum_{\vec{\omega}} e^{i k_F \vec{\omega} \vec{x}} \psi_{\vec{\omega}, \sigma}(x)$$

Let us first look at the quadratic part of the effective action of the modes localized in the shell $S_F^{(\lambda)}$. We temporarily assume that this part has the form (2.1) (there will be a finite wave function renormalization which we are neglecting at this point). Inserting the Fourier transformed fields yields

$$\begin{aligned} S_0(\psi^*, \psi) &= - \sum_{\sigma} \int d^{d+1}k d^{d+1}q \hat{\psi}_\sigma^*(k) \left(i q_0 - \frac{\hbar^2 |\vec{q}|^2}{2m} + \mu \right) \hat{\psi}_\sigma(q) (2\pi)^{d+1} \delta(k - q) \\ &= - \sum_{\vec{\omega}, \sigma} \int_{\mathbb{R} \times (B_{\vec{\omega}} - k_F \vec{\omega})} d^{d+1}p \hat{\psi}_{\vec{\omega}, \sigma}^*(p) \left(i p_0 - v_F \vec{\omega} \vec{p} + O\left(\frac{1}{\lambda^2}\right) \right) \hat{\psi}_{\vec{\omega}, \sigma}(p) \end{aligned} \quad (3.3)$$

with $v_F = \frac{k_F}{m}$. In the last line, we have put the term quadratic in p into an error term $O(\frac{1}{\lambda^2})$. The leading part of the inverse propagator in S_0 is of order $O(\frac{1}{\lambda})$. Since we always restrict our analysis to the leading order in $\frac{1}{\lambda}$, we can omit this error term in what follows.

Next, we consider the quartic term (2.13). We insert the expansion of the electron fields in terms of sector fields and get

$$S_I(\psi^*, \psi) = g_0 k_F^{1-d} \frac{1}{2} \sum_{\substack{\vec{\omega}_1, \dots, \vec{\omega}_4 \\ \sigma, \sigma'}} \int d^{d+1}x d^{d+1}y e^{-ik_F(\vec{\omega}_1 - \vec{\omega}_4)\vec{x}} e^{-ik_F(\vec{\omega}_2 - \vec{\omega}_3)\vec{y}} \\ \psi_{\vec{\omega}_1, \sigma}^*(x) \psi_{\vec{\omega}_2, \sigma'}^*(y) v(\vec{x} - \vec{y}) \delta(x_0 - y_0) \psi_{\vec{\omega}_3, \sigma'}(y) \psi_{\vec{\omega}_4, \sigma}(x) \quad (3.4)$$

In terms of the Fourier transform of the potential

$$v(\vec{x}) = \int \vec{d}^d k e^{i\vec{k}\vec{x}} \hat{v}(\vec{k})$$

we can write

$$v(\vec{x}) \delta(x^0) = \int \vec{d}^{d+1} k e^{-i(k_0 x^0 - \vec{k}\vec{x})} \hat{v}(\vec{k})$$

Since we assume that $v(\vec{x})$ is a *short-range potential*, $\hat{v}(\vec{k})$ is a smooth function of \vec{k} . Applying Fourier transformation to the sector fields in (3.4), we obtain

$$S_I = g_0 k_F^{1-d} \frac{1}{2} \sum_{\substack{\vec{\omega}_1, \dots, \vec{\omega}_4 \\ \sigma, \sigma'}} \int \vec{d}^{d+1} p_1 \dots \vec{d}^{d+1} p_4 \int \vec{d}^{d+1} k \\ 2\pi \delta(p_{1,0} - p_{4,0} - k_0) 2\pi \delta(p_{2,0} - p_{3,0} + k_0) \\ (2\pi)^d \delta(\vec{p}_1 - \vec{p}_4 - \vec{k} + k_F(\vec{\omega}_1 - \vec{\omega}_2)) (2\pi)^d \delta(\vec{p}_2 - \vec{p}_3 - \vec{k} + k_F(\vec{\omega}_2 - \vec{\omega}_3)) \\ \hat{\psi}_{\vec{\omega}_1, \sigma}^*(p_1) \hat{\psi}_{\vec{\omega}_2, \sigma'}^*(p_2) \hat{v}(\vec{k}) \hat{\psi}_{\vec{\omega}_3, \sigma'}(p_3) \hat{\psi}_{\vec{\omega}_4, \sigma}(p_4)$$

where the p_i -integrations range over the boxes $B_{\vec{\omega}_i} - k_F \vec{\omega}_i$. Performing the k -integration, we obtain the expression

$$g_0 k_F^{1-d} \frac{1}{2} \sum_{\substack{\vec{\omega}_1, \dots, \vec{\omega}_4 \\ \sigma, \sigma'}} \int \vec{d}^{d+1} p_1 \dots \vec{d}^{d+1} p_4 2\pi \delta(p_{1,0} + p_{2,0} - p_{3,0} - p_{4,0}) \\ (2\pi)^d \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4 + k_F(\vec{\omega}_1 + \vec{\omega}_2 - \vec{\omega}_3 - \vec{\omega}_4)) \\ \hat{v}(\vec{p}_1 - \vec{p}_4 + k_F(\vec{\omega}_1 - \vec{\omega}_4)) \hat{\psi}_{\vec{\omega}_1, \sigma}^*(p_1) \hat{\psi}_{\vec{\omega}_2, \sigma'}^*(p_2) \hat{\psi}_{\vec{\omega}_3, \sigma'}(p_3) \hat{\psi}_{\vec{\omega}_4, \sigma}(p_4)$$

Considering the argument of the second δ -distribution appearing in the integrand, the part containing the $\vec{\omega}_i$'s is of order k_F , while the part involving the p_i 's is only of order $\frac{k_F}{\lambda}$. Therefore, for the argument of this δ -distribution to vanish, the term containing the $\vec{\omega}_i$'s must be zero (to lowest order in $\frac{1}{\lambda}$). Furthermore, as the Fourier transform \hat{v} of the interaction is a smooth function, its value under the integral sign is well approximated by $\hat{v}(k_F(\vec{\omega}_1 - \vec{\omega}_4))$, dropping the box momenta. Consequently we get

$$S_I = g_0 k_F^{1-d} \frac{1}{2} \sum_{\substack{\vec{\omega}_1, \dots, \vec{\omega}_4 \\ \sigma, \sigma'}} \hat{v}(k_F(\vec{\omega}_1 - \vec{\omega}_4)) \int \vec{d}^{d+1} p_1 \dots \vec{d}^{d+1} p_4 \\ (2\pi)^{d+1} \delta(p_1 + p_2 - p_3 - p_4) \hat{\psi}_{\vec{\omega}_1, \sigma}^*(p_1) \hat{\psi}_{\vec{\omega}_2, \sigma'}^*(p_2) \hat{\psi}_{\vec{\omega}_3, \sigma'}(p_3) \hat{\psi}_{\vec{\omega}_4, \sigma}(p_4) \\ + \text{terms of higher order in } \frac{1}{\lambda} \quad (3.5)$$

Instead of studying the quartic term of the original (microscopic) action of the electron gas, we should actually study the quartic (and higher-degree) terms of the *effective action* at scale $\frac{k_F}{\lambda}$, with $\lambda = \lambda_0 \gg 1$ and $|g_0 \lambda_0^2| \ll 1$. Using cluster expansions to integrate out the degrees of freedom labeled by momenta $\vec{k} \notin S_F^{(\lambda_0)}$, one can show that, for weakly coupled systems, the quartic term of the effective action at scale $\frac{k_F}{\lambda_0}$ still has the form given in eq. (3.5), except that $\hat{v}(k_F(\vec{\omega}_1 - \vec{\omega}_4))$ is replaced by a coupling function $g(\vec{\omega}_1, \dots, \vec{\omega}_4) \approx \hat{v}(k_F(\vec{\omega}_1 - \vec{\omega}_4))$, with $\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4$. (Moreover, terms of degree larger than four in ψ^*, ψ are very small.)

Next, we determine the scaling dimensions of action and fields. We rescale the fields in such a way, that the supports of the Fourier transformed, rescaled sector fields are boxes of roughly cubical shape and with side length approximately equal to k_F , and such that the quadratic part of the action remains unchanged to leading order in $\frac{1}{\lambda}$. The first condition implies, that we should scale momenta according to $p = \tilde{p}/\lambda$. In configuration space this corresponds to the scaling $x = \lambda\xi$. Thus the scaled sector fields are

$$\tilde{\psi}_{\vec{\omega},\sigma}(\xi) = \lambda^\alpha \psi_{\vec{\omega},\sigma}(\lambda\xi)$$

where the *scaling dimension* α of the sector fields still has to be determined. The Fourier transformed sector fields are

$$\hat{\tilde{\psi}}_{\vec{\omega},\sigma}(\tilde{p}) = \lambda^{\alpha-d-1} \hat{\psi}_{\vec{\omega},\sigma}\left(\frac{\tilde{p}}{\lambda}\right)$$

The support of $\hat{\tilde{\psi}}_{\vec{\omega},\sigma}$ lies in $\tilde{B}_{\vec{\omega}} = \lambda(B_{\vec{\omega}} - k_F \vec{\omega})$. Indeed, $\tilde{B}_{\vec{\omega}}$ is a roughly cubical box with approximate side length k_F . Inserting the scaled, Fourier-transformed fields into the quadratic part (3.3) of the action yields

$$\begin{aligned} S_0 &= -\lambda^{2(d+1-\alpha)} \lambda^{-(d+1)} \sum_{\vec{\omega},\sigma} \int_{\mathbf{R} \times \tilde{B}_{\vec{\omega}}} \tilde{d}^{d+1} \tilde{p} \hat{\tilde{\psi}}_{\vec{\omega},\sigma}^*(\tilde{p}) \left(\frac{1}{\lambda} (i\tilde{p}_0 - v_F \vec{\omega} \vec{\tilde{p}}) + O\left(\frac{1}{\lambda^2}\right) \right) \hat{\tilde{\psi}}_{\vec{\omega},\sigma}(\tilde{p}) \\ &= -\lambda^{d-2\alpha} \sum_{\vec{\omega},\sigma} \int_{\mathbf{R} \times \tilde{B}_{\vec{\omega}}} \tilde{d}^{d+1} \tilde{p} \hat{\tilde{\psi}}_{\vec{\omega},\sigma}^*(\tilde{p}) \left(i\tilde{p}_0 - v_F \vec{\omega} \vec{\tilde{p}} + O\left(\frac{1}{\lambda}\right) \right) \hat{\tilde{\psi}}_{\vec{\omega},\sigma}(\tilde{p}) \end{aligned}$$

For the free part of the action to have the same form as for the unscaled fields, we have to choose $\alpha = \frac{d}{2}$. Thus the scaled fields are

$$\tilde{\psi}_{\vec{\omega},\sigma}(\xi) = \lambda^{\frac{d}{2}} \psi_{\vec{\omega},\sigma}(\lambda\xi) \quad (3.6)$$

$$\hat{\tilde{\psi}}_{\vec{\omega},\sigma}(\tilde{p}) = \lambda^{-(\frac{d}{2}+1)} \hat{\psi}_{\vec{\omega},\sigma}(\tilde{p}/\lambda) \quad (3.7)$$

Let us determine the scaling behaviour of a term in the action of the form

$$\begin{aligned} S_n &= \frac{1}{n!} \sum_{\substack{\vec{\omega}_1 + \dots + \vec{\omega}_n = \vec{\omega}_{n+1} + \dots + \vec{\omega}_{2n} \\ \sigma_1, \dots, \sigma_{2n}}} \int \tilde{d}^{d+1} p_1 \dots \tilde{d}^{d+1} p_{2n} \\ &\quad w(p_1, \dots, p_{2n}) \hat{\psi}_{\vec{\omega}_1, \sigma_1}^*(p_1) \dots \hat{\psi}_{\vec{\omega}_{2n}, \sigma_{2n}}(p_{2n}) (2\pi)^{d+1} \delta(p_1 + \dots - p_{2n}) \end{aligned} \quad (3.8)$$

The function w is assumed to be homogeneous of degree $\kappa \in \mathbb{R}$, i.e.

$$w(sp_1, \dots, sp_{2n}) = s^\kappa w(p_1, \dots, p_{2n})$$

Expressing the sector fields in terms of the scaled fields thus yields

$$S_n = \frac{1}{n!} \lambda^{-((n-1)d-1+\kappa)} \sum_{\substack{\vec{\omega}_1 + \dots + \vec{\omega}_n = \vec{\omega}_{n+1} + \dots + \vec{\omega}_{2n} \\ \sigma_1, \dots, \sigma_{2n}}} \int \vec{d}^{d+1} \tilde{p}_1 \dots \vec{d}^{d+1} \tilde{p}_{2n} \\ w(\tilde{p}_1, \dots, \tilde{p}_{2n}) \hat{\psi}_{\vec{\omega}_1, \sigma_1}^*(\tilde{p}_1) \dots \hat{\psi}_{\vec{\omega}_{2n}, \sigma_{2n}}(\tilde{p}_{2n}) (s\pi)^{d+1} \delta(\tilde{p}_1 + \dots - \tilde{p}_{2n}) \quad (3.9)$$

The exponent of λ is called the *scaling dimension* of S_n , i.e., the scaling dimension of S_n is $-((n-1)d-1+\kappa)$. With $n=2$ and $\kappa=0$, the quartic term (3.5) turns out to have scaling dimension $1-d$. Furthermore, we see that quartic terms of higher degree in the momenta have a smaller scaling dimension. The same is true for contributions to the effective action that are of higher degree in the fields.

