Physical Principles Underlying the FQHE

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Report on work spread over 25 years (1989-2000, 2008-2011)

Credits:

Morf

Bieri, Boyarsky, Cheianov, Graf, Kerler, Levkivskyi, Pedrini, Ruchayskiy, Schweigert, Studer, Sukhorukov, Thiran, Walcher, Zee

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2. What is the FQHE
3. Electrodynamics of Incompressible Hall Fluids
4. The Bulk of an IHF
5. Summary

General Goal (90's):

Classify states of bulk matter and their surface modes, using ideas and concepts from gauge theory and GR, such as Effective Actions (= generating functionals of current Green functions), gauge invariance and anomaly cancellation, "holography", ---

Applications (90's, 2012)

(Topological) Insulators

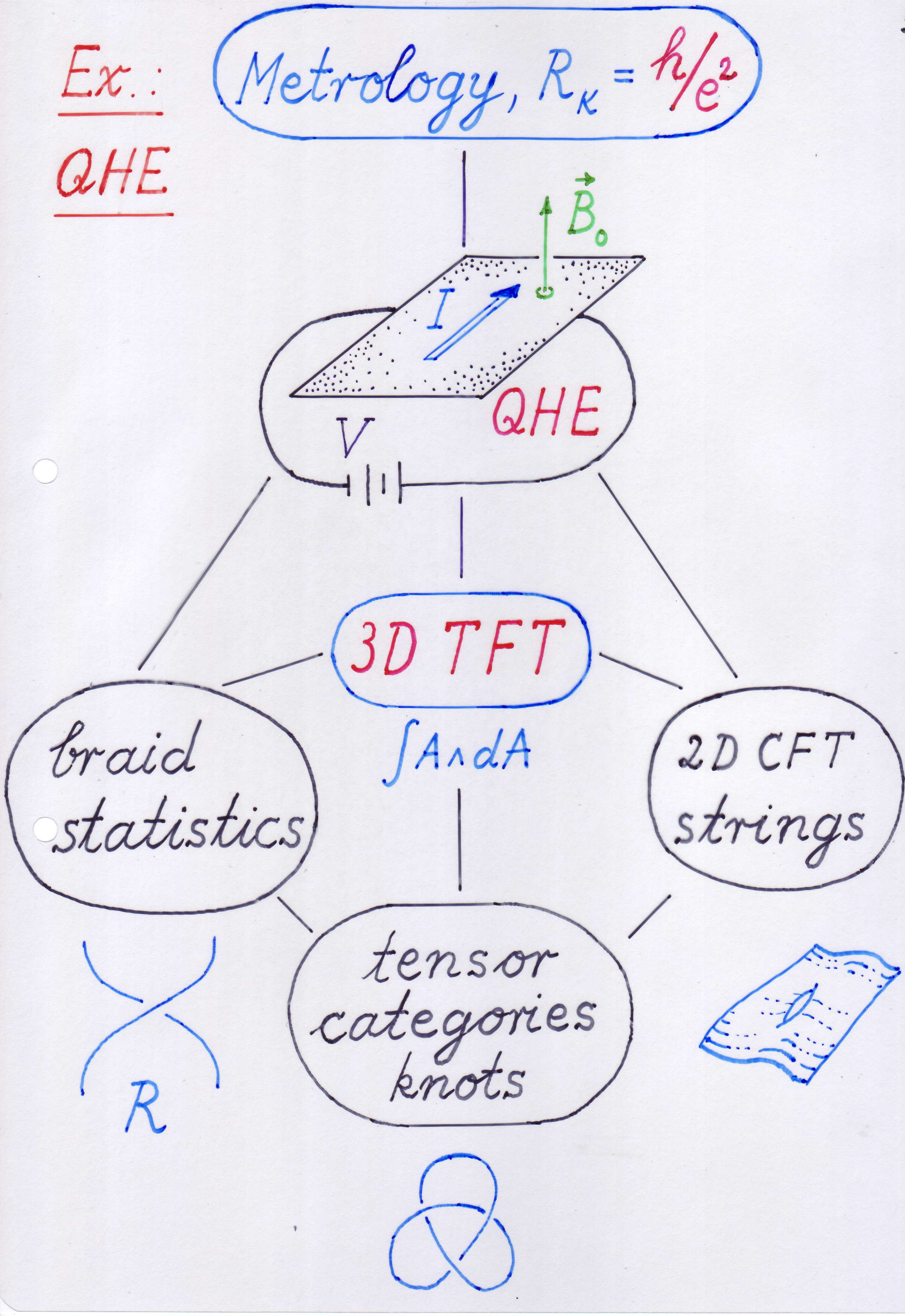
QHE

(Topological) Superconductors

Higher-dimensional cousins of QHE ➔ cosmology

Etc.

Ex.: QHE



Metrology, $R_K = h/e^2$
$\vec{B}_0$
I
QHE
V
3D TFT
$\int A \wedge dA$
braid statistics
2D CFT strings
tensor categories knots
R

1. Remarks on History

1879 : Hall

... holes in semicond.

1966 : Fowler et al. Si MOSFET ... 2DEG

1975 : Kawaji et al. dissip.-less state in Si MOSFET

1978 : Englert & v. Klitzing plateaux

1980 : v. Klitzing $\sigma_H = \frac{e^2}{h} \cdot n$

1982 : Tsui, Störmer, Gossard FQHE in GaAs/GaAlAs; idea of fract. charges

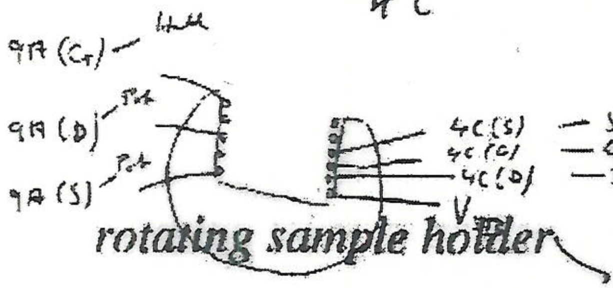
1980 – 1982 : R.B. Laughlin ... theory; $\geq$ 1982 : et al.

QHE

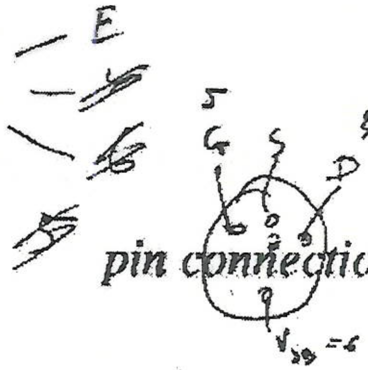
K. von Klitzing

Notes 4./5.2.1980

4 C



rotating sample holder



pin connections

$$E_H = R_H \cdot B \cdot j = \frac{1}{n e} \cdot B \cdot \frac{I}{b}$$

$$U_H = \frac{B}{n \cdot e} \cdot I \qquad N = \frac{eB}{2\pi\hbar} \qquad (g_s \cdot g_v = 1)$$

$$U_H = \frac{2\pi\hbar B}{e \cdot e \cdot B} \cdot I = \boxed{\frac{h}{e^2} \cdot I}$$

~~25,76 kΩ~~
25813

Josephson

$$\frac{\hbar c}{e^2} = \qquad \rho_{xy} = \frac{\alpha}{2} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \Rightarrow 25813\ \Omega$$

notes of the phone call to PTB (5.2.1980)

PTB 531/592-1 2240

Prof. v. Kose

$\mu_0 = 4\pi \cdot 10^{-9} \frac{Vs}{Acm}$

$\varepsilon_0 = 0.8854 \cdot 10^{-13} \frac{As}{Vcm}$

10^{-6} 12945

$\sqrt{\frac{\varepsilon_0}{\mu_0}} = 2.65 \cdot 10^{-3}\ \Omega^{-1}$ $6 \cdot 10^{-2}$ 10^{-5} 12907