The effective action in terms of the rescaled fields thus reads

$$S_{\text{eff}} = Z^{-1} \sum_{\vec{\omega}, \sigma} \int \vec{d}^{d+1} \tilde{p} \hat{\psi}_{\vec{\omega}, \sigma}^*(\tilde{p}) (i\tilde{p}_0 - v_F \vec{\omega} \vec{\tilde{p}}) \hat{\psi}_{\vec{\omega}, \sigma}(\tilde{p}) \\ + \frac{1}{2} \frac{1}{\lambda^{d-1}} Z^{-2} \sum_{\substack{\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4 \\ \sigma, \sigma'}} g(\vec{\omega}_1, \dots, \vec{\omega}_4) \int \vec{d}^{d+1} \tilde{p}_1 \dots \vec{d}^{d+1} \tilde{p}_4 \\ \hat{\psi}_{\vec{\omega}_1, \sigma}^*(\tilde{p}_1) \hat{\psi}_{\vec{\omega}_2, \sigma'}^*(\tilde{p}_2) \hat{\psi}_{\vec{\omega}_3, \sigma'}(\tilde{p}_3) \hat{\psi}_{\vec{\omega}_4, \sigma}(\tilde{p}_4) (2\pi)^{d+1} \delta(\tilde{p}_1 + \tilde{p}_2 - \tilde{p}_3 - \tilde{p}_4) \\ + \text{terms of higher order in } \frac{1}{\lambda} \quad (3.10)$$

In expression (3.10) for S_{eff} we have introduced a wave function renormalization constant Z , in order to indicate that the quadratic part of the effective action may flow under renormalization (decimation of degrees of freedom and rescaling). In the next section we shall derive renormalization group flow equations for Z , v_F and the coupling constants $g(\vec{\omega}_1, \dots, \vec{\omega}_4)$. These equations will determine the dependence of Z , v_F and $g(\vec{\omega}_1, \dots, \vec{\omega}_4)$ on the scale parameter λ .

The expression for S_{eff} on the right hand side of (3.10) shows that the inverse propagator of the sector fields $\hat{\psi}_{\vec{\omega}, \sigma}^{\#}(\tilde{p})$ is diagonal in the sector index $\vec{\omega}$ and that it only depends on p_0 and $p_{\parallel} = \vec{\omega} \vec{p}$ (but *not* on $\vec{p}_{\perp} = \vec{p} - (\vec{\omega} \vec{p}) \vec{\omega}$). These features are an aspect of the principle of *dimensional reduction* from $d+1$ to $1+1$ dimensions. Indeed, we observe a rather striking formal similarity between expression (3.10) and the action of the Gross-Neveu model of interacting, relativistic Dirac fields in $1+1$ space-time dimensions: The sector index $\vec{\omega}$ plays the rôle of the *flavour index* of the Dirac fields in the Gross-Neveu model; the number of distinct sector indices, $\approx \text{const } \lambda^{d-1}$, corresponds to the number, N , of flavours of fermions in the Gross-Neveu model. The coupling constants, $\lambda^{1-d} g(\vec{\omega}_1, \dots, \vec{\omega}_4)$, correspond to the coupling constant, $\frac{g_{GN}}{N}$, of the quartic term in the Gross-Neveu model. (The correspondence

between $\frac{g_{GN}}{N}$ and $\lambda^{1-d}g(\vec{\omega}, -\vec{\omega}, \vec{\omega}', -\vec{\omega}') \equiv \lambda^{1-d}g_{BCS}(\vec{\omega} \cdot \vec{\omega}')$ is particularly precise, as discussed in Chapter 4.)

A powerful method to analyze the Gross-Neveu model is the $\frac{1}{N}$ - expansion. This suggests to analyze non-relativistic, interacting electron gases with the help of a $\frac{1}{\lambda}$ - expansion (with $\frac{1}{\lambda} \sim \frac{1}{N}$, for $d = 2$), and this is precisely what we shall do in the remaining sections, following beautiful ideas of Feldman, Magnen, Rivasseau and Trubowitz. In the Gross-Neveu model, Z and the velocity of light (corresponding to v_F) do *not* flow under renormalization, to leading order in $\frac{1}{N}$. This suggests that, for the electron gas, Z and v_F do not flow under renormalization to leading order in $\frac{1}{\lambda}$; a prediction that will turn out to be correct!

In the following sections, we shall always work in momentum space and with rescaled sector fields. We shall thus omit the “hat” and the “tilde” from the rescaled fields on momentum space. Our analysis will be based on the assumption that $\lambda_0 \gg 1$ and that all coupling constants $g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4) = g(\vec{\omega}_1, \dots, \vec{\omega}_4)|_{\lambda=\lambda_0} \ll 1$. We shall determine the renormalization flow to leading order in $\frac{1}{\lambda_0}$ (sometimes omitting terms that are of leading order in $\frac{1}{\lambda_0}$, but of high order in $g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$).

3.2 Integrating out modes

In the last section, we have understood how the different parts of the effective action of an interacting electron gas behave under rescaling. Here we turn to the second step of the renormalization group method — the decimation of degrees of freedom. Initially, we assume the degrees of freedom in $\mathbb{R}^d \setminus S_F^{(\lambda_0)}$ to be integrated out, as discussed in Sect. 2.2. In the j 'th step, the degrees of freedom localized in $S_F^{(M^{j-1}\lambda_0)} \setminus S_F^{(M^j\lambda_0)}$ have to be eliminated, where M is a positive integer ≥ 2 . Thus the scaling factor at scale j is $\lambda_j = \lambda_0 M^j$. At scale 0 we are given $Z^{(0)}$, $v_F^{(0)}$ and functions $g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$ of the unit vectors $\vec{\omega}_1, \dots, \vec{\omega}_4$, with $\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4$. As announced, we assume that $\lambda_0 \gg 1$ and $\max_{\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3} \{g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)\} \ll 1$.

From this point on, we omit the spin indices when no confusion arises. We propose to first discuss the possible intersector scattering geometries, as we would like to better understand the structure of the coupling constants $g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$. How many independent $g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$'s exist? For $d = 3$ we suppose that $\vec{\omega}_3 \neq -\vec{\omega}_4$. On the unit sphere, there are $N^{(0)} \approx \text{const } \lambda_0^{d-1}$ different $\vec{\omega}$'s. All choices $\vec{\omega}_1, \vec{\omega}_2$ with $\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4$ lie on a cone containing $\vec{\omega}_3$ and $\vec{\omega}_4$ with symmetry axis $\vec{\omega}_3 + \vec{\omega}_4$. Therefore there are $O(\lambda_0^{d-2})$ choices. Only when $\vec{\omega}_3 = -\vec{\omega}_4$, there are $N^{(0)} \approx \text{const } \lambda_0^{d-1}$ choices. Similarly, in $d = 2$, there are exactly two choices for $\vec{\omega}_1, \vec{\omega}_2$, if $\vec{\omega}_3 \neq -\vec{\omega}_4$, and $N^{(0)} \approx \text{const } \lambda_0$ choices if $\vec{\omega}_3 = -\vec{\omega}_4$.

Couplings involving incoming states with $\vec{\omega}_3 \neq -\vec{\omega}_4$ shall be denoted by $g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$. Couplings that involve sector indices with $\vec{\omega}_3 = -\vec{\omega}_4$ or, equivalently, $\vec{\omega}_1 = -\vec{\omega}_2$ shall be denoted by $g_{BCS}^{(0)}(\vec{\omega}_1, \vec{\omega}_4)$ (for *BCS-scattering*). The latter will prove to be crucial for the understanding of the superconducting state. We observe that, because of rotation invariance, the coupling g_{BCS} is only a function of the angle between $\vec{\omega}_1$ and $\vec{\omega}_4$.

Technically, we will carry out the decimation of degrees of freedom using perturbation theory. If, under the renormalization group transformation, $g^{(j)}(\vec{\omega}_1, \dots, \vec{\omega}_4) = g(\vec{\omega}_1, \dots, \vec{\omega}_4)|_{\lambda=\lambda_j}$

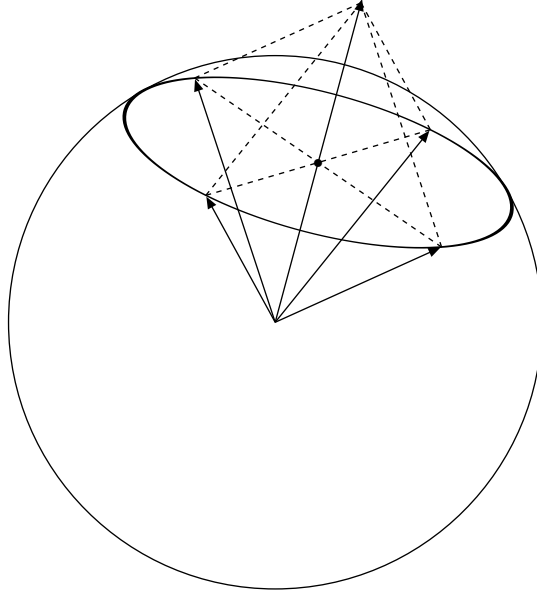


Figure 3: Sector momentum conservation in $d = 3$

increases with growing j (and decreasing distance to the Fermi surface), the validity of perturbative results eventually breaks down.

We start explicit calculations with the renormalization of the electron propagator when passing from scale λ_0 to λ_1 . We know that every interaction squiggle provides a factor $\lambda_0^{1-d} g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$. What are the dominant radiative corrections of the electron propagator? The two possible one loop diagrams are

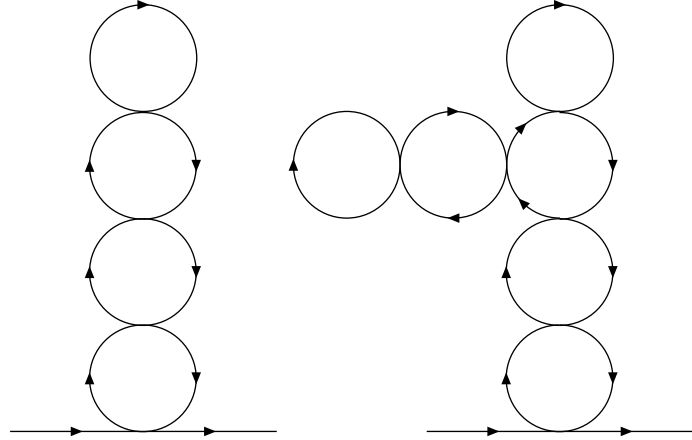
(3.11)

We call them *tadpole* and *turtle graph*. We shall estimate their amplitudes. There is one interaction squiggle of order $\frac{1}{\lambda_0^{d-1}}$ and $N^{(0)} \approx \text{const } \lambda_0^{d-1}$ choices for the inner particle sectors, denoted by $\vec{\omega}'$. Therefore, both these graphs correspond to contributions of order zero in $\frac{1}{\lambda_0}$. It will prove to be useful to introduce a simplified graphical notation. Namely, we replace the squiggle by a point, i.e., each vertex is represented by a cross with two incoming and two outgoing lines. The new vertex stands for the sum of the two old vertices that yield the same new vertex. For example, the diagrams in (3.11) are now represented by one graph,

namely


(3.12)

What about combinations of these first order corrections?

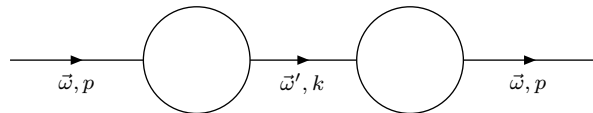

(3.13)

For every vertex corresponding to a factor $\frac{1}{\lambda_0^{d-1}}$, there are $N^{(0)} \approx \text{const } \lambda_0^{d-1}$ loop sectors to choose from. Therefore all these graphs correspond to amplitudes of order one. But, as a matter of fact, one observes with amazement that all their amplitudes vanish. The reason is that, in all these graphs, there is an oriented loop involving two particle lines with the same sector label $\vec{\omega}'$ and the same box momenta and energies. For each such loop, the integration over the loop momentum yields a factor

$$\int d\vec{k} \int d\vec{k}_0 \left(\frac{1}{ik_0 - v_F \vec{\omega}' \vec{k}} \right)^2$$

in the total amplitude. The two poles of the integrand coincide, and the integrand decays quadratically at infinity. The residue theorem then tells us that the integral vanishes.

Furthermore, we know that graphs which are not *one particle irreducible (1PI)* yield a vanishing amplitude. Indeed, a graph of this type has the following form


(3.14)

and one has to sum over $\vec{\omega}'$ and to integrate over k with $\frac{k_F}{M} \leq |\vec{\omega}' \vec{k}| \leq k_F$. The empty circles stand for arbitrary subdiagrams with the indicated external legs. Because of momentum

conservation, there is a factor $\delta_{\vec{\omega}, \vec{\omega}'} \delta(p - k)$. Hence only $\vec{\omega}' = \vec{\omega}$ contributes. But, in this case, the argument of the δ -distribution never vanishes since $|\vec{\omega}\vec{p}| < \frac{k_F}{M}$. Consequently the amplitude is zero.

The graphs not taken into account so far are of order $O(1/\lambda_0)$ or higher. As an example, let us consider the graph


(3.15)

Clearly, $\vec{\omega}_3$ is determined by $\vec{\omega}$, $\vec{\omega}_1$ and $\vec{\omega}_2$. Thus there remain two summations, over $\vec{\omega}_1$ and $\vec{\omega}_2$. If $\vec{\omega}_1$ is different from $-\vec{\omega}$, then the number of $\vec{\omega}_2$'s satisfying the relations $|\vec{\omega} + \vec{\omega}_2 - \vec{\omega}_1| = 1$ is of order $O(\lambda_0^{d-2})$. Hence the number of summands is at most of order $O(\lambda_0^{2d-3})$. Since each of the squiggles gives a factor $\frac{1}{\lambda_0^{d-1}}$, the amplitude corresponding to this graph is of order $O(1/\lambda_0)$, as claimed.

Conclusion: The corrections of lowest order in $\frac{1}{\lambda_0}$ to the electron propagator arise from the tadpole and the turtle graph. In order to identify the renormalized quantities more easily, we work with a renormalized action in its most general form (dropping terms of higher order in $\frac{1}{\lambda}$, $\lambda = \lambda_0, \lambda_1, \lambda_2, \dots$)

$$\begin{aligned}
S_{\text{eff}} = & -Z^{-1} \sum_{\vec{\omega}, \sigma} \int \tilde{d}^{d+1} k \psi_{\vec{\omega}, \sigma}^*(k) (ik_0 - v_F \vec{\omega} \vec{k}) \psi_{\vec{\omega}, \sigma}(k) \\
& - \lambda Z^{-1} \sum_{\vec{\omega}, \sigma} \int \tilde{d}^{d+1} k \psi_{\vec{\omega}, \sigma}^*(k) \delta \mu \psi_{\vec{\omega}, \sigma}(k) + O\left(\frac{1}{\lambda}\right) \\
& + \frac{1}{2} \frac{1}{\lambda^{d-1}} Z^{-2} \sum_{\substack{\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4 \\ \sigma, \sigma'}} g(\vec{\omega}_1, \dots, \vec{\omega}_4) \int \tilde{d}^{d+1} k_1 \dots \tilde{d}^{d+1} k_4
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
& \psi_{\vec{\omega}_1, \sigma}^*(k_1) \psi_{\vec{\omega}_2, \sigma'}^*(k_2) \psi_{\vec{\omega}_3, \sigma'}(k_3) \psi_{\vec{\omega}_4, \sigma}(k_4) (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4) \\
& + O\left(\frac{1}{\lambda^d}\right)
\end{aligned} \tag{3.17}$$

On the right side of (3.16), the wave vectors $k = (k_0, \vec{k})$ are constrained to belong to $\mathbb{R} \times \tilde{B}_{\vec{\omega}}$, for a given sector momentum $\vec{\omega}$. The factor λ in front of the second term expresses the fact that this term has scaling dimension 1. Denoting the inverse propagator by Γ , the sum of all quadratic terms in the effective action can be written as

$$\sum_{\sigma, \vec{\omega}} \int \tilde{d}^{d+1} k \psi_{\sigma, \vec{\omega}}^*(k) \Gamma_{\vec{\omega}}(k) \psi_{\sigma, \vec{\omega}}(k) \tag{3.18}$$

where

$$\Gamma_{\vec{\omega}}(0) = -\lambda Z^{-1} \delta \mu \tag{3.19}$$

$$\frac{\partial}{\partial p_0} \Gamma_{\vec{\omega}}(p) \big|_{p=0} = -i Z^{-1} \tag{3.20}$$

$$\vec{\omega} (\vec{\nabla}_{\vec{p}} \Gamma_{\vec{\omega}}(p) \big|_{p=0}) = v_F Z^{-1} \tag{3.21}$$

Because the tadpole and the turtle graph correspond to p -independent amplitudes, there are no corrections to Z and v_F in leading order in $\frac{1}{\lambda_0}$:

$$Z^{(1)} = Z^{(0)} + O((g^{(0)})^2/\lambda_0) \quad (3.22)$$

$$v_F^{(1)} = v_F^{(0)} + O((g^{(0)})^2/\lambda_0) \quad (3.23)$$

Let us compute the amplitude of the tadpole graph renormalizing the chemical potential:

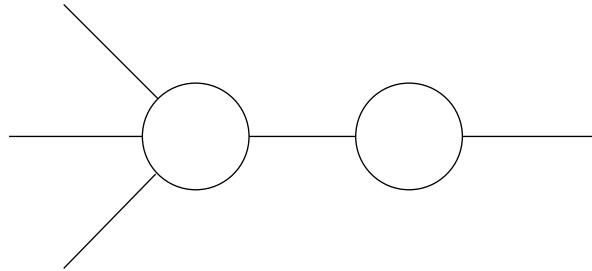
$$\frac{1}{\lambda_0^{d-1}} \sum_{\vec{\omega}', \sigma'} g^{(0)}(\vec{\omega}, \vec{\omega}', \vec{\omega}', \vec{\omega}) \int \vec{dk}_\perp \int_{\frac{k_F}{M} \leq |k_\parallel| \leq k_F} \vec{dk}_\parallel \lim_{\tau \downarrow 0} \int \vec{dk}_0 \frac{e^{-i\tau k_0}}{ik_0 - v_F k_\parallel} \quad (3.24)$$

Applying the residue theorem, the k_0 -integration yields a constant equal to -1 for $k_\parallel > 0$ and zero for $k_\parallel < 0$. Thus the k -integration gives a result of order 1. The factor λ_0^{1-d} is cancelled by the summation over sectors. Therefore the tadpole amplitude is $O(g^{(0)})$. Direct computation shows that the turtle diagram is of the same order. Comparing with (3.16) we obtain

$$\delta\mu^{(1)} = O(g^{(0)}/\lambda_0) \quad (3.25)$$

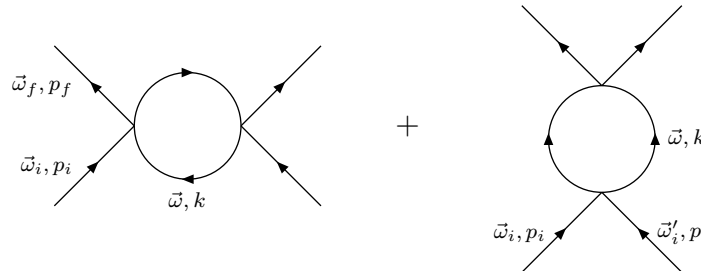
The renormalization of the chemical potential deforms the singular surface in the propagator — the Fermi surface — and modifies the original form (3.10) of the effective action. This will be cured by a change of variables discussed at the end of this section.

Let us now turn to the renormalization of the couplings g . To determine $g^{(1)}$, we have to sum over all connected graphs with four external legs. In fact, the graphs that are not 1PI give a vanishing contribution. These graphs are of the form

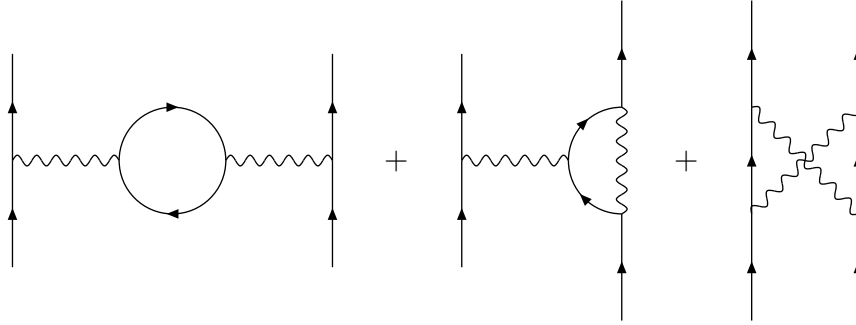

(3.26)

As in the renormalization of the electron propagator, the amplitude of such a graph vanishes, as a consequence of momentum conservation.

The tree level contribution to $g^{(1)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$ is just $g^{(0)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$. The one-loop correction is described by the diagrams


(3.27)

The first diagram corresponds to the sum of the following three graphs



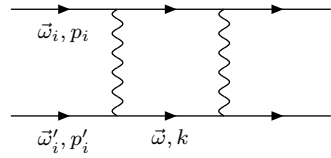
$$(3.28)$$

The amplitude is of order at most $O((g^{(0)})^2/\lambda_0^d)$. In fact, for $\vec{\omega}_i \neq \vec{\omega}_f$, the number of $\vec{\omega}$'s contributing to the $\vec{\omega}$ -summation is $O(\lambda_0^{d-2})$, due to sector momentum conservation. Consequently the amplitude is of order $O((g^{(0)})^2/\lambda_0^d)$. But when $\vec{\omega}_i = \vec{\omega}_f$ the number of terms in the sum is actually $O(\lambda_0^{d-1})$. We thus must take a closer look at the integration involved in the calculation of the amplitudes. The \vec{k} -integration extends over \vec{k} 's satisfying $\frac{k_F}{M} < |\vec{\omega}\vec{k}| < k_F$ and $\frac{k_F}{M} < |\vec{\omega}(\vec{k} + \vec{p}_i - \vec{p}_f)| < k_F$. Explicitly the loop integral is given by

$$\int \vec{d}^d k \int \vec{d} k_0 \frac{1}{ik_0 - v_F \vec{\omega}\vec{k}} \frac{1}{i(k_0 + p_{i,0} - p_{f,0}) - v_F \vec{\omega}(\vec{k} + \vec{p}_i - \vec{p}_f)}$$

In order for the integral to be different from zero, the two poles must not lie in the same (upper or lower) half plane, i.e., $\vec{\omega}\vec{k}$ and $\vec{\omega}(\vec{k} + \vec{p}_i - \vec{p}_f)$ must have opposite signs. The set of \vec{k} 's satisfying all three conditions is of measure zero since $|\vec{\omega}(\vec{p}_i - \vec{p}_f)| < \frac{2k_F}{M}$. Hence the integral vanishes. Thus the amplitude of the graph is of order $O((g^{(0)})^2/\lambda_0^d)$, as claimed, except for exceptional configurations of external sector momenta ($|\vec{\omega}_i - \vec{\omega}_f| \sim \frac{1}{\lambda_0}$).

Next, we turn to the second diagram in (3.27) corresponding to the graph



$$(3.29)$$

When $\vec{\omega}_i \neq -\vec{\omega}'_i$, the number of nonvanishing terms in the $\vec{\omega}$ -summation is $O(\lambda_0^{d-2})$, and the amplitude is at most of order $O((g^{(0)})^2/\lambda_0^d)$. But, for the BCS configuration, $\vec{\omega}_i = -\vec{\omega}'_i$, the situation is completely different. There are now $O(\lambda_0^{d-1})$ possible choices for the internal particle momenta, and no miracle makes the amplitude vanish. In fact, the loop integration for $\vec{p}_i = \vec{p}_f = 0$ is

$$\int_{\vec{B}_{\vec{\omega}}} \vec{d}^d k \int \vec{d} k_0 \frac{1}{k_0^2 + (v_F \vec{\omega}\vec{k})^2} \quad (3.30)$$

which is strictly positive. Thus the amplitude is of order $O((g^{(0)})^2/\lambda_0^{d-1})$.

Inserting the correct scale factors, we get the following flow equations for the quartic couplings:

$$g^{(1)} = g^{(0)} + O((g^{(0)})^2/\lambda_0) , \text{ for } \vec{\omega}_i \neq -\vec{\omega}'_i \quad (3.31)$$

$$g_{\text{BCS}}^{(1)} = g_{\text{BCS}}^{(0)} + O((g_{\text{BCS}}^{(0)})^2) , \text{ for } \vec{\omega}_i = -\vec{\omega}'_i \quad (3.32)$$

Thus, for the BCS couplings, the tree level does not yield the complete contribution to lowest order in $\frac{1}{\lambda_0}$! In order to understand the flow of the BCS couplings, we have to investigate the loop corrections in more detail. This will be done in the next section.