$\sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.7\ \Omega$

25813 Ω : N
1 MΩ parallel

25813 →	25163.46
12906.5	12742.04
6453.25	6411.87
3226.63	3216.25
2151.08	2146.47

quantized resistances with and without the input resistance of the x-y recorder

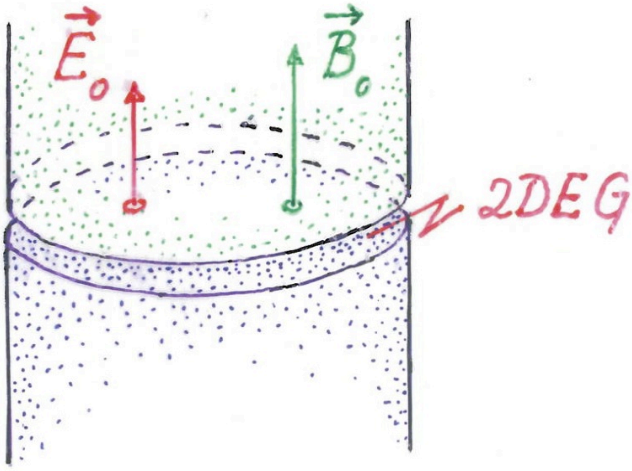
2. What is the FQHE ?

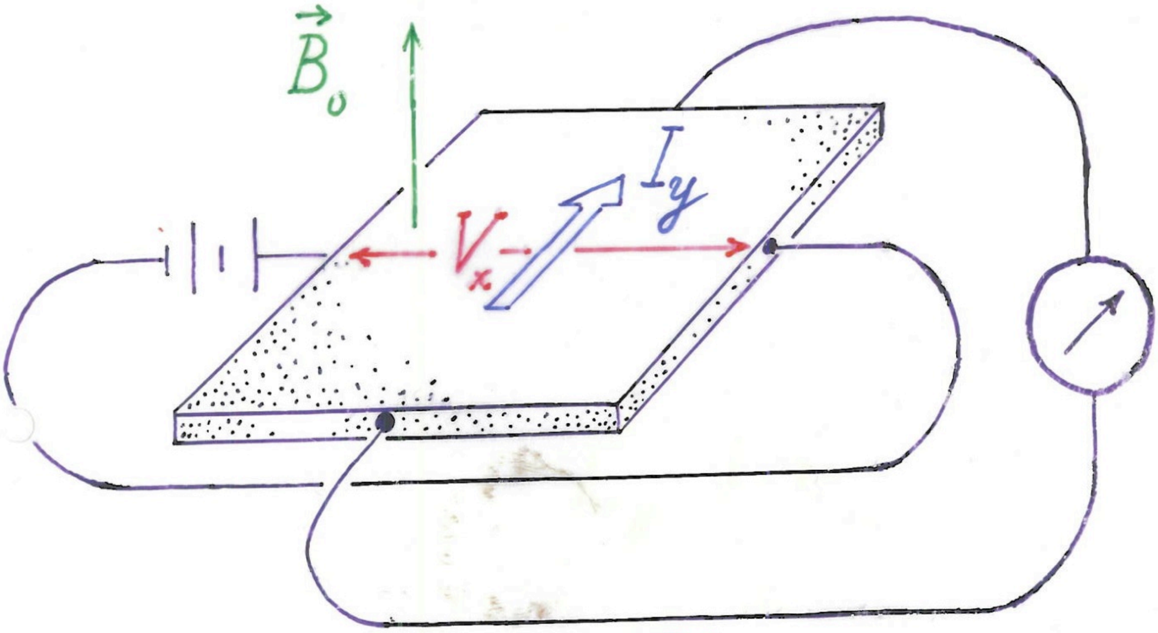
$Ga_x Al_{1-x} As$

Insulator

$GaAs$

Semi-cond





$$\left.\begin{aligned} R_H &= \rho_H = -\frac{V_x}{I_y} \\ R_L &= \frac{V_y}{I_y} \end{aligned}\right\} \text{measured}$$

n : e^- density in 2DEG

$\phi_0 = \frac{hc}{e}$: magn. flux qu.