The iteration step from scale j to scale $j+1$ is analyzed in a similar manner as from scale 0 to scale 1. Neglecting vertices of degree > 4 in ψ^* and ψ , we obtain

$$Z^{(j+1)} = Z^{(j)} + O((g^{(j)})^2/\lambda_j) \quad (3.33)$$

$$v_F^{(j+1)} = v_F^{(j)} + O((g^{(j)})^2/\lambda_j) \quad (3.34)$$

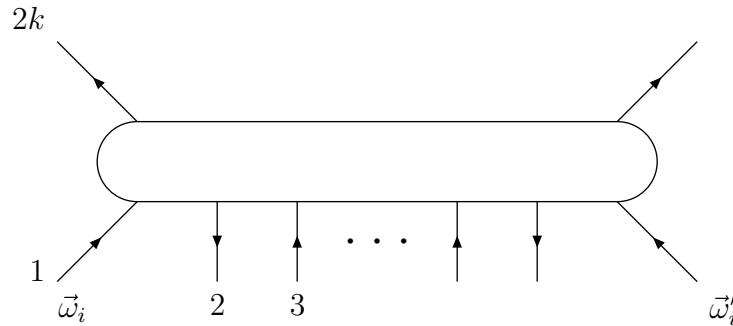
$$\delta\mu^{(j+1)} = O(g^{(j)}/\lambda_j) \quad (3.35)$$

$$g^{(j+1)} = g^{(j)} + O((g^{(j)})^2/\lambda_j) , \text{ for } \vec{\omega}_i \neq -\vec{\omega}'_i \quad (3.36)$$

$$g_{\text{BCS}}^{(j+1)} = g_{\text{BCS}}^{(j)} + O((g_{\text{BCS}}^{(j)})^2) , \text{ for } \vec{\omega}_i = -\vec{\omega}'_i \quad (3.37)$$

We observe that the couplings $g^{(j)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$, for $\vec{\omega}_i \neq -\vec{\omega}'_i$, do essentially not flow. But the BCS couplings may change considerably under the renormalization group flow. If the BCS channel is turned off, perturbation theory is valid, and the system approaches a Landau-Fermi liquid. If $g_{\text{BCS}}^{(j)}$ grows in j then perturbation theory breaks down. In this situation the system becomes a superconductor, as studied in more detail in the next section and in Chapter 4. The unlimited growth of $g_{\text{BCS}}^{(j)}$ in j then reflects the fact that we are performing a perturbative analysis about a state that is not a ground state. As a matter of fact, superconductors do not possess a Fermi surface.

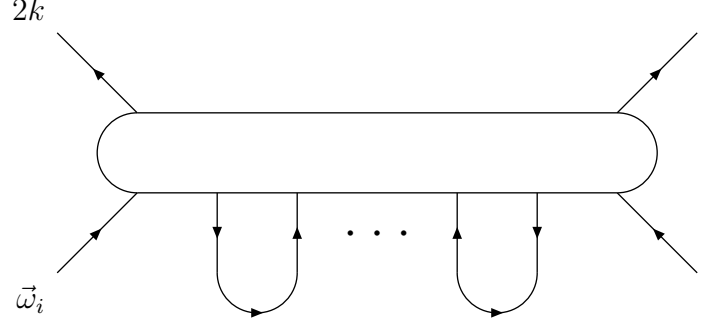
At this point, a comment on contributions of degree > 4 in ψ^* , ψ to the effective action is appropriate. The decimation of degrees of freedom (i.e., the integration over degrees of freedom) in an iteration step j , $j = 0, 1, 2, \dots$, of the renormalization group procedure produces terms like



$$(3.38)$$

of scaling dimension $-(d-1)(k-1)$ in the effective action $S_{\text{eff}}^{(j)}$. The form and renormalization flow of the dominant contributions to these terms can be studied quite explicitly. In the next

iteration step, from j to $j+1$, these terms yield contributions to the dimensionless coupling constants $g^{(j+1)}$ of the terms of degree 4 in $S_{\text{eff}}^{(j+1)}$ that turn out to be of order $1 = (\frac{1}{\lambda_j})^0$, even for incoming sector momenta $\vec{\omega}_i \neq -\vec{\omega}'_i$,


(3.39)

and hence may be important. They are, however, of higher order in $g^{(j)}, g^{(j-1)}, \dots$. A careful analysis (presented elsewhere) reveals that these contributions induce a finite flow of the couplings $g^{(j)}$ towards RPA-(random phase approximation) type fixed points. For short-range two-body interactions, and if $g^{(0)} \ll 1$, this represents an *unimportant* modification of (3.36). However, for long-range (e.g. Coulomb) two-body interactions, or if $g^{(0)}$ is not small, the modification in the renormalization flow of the $g^{(j)}$'s due to the terms of degree > 4 in ψ^*, ψ in the effective action is *essential* and is intimately connected with the phenomenon of *screening*. A detailed discussion of these matters would go beyond the scope of these notes.

Next, we analyze the renormalization of the chemical potential. At scale j , the effective action is given by (we use unscaled fields)

$$S_{\text{eff}}^{(j)} = -Z^{(j)-1} \sum_{\sigma, \vec{\omega}} \int \tilde{d}^{d+1} k \psi_{\vec{\omega}, \sigma}^*(k) (ik_0 - v_F^{(j)} \vec{\omega} \vec{k} + \delta\mu^{(j)}) \psi_{\vec{\omega}, \sigma}(k) + \text{higher degree terms}$$

We may perform a shift, $k_{\parallel} \rightarrow k_{\parallel} - \frac{\delta\mu^{(j)}}{v_F^{(j)}}$, of the k_{\parallel} -variable. Note that $\frac{\delta\mu^{(j)}}{v_F^{(j)}} = k_F O(\frac{g^{(j)}}{\lambda_j})$. Thus, if the coupling $g^{(j)}$ remains approximately constant under the RG flow and sufficiently small, the shift of the k_{\parallel} -variable will always be smaller than $\frac{k_F}{\lambda_j}$. The integration measure is invariant under this coordinate transformation. The transformed action reads

$$S_{\text{eff}}^{(j)} = -Z^{(j)-1} \sum_{\sigma, \vec{\omega}} \int \tilde{d}^{d+1} k \psi_{\vec{\omega}, \sigma}^*(k) (ik_0 - v_F^{(j)} k_{\parallel}) \psi_{\vec{\omega}, \sigma}(k) + \text{higher order terms}$$

Hence $S_{\text{eff}}^{(j)}$ is again of the form (3.10) (except that the domain over which \vec{k} is integrated has changed slightly).

Our results motivate the following interpretation of the renormalization group transformation: Assuming that the interacting system has a Fermi surface (i.e., that the BCS channel is turned off), the RG procedure enables us to approach the renormalized Fermi surface stepwise, starting from the Fermi surface of the noninteracting system. In the limit

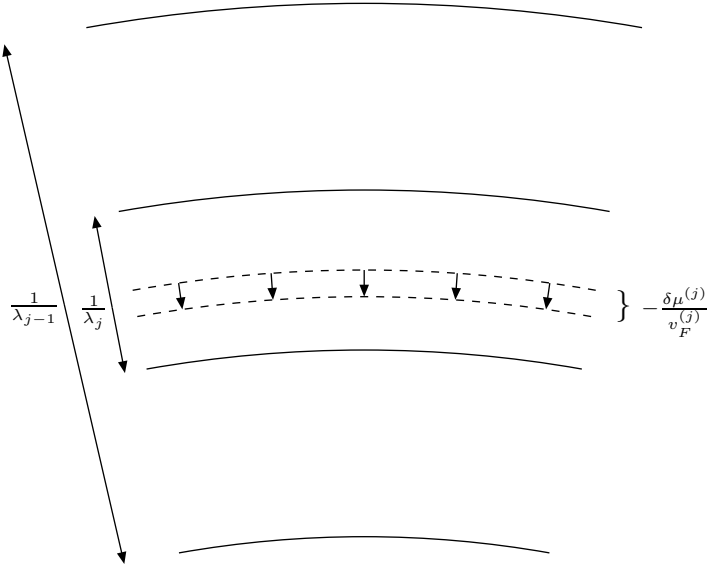


Figure 4: Renormalization of the Fermi surface

$j \rightarrow \infty$, we reach the physical Fermi surface of the interacting system. Imposing renormalization conditions on $Z^{(\infty)}$, $v_F^{(\infty)}$, $g^{(\infty)}$ amounts to solving a “final condition problem” in the space of running coupling constants. The RG recursion generates the discrete dynamics of coupling constants, as the scale parameter j is varied, but instead of initial conditions, experiment fixes final conditions. What we actually do is that we integrate the equations for the running couplings backwards from the Landau-Fermi liquid fixed point ($j = \infty$) down to $j = 0$. The values $\mu^{(0)}$, $Z^{(0)}$, $v_F^{(0)}$, and $g^{(0)}$ parametrize a *microscopic system*, whereas $\mu^{(\infty)}$, $Z^{(\infty)}$, $v_F^{(\infty)}$ and $g^{(\infty)}$ parametrize a *universality class of scaling limits describing macroscopic states*.

3.3 The BCS channel

In the previous section we observed that the one-loop correction in the renormalization group flow of g_{BCS} is of order zero in $\frac{1}{\lambda}$; g_{BCS} might be a *relevant coupling*. To reach a better understanding of g_{BCS} , we determine the flow of g_{BCS} , taking into account all corrections of order zero in $\frac{1}{\lambda}$ (but ignoring terms of degree > 4 in ψ^* and ψ in the effective actions; see (3.38), (3.39)). We omit the scale index when no confusion may arise.

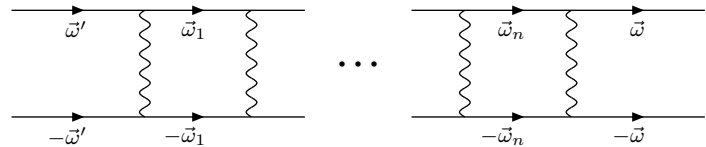
First, we have to determine all four-legged diagrams corresponding to corrections to g_{BCS} of order zero in $\frac{1}{\lambda}$. These are precisely those diagrams whose amplitude is of order $O(\frac{1}{\lambda^{d-1}})$. Every graph with four or more external legs can be constructed from a uniquely determined diagram containing no two-legged subgraphs. This is done by replacing each inner line by

an appropriate graph with two external legs. We denote by


(3.40)

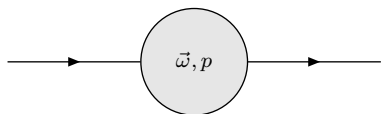
the sum of all graphs with two external legs; i.e., in a diagram we have to replace each such symbol by the sum of all two-legged diagrams. Let us consider an arbitrary four- or more legged diagram with no two-legged subdiagrams, with all inner lines carrying this modified electron propagator line. We shall compute the amplitude of such a graph by just replacing the electron propagator by the amplitude of the modified electron propagator line. (To see why this is correct, we expand each modified propagator into the corresponding sum of terms. Every summand is the amplitude of a specific diagram to which a combinatorial factor is assigned that is, in general, smaller than the combinatorial factor belonging to that diagram. But one easily proves that summing up all terms corresponding to a given diagram yields the correct factor.)

What are the connected diagrams with four external legs whose amplitude is of order $O(\frac{1}{\lambda^{d-1}})$? Consider a four-legged graph with n interaction squiggles. The squiggles provide a factor of order $\lambda^{-n(d-1)}$. There are $2n-2$ inner lines. But the δ -functions of sector momentum conservation restrict the sum over the $\vec{\omega}$'s to $n-1$ independent sector momenta. Thus, in order for the amplitude to be of order $O(\frac{1}{\lambda^{d-1}})$, *all* the $n-1$ sector momentum summations must extend over the entire $(d-1)$ -dimensional Fermi sphere! In this case, the diagrams


(3.41)

with $n \geq 0$, are of order $O(\frac{1}{\lambda^{d-1}})$, unless a miracle happens that makes some of them vanish. Using the above argument, one can prove that the graphs (3.41) are the *only* four-legged diagrams with no two-legged subdiagrams which are of order $O(\frac{1}{\lambda^{d-1}})$.

We conclude that if we do not include terms of degree > 4 in ψ^* , ψ in the effective action then all the diagrams that yield a correction to g_{BCS} of order zero in $\frac{1}{\lambda}$ are contained in the set of graphs built from (3.41) by replacing all the inner lines by (3.40). In fact, in (3.40) only the graphs of order zero in $\frac{1}{\lambda}$ have to be taken into account. Let us consequently define


(3.42)

In order to get a more explicit expression for the flow equation (3.47), it is useful to expand the BCS couplings, $g_{\text{BCS}}(\vec{\omega}, \vec{\omega}') \equiv g_{\text{BCS}}(\angle(\vec{\omega}, \vec{\omega}'))$, into spherical harmonics

$$g_{\text{BCS}}(\vec{\omega}, \vec{\omega}') = \sum_{l=0}^{\infty} g_l h_l(\vec{\omega}, \vec{\omega}') \quad (3.48)$$

with

$$h_l(\vec{\omega}, \vec{\omega}') := \begin{cases} \frac{1}{\pi(1+\delta_{l,0})} \cos(l\angle(\vec{\omega}, \vec{\omega}')) & d = 2 \\ \frac{2l+1}{4\pi} P_l\left(\frac{\vec{\omega} \cdot \vec{\omega}'}{|\vec{\omega}||\vec{\omega}'|}\right) & d = 3 \end{cases} \quad (3.49)$$

P_l , $l \in \mathbb{N}$, are the Legendre polynomials. For $d = 2$, we assume $O(2)$ -invariance of the potentials (instead of only $SO(2)$ -invariance). The normalizations of the functions h_l have been chosen in such a way that

$$\int_{|\vec{\omega}|=1} d\sigma(\vec{\omega}_1) h_l(\vec{\omega}, \vec{\omega}_1) h_{l'}(\vec{\omega}_1, \vec{\omega}') = \delta_{l,l'} h_l(\vec{\omega}, \vec{\omega}') \quad (3.50)$$

Because

$$\frac{1}{\lambda^{d-1}} \sum_{\vec{\omega}} (\cdot) = \int_{|\vec{\omega}|=1} d\sigma(\vec{\omega}) (\cdot) + O\left(\frac{1}{\lambda}\right) \quad (3.51)$$

the r.h.s. of the flow equation (3.47) becomes (we omit the sub- and superscripts j)

$$\begin{aligned} g_{\text{BCS}}(\vec{\omega}, \vec{\omega}') &+ \sum_{n=1}^{\infty} (-1)^n \beta^n \int_{|\vec{\omega}_i|=1} d\sigma(\vec{\omega}_1) \cdots d\sigma(\vec{\omega}_n) g_{\text{BCS}}(\vec{\omega}, \vec{\omega}_n) \cdots g_{\text{BCS}}(\vec{\omega}_1, \vec{\omega}') \\ &= \sum_{l=0}^{\infty} g_l h_l(\vec{\omega}, \vec{\omega}') + \sum_{n=1}^{\infty} (-1)^n \beta^n \sum_{l_i \geq 0} g_{l_0} \cdots g_{l_n} \delta_{l_0, l_1} \cdots \delta_{l_{n-1}, l_n} h_{l_0}(\vec{\omega}, \vec{\omega}') \\ &= \sum_{l=0}^{\infty} \left(\sum_{n=0}^{\infty} (-1)^n \beta^n g_l^{n+1} \right) h_l(\vec{\omega}, \vec{\omega}') \\ &= \sum_{l=0}^{\infty} \frac{g_l}{1 + \beta g_l} h_l(\vec{\omega}, \vec{\omega}') \end{aligned}$$

up to terms of order $\frac{1}{\lambda}$. The flow equation for the BCS couplings hence takes the form

$$g_l^{(j+1)} = \frac{g_l^{(j)}}{1 + \beta_j g_l^{(j)}} + O\left(\frac{1}{\lambda_j}\right), \quad l \in \mathbb{N} \quad (3.52)$$

where β_j is positive and approximately independent of j ; it depends on $Z^{(j)}$, $v_F^{(j)}$ and $g^{(j)}$. The flow equations for different angular momenta, l , decouple to lowest order in $\frac{1}{\lambda}$. But at order $\frac{1}{\lambda}$, the flow equations for different l 's are coupled (*Kohn-Luttinger effect*).

Before we analyze the flow of the couplings, we study the rôle played by electron spin. The potential $v(\vec{x})$ was supposed to be independent of spin. Thus the couplings $g^{(j)}(\vec{\omega}_1, \dots, \vec{\omega}_4)$ are spin-independent, too. We thus divide the quartic term in the effective action into a spin-singlet and a spin-triplet part. We define

$$\phi_{\vec{\omega}_1, \vec{\omega}_2, \sigma, \sigma'}^s(k_1, k_2) := \frac{1}{2} (\psi_{\vec{\omega}_1, \sigma}(k_1) \psi_{\vec{\omega}_2, \sigma'}(k_2) \mp \psi_{\vec{\omega}_1, \sigma'}(k_1) \psi_{\vec{\omega}_2, \sigma}(k_2)) \quad (3.53)$$

Clearly $\phi_{\vec{\omega}_1, \vec{\omega}_2, \sigma, \sigma'}^t = \mp \phi_{\vec{\omega}_1, \vec{\omega}_2, \sigma', \sigma}^t$, and hence $\phi_{\vec{\omega}_1, \vec{\omega}_2, \sigma, \sigma}^s = 0$ (ϕ^s will correspond to spin-singlet pairing, and ϕ^t to spin-triplet pairing). The quartic term in the action becomes

$$\begin{aligned} & \frac{1}{2} \frac{1}{\lambda^{d-1}} Z^{-2} \sum_{\substack{\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4 \\ \sigma, \sigma'}} \int \vec{d}^{d+1} k_1 \cdots \vec{d}^{d+1} k_4 (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4) \\ & \left\{ \frac{1}{2} (g(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) + g(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_4, \vec{\omega}_3)) \psi_{\vec{\omega}_1, \sigma}^* (k_1) \psi_{\vec{\omega}_2, \sigma'}^* (k_2) \phi_{\vec{\omega}_3, \vec{\omega}_4, \sigma', \sigma}^s (k_3, k_4) \right. \\ & \left. + \frac{1}{2} (g(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) - g(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_4, \vec{\omega}_3)) \psi_{\vec{\omega}_1, \sigma}^* (k_1) \psi_{\vec{\omega}_2, \sigma'}^* (k_2) \phi_{\vec{\omega}_3, \vec{\omega}_4, \sigma', \sigma}^t (k_3, k_4) \right\} \end{aligned}$$

Thus, we define the singlet and the triplet couplings as

$$g^{\bar{s}}(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) := \frac{1}{2} (g(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) \pm g(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_4, \vec{\omega}_3)) \quad (3.54)$$

Because of the property that $g(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) = g(\vec{\omega}_2, \vec{\omega}_1, \vec{\omega}_4, \vec{\omega}_3)$ we see that

$$g^{\bar{s}}(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) = \pm g^{\bar{s}}(\vec{\omega}_2, \vec{\omega}_1, \vec{\omega}_3, \vec{\omega}_4) = \pm g^{\bar{s}}(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_4, \vec{\omega}_3) \quad (3.55)$$

Using these symmetry properties, the quartic term in the effective action is found to be

$$\begin{aligned} & \frac{1}{2} \frac{1}{\lambda^{d-1}} Z^{-2} \sum_{\substack{\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4 \\ \sigma, \sigma'}} \int \vec{d}^{d+1} k_1 \cdots \vec{d}^{d+1} k_4 (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4) \\ & \left\{ g^s(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) \phi_{\vec{\omega}_2, \vec{\omega}_1, \sigma', \sigma}^{s*} (k_1, k_2) \phi_{\vec{\omega}_3, \vec{\omega}_4, \sigma', \sigma}^s (k_3, k_4) \right. \\ & \left. + \{ g^t(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) \phi_{\vec{\omega}_2, \vec{\omega}_1, \sigma', \sigma}^{t*} (k_1, k_2) \phi_{\vec{\omega}_3, \vec{\omega}_4, \sigma', \sigma}^t (k_3, k_4) \} \right\} \end{aligned} \quad (3.56)$$

i.e., the quartic term is the sum of a singlet and a triplet part. The coupling of the singlet part is g^s , and the coupling of the triplet part g^t . In terms of the original BCS couplings, g^s and g^t are given by

$$g_{\text{BCS}}^{\bar{s}}(\vec{\omega}, \vec{\omega}') = \frac{1}{2} (g_{\text{BCS}}(\vec{\omega}, \vec{\omega}') \pm g_{\text{BCS}}(\vec{\omega}, -\vec{\omega}')) \quad (3.57)$$

Using that $\angle(\vec{\omega}, -\vec{\omega}') = \angle(\vec{\omega}, \vec{\omega}') + \pi$, the functions h_l can be seen to satisfy $h_l(\vec{\omega}, -\vec{\omega}') = (-1)^l h_l(\vec{\omega}, \vec{\omega}')$. Consequently the singlet and triplet BCS couplings are

$$g_{\text{BCS}}^s(\vec{\omega}, \vec{\omega}') = \sum_{q=0}^{\infty} g_{2q} h_{2q}(\vec{\omega}, \vec{\omega}') \quad (3.58)$$

$$g_{\text{BCS}}^t(\vec{\omega}, \vec{\omega}') = \sum_{q=0}^{\infty} g_{2q+1} h_{2q+1}(\vec{\omega}, \vec{\omega}') \quad (3.59)$$

In the expansion of g_{BCS}^s , only *even* angular momenta appear, and the expansion of g_{BCS}^t only involves *odd* angular momenta, as required by the Pauli principle.

We now return to the analysis of the renormalization group flow. The flow equations (3.52) can be written in the form

$$g_l^{(j+1)} - g_l^{(j)} = -\frac{\beta_j g_l^{(j)2}}{1 + \beta_j g_l^{(j)}} + O\left(\frac{g^{(j)2}}{\lambda_j}\right) \quad (3.60)$$

Instead of studying this difference equation we propose to investigate the corresponding differential equation. The differential flow equation is obtained in the limit where the size, $M - 1$, of the scale change in an iteration step tends to zero. Let us write $g_l^{(j)} \equiv g_l^{(\lambda_j)}$, and similarly for all the other running coupling constants. We define

$$g_l(t) := g_l^{(e^t \lambda_0)} \quad , \quad g(t) := g^{(e^t \lambda_0)} \quad , \quad Z(t) := Z^{(e^t \lambda_0)} \quad , \quad v_F(t) := v_F^{(e^t \lambda_0)} \quad (3.61)$$

Consider a scale $\lambda = e^t \lambda_0$ and set $M = e^{t'-t}$, $t' > t$. We divide both sides of (3.60) by $t' - t$ and take the limit $t' \searrow t$. The l.h.s. yields $\frac{d}{dt} g_l(t)$. The coefficient $\beta = \beta(t', t)$ vanishes in the limit $t' \searrow t$, and only terms linear in $t' - t$ have to be kept on the r.h.s. of (3.60). Thus we linearize $\beta(t', t)$:

$$\beta(t', t) = (t' - t)\gamma(t) + O((t' - t)^2) \quad (3.62)$$

$$\gamma(t) = \frac{\partial}{\partial t'} \beta(t', t) |_{t'=t} > 0 \quad (3.63)$$

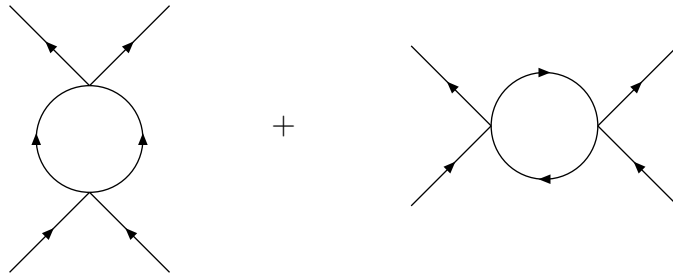
The positivity of γ follows from the monotone growth of $\beta(t', t)$ in t' . Contributions to β corresponding to diagrams with two or more loops are of order $O((t' - t)^2)$. Thus, for the calculation of γ , only one-loop diagrams have to be taken into account; in particular, we don't have to calculate corrections to the electron propagator as we did in the calculation of β . Taking the limit $t' \searrow t$, the r.h.s. of (3.60), divided by $t' - t$ becomes

$$-\gamma g_l(t)^2 + O(e^{-t} g(t)^2)$$

For the error term the same argument as above applies, and we conclude that only one-loop diagrams contribute. We thus obtain the flow equations

$$\frac{d}{dt} g_l(t) = -\gamma g_l(t)^2 + O(e^{-t} g(t)^2) \quad (3.64)$$

where $\gamma = \gamma(t, Z(t), v_F(t), g(t))$ is positive, independent of l , and approximately independent of t . Of course, we could have found these flow equations by just looking at all possible one-loop graphs, namely



The first one yields the first term on the right side in the flow equation and the second graph the error term.

We propose to determine the flow described by (3.64), neglecting the error term, and, in accordance with the results of the previous section, assuming that v_F , Z and g essentially do not flow. Then γ may be approximated by a positive constant γ_0 . Thus we have to solve the differential equation

$$\frac{d}{dt}g_l = -\gamma_0 g_l^2 \quad (3.65)$$

The solution of this equation is easily found to be

$$g_l(t) = \frac{g_l(0)}{1 + \gamma_0 g_l(0)t} \quad (3.66)$$

Note that if $g_l(0) < 0$ the solution blows up at a scale

$$t = -\frac{1}{\gamma_0 g_l(0)} \quad , \text{i.e.} \quad , \text{ for } \quad \lambda = e^{-\frac{1}{\gamma_0 g_l(0)}} \lambda_0 \quad (3.67)$$

Thus if there is an angular momentum channel, l , with attractive interactions ($g_l(0) < 0$) the flow seems to diverge at a finite value of t . But this just means that perturbation theory breaks down when $t \approx -(\gamma_0 g_l(0))^{-1}$, and we shall have to employ nonperturbative methods. The failure of perturbative methods is due to the circumstance that we are expanding around the wrong state !

Seemingly, everything is fine if $g_l(0) \geq 0 \quad \forall l$. However, we have to remember that in (3.65) we have omitted the error term which couples the flows of the running couplings g_l for different values of l . Without fine-tuning of the microscopic two-body potential, it will typically happen that $g_l(t) < 0$, for some l , at some scale t . At that point, $g_l(t)$ will start to grow untill perturbation theory breaks down, and the ground state of the system will be superconducting. Thus, generically, a rotationally invariant system of non-relativistic interacting electrons will be a superconductor for small enough temperatures. This is the celebrated Kohn-Luttinger effect first studied in this fashion by Feldman, Magnen, Rivasseau and Trubowitz.

4 Spontaneous breaking of gauge invariance, and superconductivity

In the last chapter, we have argued that if some BCS coupling $g_l^{(j_0)}$ becomes *negative* at some scale λ_{j_0} then $g_l^{(j)}$ grows in j , for $j > j_0$, untill it becomes so large that our perturbative treatment breaks down. This phenomenon is the signal for an instability of the RG fixed point around which we are doing perturbation theory. Indeed, the Landau-Fermi liquid state is *not* the true ground state of the system anymore, and we expect that the RG flow drives

the system towards a new RG fixed point describing a *superconductor*. The global U(1)-symmetry of non-relativistic many-body theory (gauge invariance of the first kind) turns out to be spontaneously broken in the new (stable) ground state of the system.

The purpose of this chapter is to analyze superconducting ground states and the associated breaking of gauge invariance. This is not an entirely simple story, and we therefore focus our attention on the simplest example, that of an *s-wave (BCS) superconductor*.

Thus, we consider a system with the property that, at some scale $\lambda_0 \gg 1$ (with $|g_0 \lambda_0^2| \ll 1$), $g_{l=0}^{(0)} < 0$ and $|g_l^{(0)}| \ll -g_0^{(0)}$, for $l = 1, 2, 3, \dots$. According to the results of Chapter 3, there is then some $j_{sc} \approx -\frac{1}{\gamma_0 g_0^{(0)} \ln M}$, see (3.67), such that at scale $\lambda_{j_{sc}} > \lambda_0$ $g_0^{(j_{sc})} \ll 0$, $|g_l^{(j_{sc})}| \ll -g_0^{(j_{sc})}$, for $l = 1, 2, 3, \dots$, and (for $\vec{\omega}_1 \neq -\vec{\omega}_2$) $|g^{(j_{sc})}(\vec{\omega}_1, \dots, \vec{\omega}_4)| \ll -g_0^{(j_{sc})}$. This suggests that we neglect all terms of degree ≥ 4 (in ψ^* , ψ) in the effective action of the system at scale $\lambda_{j_{sc}}$, *except the s-wave BCS term* with coupling constant $\frac{g_0^{(j_{sc})}}{\lambda_{j_{sc}}^{d-1}}$. The resulting effective field theory is the one first considered by Nambu and Gorkov.

There is some useful notation to be introduced: Let $[\vec{\omega}]$ be the ray through the origin containing $\vec{\omega}$ and $-\vec{\omega}$. We might think of $\vec{\omega}$ and $-\vec{\omega}$ as the two chiralities of a 1+1 dimensional system of relativistic fermions. We define field variables

$$\psi_{[\vec{\omega}]\uparrow} := \begin{pmatrix} \psi_{\vec{\omega}\uparrow} \\ \psi_{-\vec{\omega}\uparrow} \end{pmatrix} \quad , \quad \psi_{[\vec{\omega}]\downarrow} := \begin{pmatrix} \psi_{\vec{\omega}\downarrow}^* \\ \psi_{-\vec{\omega}\downarrow}^* \end{pmatrix} \quad (4.1)$$

$$\bar{\psi}_{[\vec{\omega}]\uparrow} := (\psi_{-\vec{\omega}\uparrow}^*, \psi_{\vec{\omega}\uparrow}^*) \quad , \quad \bar{\psi}_{[\vec{\omega}]\downarrow} := (\psi_{-\vec{\omega}\downarrow}, \psi_{\vec{\omega}\downarrow}) \quad (4.2)$$

We thus group together field variables belonging to sectors on the same ray. Ordered according to spin indices, we consider them as entries of a four-component quasi-particle field

$$\psi_{[\vec{\omega}]} := \begin{pmatrix} \psi_{[\vec{\omega}]\uparrow} \\ \psi_{[\vec{\omega}]\downarrow} \end{pmatrix} \quad , \quad \bar{\psi}_{[\vec{\omega}]} := (\bar{\psi}_{[\vec{\omega}]\uparrow}, \bar{\psi}_{[\vec{\omega}]\downarrow}) \quad (4.3)$$

named after Nambu and Gorkov. Let $\mathcal{V}_{[\vec{\omega}]}$ denote the two-dimensional, complex vector space whose elements are of the form $\begin{pmatrix} \psi_{\vec{\omega}} \\ \psi_{-\vec{\omega}} \end{pmatrix}$, and let $\mathcal{V}_{\text{spin}}$ denote the two-dimensional, complex vector space whose elements are SU(2) spinors $\begin{pmatrix} \psi_{\uparrow} \\ \psi_{\downarrow} \end{pmatrix}$. We can think of the four-component object $\psi_{[\vec{\omega}]}$ defined in (4.3) as being an element of $\mathcal{V}_{\text{spin}} \otimes \mathcal{V}_{[\vec{\omega}]}$. Stressing analogies to 1+1 dimensional, relativistic models (“dimensional reduction”), we also define gamma matrices

$$\gamma^0 := 1_2 \otimes \sigma_1 \quad , \quad \gamma^1 := 1_2 \otimes \sigma_2 \quad (4.4)$$

($\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ are Pauli matrices). Then we see that

$$\bar{\psi}_{[\vec{\omega}]\uparrow} = \psi_{[\vec{\omega}]\uparrow}^* \sigma_1 \quad , \quad \bar{\psi}_{[\vec{\omega}]\downarrow} = \psi_{[\vec{\omega}]\downarrow}^* \sigma_1 \quad (4.5)$$

$$\bar{\psi}_{[\vec{\omega}]} = \psi_{[\vec{\omega}]}^* \gamma^0 \quad (4.6)$$

The effective action at scale j_{sc} , simplified by omitting all subleading terms of degree 2 (curvature of Fermi surface) and ≥ 4 ($g^{(j_{sc})}$, $g_{l>0}^{(j_{sc})} \approx 0 \ll -g_{l=0}^{(j_{sc})}, \dots$) is given by

$$\begin{aligned}
S_{\text{eff}}(\bar{\psi}, \psi) &= \sum_{[\vec{\omega}]} \int [\bar{\psi}_{[\vec{\omega}]}(\gamma^0 \partial_t - v_F \gamma^1 \vec{\omega} \vec{\nabla}) \psi_{[\vec{\omega}]}] d^{d+1}x \\
&+ \frac{g_0^{(j_{sc})}}{4\lambda_{j_{sc}}^{d-1}} \sum_{[\vec{\omega}], [\vec{\omega}']} \int [\bar{\psi}_{[\vec{\omega}]}(\sigma_1 \otimes 1_2) \psi_{[\vec{\omega}]} \bar{\psi}_{[\vec{\omega}']}(\sigma_1 \otimes 1_2) \psi_{[\vec{\omega}']} \\
&+ \bar{\psi}_{[\vec{\omega}]}(\sigma_2 \otimes 1_2) \psi_{[\vec{\omega}]} \bar{\psi}_{[\vec{\omega}']}(\sigma_2 \otimes 1_2) \psi_{[\vec{\omega}']}] d^{d+1}x
\end{aligned} \tag{4.7}$$

It only includes the s-wave BCS interactions and has the following basic features:

- 1) The action strongly resembles the one of a $1 + 1$ dimensional, relativistic quantum field theory with $N = \frac{N^{(j_{sc})}}{2}$ fermion flavours and quartic self-interaction, such as the chiral Gross-Neveu model. Its infrared properties are closely related to those of the chiral Gross-Neveu model. The perturbative infrared renormalization around the Fermi surface and the $\frac{1}{N}$ -expansion are virtually identical.
- 2) The action, written as in (4.7), exhibits a manifest global $U(1)$ -symmetry, given by

$$\begin{aligned}
\psi_{[\vec{\omega}]} &\rightarrow e^{i\alpha(\sigma_3 \otimes 1_2)} \psi_{[\vec{\omega}]} \\
\bar{\psi}_{[\vec{\omega}]} &\rightarrow \bar{\psi}_{[\vec{\omega}]} e^{-i\alpha(\sigma_3 \otimes 1_2)}
\end{aligned}$$

with $\alpha \in [0, 2\pi)$.

The generator of this symmetry is the particle number operator N ; it is the usual gauge symmetry of the first kind. In the superconducting phase of the system, this symmetry is spontaneously broken. We recall that the *Mermin-Wagner theorem* states that, for a field theory model in $d + 1 = 2$ dimensions with a continuous symmetry such as the chiral Gross-Neveu model, the continuous symmetry *cannot* be broken spontaneously. In the chiral Gross-Neveu model, continuous symmetry breaking only occurs in the $N \rightarrow \infty$ limit. In spite of the formal similarities between nonrelativistic many-body theory in $d \geq 2$ space dimensions, in the Nambu-Gorkov approximation, and the chiral Gross-Neveu model in one space dimension (“dimensional reduction”), the Mermin-Wagner theorem does actually *not* apply to the former, and spontaneous breaking of the $U(1)$ gauge symmetry *is* possible in many-body theory (for $d = 2$ at temperature $T = 0$, and for $d \geq 3$ at sufficiently small temperatures) !

By direct calculation we see that

$$\begin{aligned}
\bar{\psi}_{[\vec{\omega}]}(\sigma_1 \otimes 1_2) \psi_{[\vec{\omega}]} &= \{\psi_{-\vec{\omega}\uparrow}^* \psi_{\vec{\omega}\downarrow}^* + \psi_{\vec{\omega}\uparrow}^* \psi_{-\vec{\omega}\downarrow}^* + \psi_{-\vec{\omega}\downarrow} \psi_{\vec{\omega}\uparrow} + \psi_{\vec{\omega}\downarrow} \psi_{-\vec{\omega}\uparrow}\} \\
&= 2\{\phi_{\vec{\omega}, -\vec{\omega}, \downarrow, \uparrow}^s(x, x) + \phi_{\vec{\omega}, -\vec{\omega}, \uparrow, \downarrow}^{s*}(x, x)\}
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}\bar{\psi}_{[\vec{\omega}]}(\sigma_2 \otimes 1_2)\psi_{[\vec{\omega}]} &= -i\{\psi_{-\vec{\omega}\uparrow}^*\psi_{\vec{\omega}\downarrow}^* + \psi_{\vec{\omega}\uparrow}^*\psi_{-\vec{\omega}\downarrow}^* - \psi_{-\vec{\omega}\downarrow}\psi_{\vec{\omega}\uparrow} - \psi_{\vec{\omega}\downarrow}\psi_{-\vec{\omega}\uparrow}\} \\ &= 2i\{\phi_{\vec{\omega},-\vec{\omega},\downarrow,\uparrow}^s(x,x) - \phi_{\vec{\omega},-\vec{\omega},\downarrow,\uparrow}^{s,*}(x,x)\}\end{aligned}\quad (4.9)$$

where ϕ^s has been defined in (3.53). Thus, these quadratic expressions correspond to real and imaginary part of the BCS order parameter, respectively.

The action $S_{\text{eff}}(\bar{\psi}, \psi)$ defined in (4.7) can be replaced by a more convenient, *equivalent* action, \tilde{S} , that is quadratic in $\bar{\psi}$ and ψ and depends on a complex Lagrange multiplier field $\phi = \phi_1 + i\phi_2$. The new action is given by

$$\begin{aligned}\tilde{S}(\bar{\psi}, \psi, \bar{\phi}, \phi) &= \sum_{[\vec{\omega}]} \int [\bar{\psi}_{[\vec{\omega}]}(\gamma^0 \partial_t - v_F \gamma^1 \vec{\omega} \vec{\nabla})\psi_{[\vec{\omega}]}] d^{d+1}x \\ &+ g \sum_{[\vec{\omega}]} \int [\bar{\psi}_{[\vec{\omega}]}(\sigma_1 \otimes 1_2)\psi_{[\vec{\omega}]} \phi_1 - \bar{\psi}_{[\vec{\omega}]}(\sigma_2 \otimes 1_2)\psi_{[\vec{\omega}]} \phi_2] d^{d+1}x \\ &+ \frac{1}{2} \int (\phi_1^2 + \phi_2^2) d^{d+1}x\end{aligned}\quad (4.10)$$

where $2g^2 = -\frac{g_0^{(jsc)}}{\lambda_{jsc}^{d-1}} > 0$. We note that, under the U(1) symmetry discussed in remark 2) above, the field ϕ transforms as $\phi \rightarrow e^{2i\alpha}\phi$. We emphasize that $S_{\text{eff}}(\bar{\psi}, \psi)$ and $\tilde{S}(\bar{\psi}, \psi, \bar{\phi}, \phi)$ are equivalent in terms of their physical content: It is easily checked that, after functionally integrating out the ϕ -field,

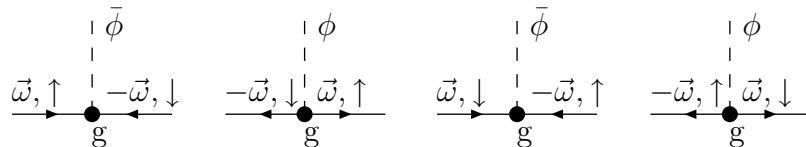
$$\int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-\tilde{S}(\bar{\psi}, \psi, \bar{\phi}, \phi)} = \text{const } e^{-S_{\text{eff}}(\bar{\psi}, \psi)}$$

the original action $S_{\text{eff}}(\bar{\psi}, \psi)$ is restored in the exponent. But $\tilde{S}(\bar{\psi}, \psi, \bar{\phi}, \phi)$ is much easier to work with, because it is quadratic in the fields $\bar{\psi}$ and ψ . The Bose field ϕ will turn out to describe the *Cooper pairs* of electrons.

The interaction between fermion- and Lagrange multiplier fields is described by

$$g \sum_{[\vec{\omega}]} \int [\phi(x) \bar{\psi}_{[\vec{\omega}],\uparrow} \psi_{[\vec{\omega}],\downarrow} + \bar{\phi}(x) \bar{\psi}_{[\vec{\omega}],\downarrow} \psi_{[\vec{\omega}],\uparrow}] d^{d+1}x \quad (4.11)$$

This expression shows how interaction vertices between electrons (holes) and bosons $\phi, \bar{\phi}$ are organized in the Nambu-Gorkov theory. One Nambu-Gorkov vertex is equivalent to the following four vertices (each arrow points to a $\psi_{\vec{\omega},\sigma}$):



$$(4.12)$$

One can join only the first two vertices with each other and the second two vertices with each other. There are two possibilities of joining the first two vertices :

$$(4.13)$$

and two possibilities of joining the second two vertices, obtained by exchanging \uparrow and \downarrow . Along a string of electron (hole) propagator lines, the different sector labels $\vec{\omega}$, $-\vec{\omega}$ and boson fields ϕ , $\bar{\phi}$ have to occur in an alternating pattern. Because the first two and the second two vertices never intercombine, the spins can be omitted if a factor of 2 is attached to each loop.

In the following, we focus our attention on the boson fields, and we will attempt to eliminate the Nambu-Gorkov fermions. Because $\tilde{S}(\bar{\psi}, \psi, \bar{\phi}, \phi)$ is quadratic in $\bar{\psi}$ and ψ , one can perform the fermionic functional integration explicitly:

$$\int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{-\tilde{S}(\bar{\psi}, \psi, \bar{\phi}, \phi)} = \exp \left(-\frac{1}{2} \int d^{d+1}x |\phi(x)|^2 \right) \cdot \det[(\gamma^0 \partial_t - v_F \gamma^1 \vec{\omega} \vec{\nabla}) + g \begin{pmatrix} 0 & \phi(x) \\ \bar{\phi}(x) & 0 \end{pmatrix} \otimes 1_2] \quad (4.14)$$

After normalizing the right hand side by division through $\det[\gamma^0 \partial_t - v_F \gamma^1 \vec{\omega} \vec{\nabla}]$, the determinant

$$\det[1 + \frac{1}{\gamma^0 \partial_t - v_F \gamma^1 \vec{\omega} \vec{\nabla}} \{g \begin{pmatrix} 0 & \phi(x) \\ \bar{\phi}(x) & 0 \end{pmatrix} \otimes 1_2\}] \quad (4.15)$$

can be evaluated using the identity

$$\det(1 + A) = \exp \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \text{Tr}(A^n) \right] \quad (4.16)$$

The term $\text{Tr}(A^n)$ is the amplitude of the n-th order one-loop diagram

$$(4.17)$$

with a factor of $\{g \begin{pmatrix} 0 & \phi(x) \\ \bar{\phi}(x) & 0 \end{pmatrix} \otimes 1_2\}$ on each external line and a Nambu-Gorkov propagator on each segment of the loop. From our discussion of the possible pairings of Nambu-Gorkov vertices we conclude that loops with an odd number of vertices vanish. The expression in the exponent on the right side of (4.16) reduces to $-\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}(A^{2n})$. Calculating $\text{Tr}(A^{2n})$ in terms of the original electron (hole) propagators, the spin summation can be absorbed in an overall factor of 2 cancelling the $\frac{1}{2}$ in the exponent. For a fixed $[\vec{\omega}]$, a $2n$ -loop looks like

inserting alternately electron (hole) sector labels $\vec{\omega}$, $-\vec{\omega}$ and boson fields ϕ and $\bar{\phi}$ along the loop line. In order to compute the full amplitude of such a graph in momentum space, we have to integrate over the loop momentum (k_0, \vec{k}) for arbitrary external momenta $(p_{j,0}, \vec{p}_j)$, $j = 1, \dots, 2n$, which is a difficult task.

It will turn out to be useful to expand the amplitude corresponding to (4.18) into a sum of terms that look somewhat more manageable: Anticipating spontaneous symmetry breaking, we assume that, in a (superconducting, extremal) ground state of the system, the Bose field ϕ has a *non-zero* expectation value ϕ_c . The modulus of $|\phi_c|$ is determined by the values of physical parameters (the density of the system, the strength of $g_0^{(j_{sc})}$, etc.), while the *phase* of ϕ_c , an angle in $[0, 2\pi)$, is only fixed after suitable symmetry breaking *boundary conditions* have been imposed — as usual in the study of systems with spontaneously broken symmetries. We thus decompose the Bose field ϕ into a constant part, ϕ_c , and a fluctuation field, $\chi(x)$:

$$\begin{aligned} \phi(x) &= \phi_c + \chi(x) \\ \bar{\phi}(x) &= \bar{\phi}_c + \bar{\chi}(x) \end{aligned} \tag{4.19}$$

The field $\chi(x)$ describes small fluctuations of the Cooper-pair condensate around ϕ_c . The decomposition of $\phi(x)$ induces a decomposition of the amplitude corresponding to (4.18) into a sum of monomials in χ and $\bar{\chi}$. For each fixed n and $[\vec{\omega}]$, this decomposition, a binomial

series in $\bar{\chi}(p)$ and $\chi(p)$, can be described pictorially, as follows:

$$\begin{aligned}
& \text{Diagram 1} = \text{Diagram 2} \\
& + \sum_{\text{all positions of } \chi \text{ on the loop}} \phi_c \text{ Diagram 3} + \sum_{\text{all positions of } \chi \text{ on the loop}} \bar{\phi}_c \text{ Diagram 4} \\
& + \sum_{\text{all positions of two external } \chi \text{-legs on the loop}} \chi(p) \text{ Diagram 5} + \sum_{\text{all positions of two external } \bar{\chi} \text{-legs on the loop}} \bar{\chi}(p) \text{ Diagram 6} \\
& + \sum_{\text{all positions of an external } \chi_c \text{ and an external } \bar{\chi}_c \text{ leg on the loop}} \bar{\chi}(p) \text{ Diagram 7} \\
& + \text{terms of degree } \geq 3 \text{ in } \chi \text{ and } \bar{\chi}.
\end{aligned}$$

Each dot without an external leg stands for a factor $g\phi_c(2\pi)^{d+1}\delta(p)$ or for its complex conjugate. The external momenta flowing into the diagram at all such dots *vanish*, because

ϕ_c is constant in x -space. The graphs with one external leg are summed over all $2n$ possible positions of either $\chi(0)$ or $\bar{\chi}(0)$ on the loop. The corresponding amplitudes (which are linear in $\bar{\chi}$ or χ) vanish if ϕ_c is the expectation value of ϕ in the ground state of the system. The effective action for the field $\chi(p)$ is thus given by a power series in $\bar{\chi}$ and χ :

$$S(\bar{\chi}, \chi) = S^{(0)} + S^{(1)}(\bar{\chi}, \chi) + S^{(2)}(\bar{\chi}, \chi) + \dots \quad (4.20)$$

where $S^{(r)}$, $r = 0, 1, 2, \dots$, is a sum of monomials of degree r in $\bar{\chi}$ and χ . We remark that the first three terms on the right side of (4.20) also contain the contribution

$$\begin{aligned} & \frac{1}{2} \int \bar{d}^{d+1} p (\bar{\phi}_c \phi_c ((2\pi)^{d+1} \delta(p))^2 \\ & + \phi_c \bar{\chi}(p) (2\pi)^{d+1} \delta(p) + \bar{\phi}_c \chi(p) (2\pi)^{d+1} \delta(p) + \bar{\chi}(p) \chi(p)) \end{aligned} \quad (4.21)$$

from the part of the original action \tilde{S} (defined in (4.10)) quadratic in $\bar{\phi}$ and ϕ .

After dividing by the total volume of the system, i.e., by $(2\pi)^{d+1} \delta(0)$, the amplitude of the $2n$ -th order loop without any external $\bar{\chi}$ - and χ -legs, and for a fixed ray $[\vec{\omega}]$,



$$\quad (4.22)$$

is given by

$$2 \int_{\mathbf{R} \times \tilde{B}_{\vec{\omega}}} \bar{d}^{d+1} k \left(\frac{-g^2 |\phi_c|^2}{k_0^2 + v_F^2 (\vec{\omega} \vec{k})^2} \right)^n \quad (4.23)$$

Summing over all orders $2n$ (see (4.16)) and all rays $[\vec{\omega}]$, our result for $S^{(0)}$ is found to be

$$\begin{aligned} S^{(0)} &= \frac{1}{2} \int_{\mathbf{R} \times \tilde{B}_{\vec{\omega}}} \bar{d}^{d+1} k |\phi_c|^2 ((2\pi) \delta(k))^2 \\ &+ 2 \sum_{[\vec{\omega}]} \int_{\mathbf{R} \times \tilde{B}_{\vec{\omega}}} \bar{d}^{d+1} k \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{-g^2 |\phi_c|^2}{k_0^2 + v_F^2 (\vec{\omega} \vec{k})^2} \right)^n (2\pi)^{d+1} \delta(p=0) \\ &= (2\pi)^{d+1} \delta(p=0) \left(\frac{1}{2} |\phi_c|^2 - 2 \sum_{[\vec{\omega}]} \int_{\mathbf{R} \times \tilde{B}_{\vec{\omega}}} \bar{d}^{d+1} k \log \left(1 + \frac{g^2 |\phi_c|^2}{k_0^2 + v_F^2 (\vec{\omega} \vec{k})^2} \right) \right) \end{aligned} \quad (4.24)$$

The effective potential, $U_{\text{eff}}(\bar{\phi}_c, \phi_c)$, is defined as the density of $S^{(0)}$, i.e., as $S^{(0)}$ divided by the total volume of the system. Thus

$$U_{\text{eff}}(\phi_c) = \frac{1}{2} |\phi_c|^2 - 2 \sum_{[\vec{\omega}]} \int_{\mathbf{R} \times \tilde{B}_{\vec{\omega}}} \bar{d}^{d+1} k \log \left(1 + \frac{g^2 |\phi_c|^2}{k_0^2 + v_F^2 (\vec{\omega} \vec{k})^2} \right) \quad (4.25)$$

In the analysis of spontaneous symmetry breaking, the effective potential plays an important rôle. In the approximation of mean field theory, the expectation value of the field ϕ in an arbitrary, extremal ground state of the system is given by a minimum of $U_{\text{eff}}(\bar{\phi}_c, \phi_c)$. Thanks to the minus sign in front of the integral on the right side of (4.25), the graph of the effective potential has the shape of a Mexican hat. The minima of U_{eff} are obtained by setting the derivative of U_{eff} with respect to $|\phi_c|^2$ to zero. The result is that (for small values of $g^2 k_F^{d-1}$)

$$|\phi_c| \approx \frac{k_F v_F}{g} \exp\left(-\frac{\pi v_F}{g^2 (k_F \lambda_{j_{sc}})^{d-1}}\right) \quad (4.26)$$

where v_F is dimensionless (in our units), k_F has a dimension of inverse length, and g has a dimension of $(\text{length})^{\frac{d-1}{2}}$, so that the dimension of $|\phi_c|$ is that of $(\text{length})^{-\frac{d+1}{2}}$, as it should be in view of the last term on the right side of (4.10).

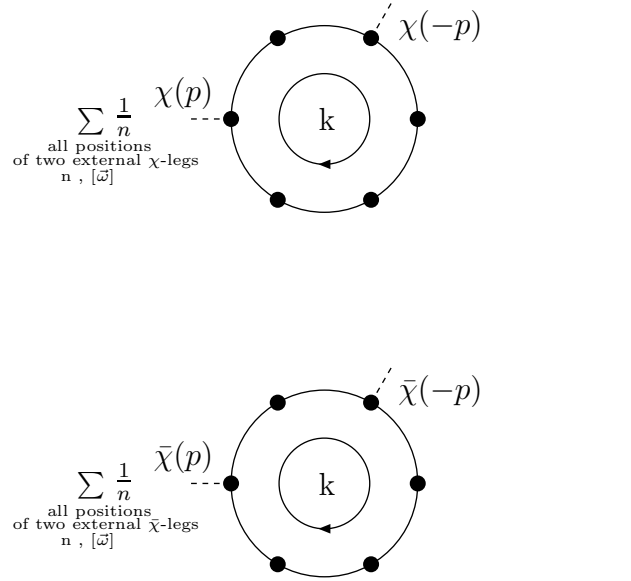
We recall that $2g^2 = -\frac{g_0^{(j_{sc})}}{\lambda_{j_{sc}}^{d-1}}$ (see (4.10)) and hence

$$|\phi_c| \approx \frac{\sqrt{2} k_F v_F \lambda_{j_{sc}}^{\frac{d-1}{2}}}{\sqrt{-g_0^{j_{sc}}}} \exp\left(-\frac{2\pi v_F}{g_0^{(j_{sc})} k_F^{d-1}}\right) \quad (4.27)$$

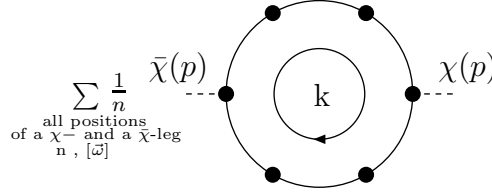
for small values of $|g_0^{(j_{sc})}| \left(\frac{k_F}{\lambda_{j_{sc}}}\right)^{d-1}$.

At values of ϕ_c minimizing $U_{\text{eff}}(\bar{\phi}_c, \phi_c)$, the terms in $S(\bar{\chi}, \chi)$ linear in $\bar{\chi}$ or χ must vanish, i.e. $S^{(1)} = 0$. (The equations $\frac{\partial U_{\text{eff}}}{\partial(|\phi_c|^2)} = 0$ and $S^{(1)} = 0$ are, of course, equivalent; the solution is given by (4.26).) From now on, ϕ_c will denote a minimum of U_{eff} .

The term $S^{(2)}(\bar{\chi}, \chi)$ consists of three different contributions, proportional to χ^2 , $\bar{\chi}^2$ and $\bar{\chi}\chi$, respectively. They can be found by calculating the amplitudes corresponding to the following sums of diagrams:



and



We are interested in calculating the Taylor series expansions of the amplitudes corresponding to these sums of diagrams in p around $p = 0$. The terms constant in p yield the coefficients of the “mass terms” of the fluctuation field χ . These coefficients are found to be given by

$$2g^4 \bar{\phi}_c^2 \sum_{[\vec{\omega}]} \int_{\mathbf{R} \times \bar{B}_{\vec{\omega}}} d^{d+1}k \frac{1}{(k_0^2 + v_F^2(\vec{\omega} \cdot \vec{k})^2 + g^2|\phi_c|^2)^2} \quad (4.28)$$

(coefficient of term proportional to χ^2), the complex conjugate of this expression (coefficient of term proportional to $\bar{\chi}^2$), and by

$$\frac{1}{2} - 2g^2 \sum_{[\vec{\omega}]} \int d^{d+1}k \frac{k_0^2 + v_F^2(\vec{\omega} \cdot \vec{k})^2}{(k_0^2 + v_F^2(\vec{\omega} \cdot \vec{k})^2 + g^2|\phi_c|^2)^2} \quad (4.29)$$

(coefficient of term proportional to $\bar{\chi}\chi$), where the “ $\frac{1}{2}$ ” comes from the term $\frac{1}{2} \int d^{d+1}x \bar{\chi}\chi$ in the original action \tilde{S} (see (4.10)).

It is convenient to introduce polar coordinates in field space by setting

$$\phi(x) = |\phi_c| e^{i\theta} + (\chi_t(x) + i\chi_l(x)) e^{i\theta} \quad (4.30)$$

where the phase θ of the ground state expectation value of $\phi(x)$, $\approx \phi_c$, is fixed by the boundary conditions imposed on the system. The component χ_t of χ is parallel to ϕ_c , i.e., it is *transversal* to the manifold of minima of U_{eff} , a circle, at ϕ_c . The component χ_l is perpendicular to ϕ_c , hence *tangential* to the manifold of minima of U_{eff} at ϕ_c .

A straightforward calculation (involving (4.26), i.e., the fact that ϕ_c minimizes U_{eff}) now shows that the mass terms for the field χ combine to

$$\frac{M^2}{2} \int d^{d+1}x \chi_t^2 \quad (4.31)$$

where

$$M^2 = 16g^4 |\phi_c|^2 \sum_{[\vec{\omega}]} \int_{\mathbf{R} \times \bar{B}_{\vec{\omega}}} d^{d+1}k \frac{1}{(k_0^2 + v_F^2(\vec{\omega} \cdot \vec{k})^2 + g^2|\phi_c|^2)^2} \quad (4.32)$$

In accordance with the Goldstone theorem, the effective field theory for χ contains a massless field χ_l describing the Goldstone bosons and a seemingly massive field χ_t of mass M , where

M is equal to the square root of the curvature of U_{eff} in the radial direction (parallel to ϕ_c) at ϕ_c . Actually, it will turn out that the degree 3 terms, $S^{(3)}$, of the effective action of χ couple χ_t to the second power of χ_l . Thus the field quanta of χ_t are resonances that decay into pairs of Goldstone bosons.

We note that *all* terms in the effective action for the field χ that are local and do not contain any derivatives of χ and $\bar{\chi}$ can be determined from $U_{\text{eff}}(\bar{\phi}_c + \bar{\chi}, \phi_c + \chi)$, with $\chi(x) = \text{const}$, by expanding in powers of χ and $\bar{\chi}$.

We can also compute the kinetic terms of the effective action of χ . Let us denote their inverse propagators by $\Pi_{\chi,\chi}(p)$, $\Pi_{\bar{\chi},\bar{\chi}}(p)$, $\Pi_{\bar{\chi},\chi}(p)$ and $\Pi_{\chi,\bar{\chi}}(p)$, where p is the external momentum, and the indices stand for the external legs. For example, the coefficients, $\alpha_{\bar{\chi},\chi}$ and $\beta_{\bar{\chi},\chi}$, of the contribution $\int \bar{d}^{d+1}p \bar{\chi}(p)(\alpha_{\bar{\chi},\chi}p_0^2 + \beta_{\bar{\chi},\chi}|\vec{p}|^2)\chi(p)$ to the effective action are given by

$$\alpha_{\bar{\chi},\chi} = \frac{1}{2} \frac{\partial^2}{\partial p_0^2} \big|_{p=0} \Pi_{\bar{\chi},\chi}(p) \quad (4.33)$$

$$\beta_{\bar{\chi},\chi} = \frac{1}{2} \frac{\partial^2}{\partial p_i^2} \big|_{p=0} \Pi_{\bar{\chi},\chi}(p) \quad (4.34)$$

p_i ($i = 1, \dots, d$) is the i -th component of \vec{p} . Due to rotational symmetry, $\beta_{\bar{\chi},\chi}$ is independent of the choice of i . All other coefficients can be expressed in terms of derivatives of $\Pi_{\chi,\chi}$ and $\Pi_{\bar{\chi},\bar{\chi}}$. In $d = 2$ space dimensions and for small values of $g^2 k_F$, we obtain the following results:

$$\alpha_{\bar{\chi},\chi} \equiv \alpha_{\chi,\bar{\chi}} \approx \frac{k_F \lambda_{j_{sc}}}{12\pi v_F |\phi_c|^2} \quad (4.35)$$

$$\beta_{\bar{\chi},\chi} \equiv \beta_{\chi,\bar{\chi}} \approx \frac{v_F k_F \lambda_{j_{sc}}}{48\pi |\phi_c|^2} \quad (4.36)$$

$$\alpha_{\chi,\chi} \equiv \alpha_{\bar{\chi},\bar{\chi}} \approx -\frac{k_F \lambda_{j_{sc}}}{24\pi v_F |\phi_c|^2} \quad (4.37)$$

$$\beta_{\chi,\chi} \equiv \beta_{\bar{\chi},\bar{\chi}} \approx -\frac{v_F k_F \lambda_{j_{sc}}}{96\pi |\phi_c|^2} \quad (4.38)$$

Splitting χ into a transversal and a tangential mode, we arrive at the expression

$$\begin{aligned} & \int \bar{d}^3 p \chi_t(p) \left(\frac{k_F \lambda_{j_{sc}}}{12\pi v_F |\phi_c|^2} p_0^2 + \frac{v_F k_F \lambda_{j_{sc}}}{48\pi |\phi_c|^2} |\vec{p}|^2 \right) \chi_t(p) \\ & + \int \bar{d}^3 p \chi_l(p) \left(\frac{k_F \lambda_{j_{sc}}}{4\pi v_F |\phi_c|^2} p_0^2 + \frac{v_F k_F \lambda_{j_{sc}}}{16\pi |\phi_c|^2} |\vec{p}|^2 \right) \chi_l(p) \end{aligned} \quad (4.39)$$

for the kinetic part of the effective action. For $g^2 k_F$ small and $d = 2$, the contributions to $S^{(3)}(\bar{\chi}, \chi)$ of order zero in the momenta turn out to be

$$\begin{aligned} & \int \bar{d}^3 p_1 \bar{d}^3 p_2 \frac{1}{3!} \frac{k_F^3 v_F \lambda_{j_{sc}}}{\pi |\phi_c|^3} (-\chi_l(p_1) \chi_t(-p_2) \chi_l(p_2 - p_1) \\ & + \chi_t(p_1) \chi_t(-p_2) \chi_t(p_2 - p_1)) \end{aligned} \quad (4.40)$$

Rescaling χ_t by a factor $(\frac{k_F \lambda_{jsc}}{12\pi v_F |\phi_c|^2})^{\frac{1}{2}}$ and χ_l by a factor $(\frac{k_F \lambda_{jsc}}{4\pi v_F |\phi_c|^2})^{\frac{1}{2}}$, the lowest order terms in the effective action for the χ -field take the following standard form :

$$\begin{aligned}
S(\chi, \bar{\chi}) = & \int \bar{d}^3 p \chi_t(p) (p_0^2 + \frac{1}{4} v_F^2 |\vec{p}|^2) \chi_t(-p) \\
& + \int \bar{d}^3 p \chi_l(p) (p_0^2 + \frac{1}{4} v_F^2 |\vec{p}|^2) \chi_l(-p) \\
& + \int \bar{d}^3 p \quad 12 v_F^2 k_F^2 \quad \chi_t(p) \chi_t(-p) \\
& + \int \bar{d}^3 p_1 \bar{d}^3 p_2 \quad 4 v_F^2 k_F^2 \quad \sqrt{\frac{\pi v_F}{3 k_F \lambda_{jsc}}} \\
& \left[-\chi_l(p^{(1)}) \chi_t(-p^{(2)}) \chi_l(p^{(2)} - p^{(1)}) + 3 \chi_t(p^{(1)}) \chi_t(-p^{(2)}) \chi_t(p^{(2)} - p^{(1)}) \right]
\end{aligned} \tag{4.41}$$

At tree level, the propagator $\langle \chi_t(0) \chi_t(x) \rangle$ of the χ_t -field in x -space decays exponentially in $|x|$, with decay rate M . Using expression (4.41) to calculate radiative corrections, the behaviour of the propagator of χ_t changes drastically. This is due to the vertex in $S(\bar{\chi}, \chi)$ proportional to $\chi_l^2 \chi_t$. The dominant one-loop radiative correction to the propagator of χ_t is proportional to

$$\int d^{d+1} y d^{d+1} z \langle \chi_t(0) \chi_t(y) \rangle_0 \langle \chi_l(y) \chi_l(z) \rangle_0^2 \langle \chi_t(z) \chi_t(x) \rangle_0 \tag{4.42}$$

where $\langle (\cdot) \rangle_0$ indicates that the expectation value is calculated at tree level. Because the propagator $\langle \chi_t(0) \chi_t(x) \rangle_0$ decays exponentially in $|x|$, while $\langle \chi_l(0) \chi_l(x) \rangle_0 \approx |x|^{1-d}$, expression (4.42) is proportional to

$$\langle \chi_l(0) \chi_l(x) \rangle_0 \sim |x|^{2-2d} \tag{4.43}$$

Thus the seemingly massive field quanta of the field χ_t are resonances that decay into pairs of massless Goldstone bosons, as announced.

The form (4.41) of the effective action of the χ -field shows that in *one* space dimension the fluctuations of χ_l are *logarithmically divergent* (logarithmic infrared divergence of $\int \bar{d}^2 p \frac{1}{p_0^2 + \text{const } p_1^2}$). The prediction of mean field theory that the continuous U(1) gauge symmetry is spontaneously broken is therefore wrong (Mermin-Wagner theorem). The same conclusion is reached in two space dimensions at *positive* temperature, T . The propagator of χ_l in momentum space is then proportional to

$$\frac{1}{(k_B T n)^2 + \text{const } (p_1^2 + p_2^2)} \tag{4.44}$$

where the *Matsubara frequencies* n are integers. The term corresponding to $n = 0$ yields logarithmically divergent fluctuations of χ_l , and, again, the U(1) symmetry is restored. However, these systems exhibit a *Kosterlitz-Thouless transition*, as g is varied.

But, for $d = 2$ and $T = 0$, or for $d \geq 3$ and at sufficiently small temperatures, the predictions of mean field theory concerning spontaneous symmetry breaking are qualitatively correct !

The form (4.10) of the effective action $\tilde{S}(\bar{\psi}, \psi, \bar{\phi}, \phi)$ and Eqs. (4.8) and (4.9) show that the field quanta of the χ -field have electric charge $\pm 2e$, (where e is the elementary electric charge). If the Coulomb interactions between charged quasi-particles are incorporated in our theory then the *Goldstone bosons acquire a positive mass* (“*Anderson-Higgs mechanism*”), as briefly discussed in Chapter 5 of Part I.

The prediction of our theory for the mass, $\approx g|\phi_c|$ with $|\phi_c|$ given by (4.26), of quasi-particles of charge $\pm e$ agrees with the solution of the standard BCS gap equation.