$\nu = n \cdot \left(|\vec{B}_{0\perp}| / \phi_0 \right)^{-1}$: dim. less

filling factor; # filled Landau levels

Classical theory:

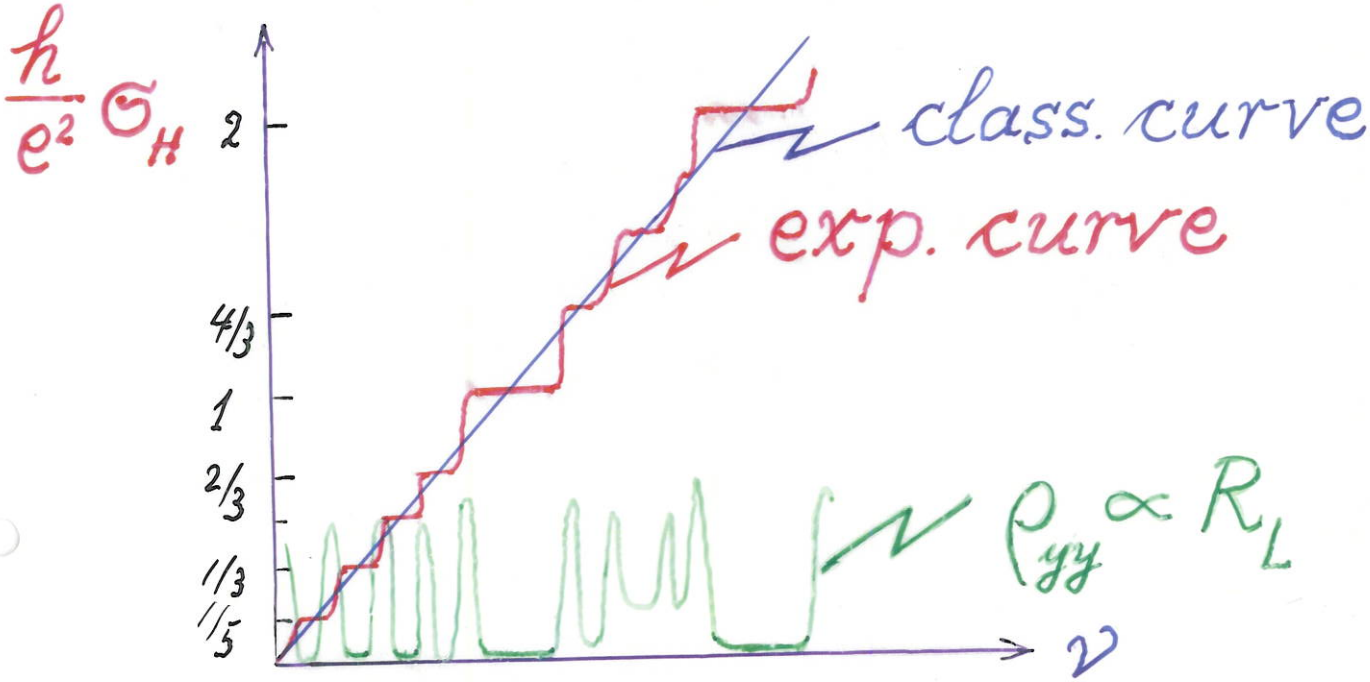
$$\vec{a}_{e^-} = 0 \Leftrightarrow e\vec{E}_{\parallel} = -e \frac{\vec{v}_{e^-}}{c} \wedge \vec{B}_{0\perp}$$

$$\rightarrow \vec{E}_{\parallel} \perp \vec{v}_{e^-}$$

$$\vec{j} = -en\vec{v}_{e^-} =: \sigma_H (\vec{e}_z \wedge \vec{E})$$

$$\Rightarrow \sigma_H = \frac{enc}{|\vec{B}_{0\perp}|} = \frac{e^2}{h} \nu$$

Experiment:



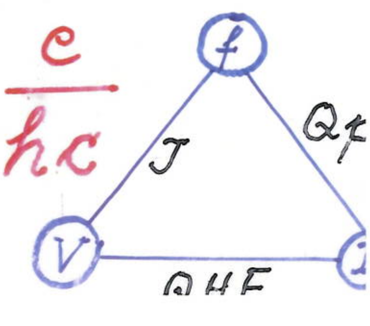
Observations:

I. $R_L = 0 \Leftrightarrow (\nu, \sigma_H) \in$ plateau

II. Plateau heights $\in \frac{e^2}{h}\mathbb{Q}$

Precision - IQHE : $1:10^9$

- QHE $\rightarrow R_K^{-1} = \frac{e^2}{h}$
- Josephson $\rightarrow K_J = \frac{e}{hc}$
- Q - pumps $\rightarrow e$

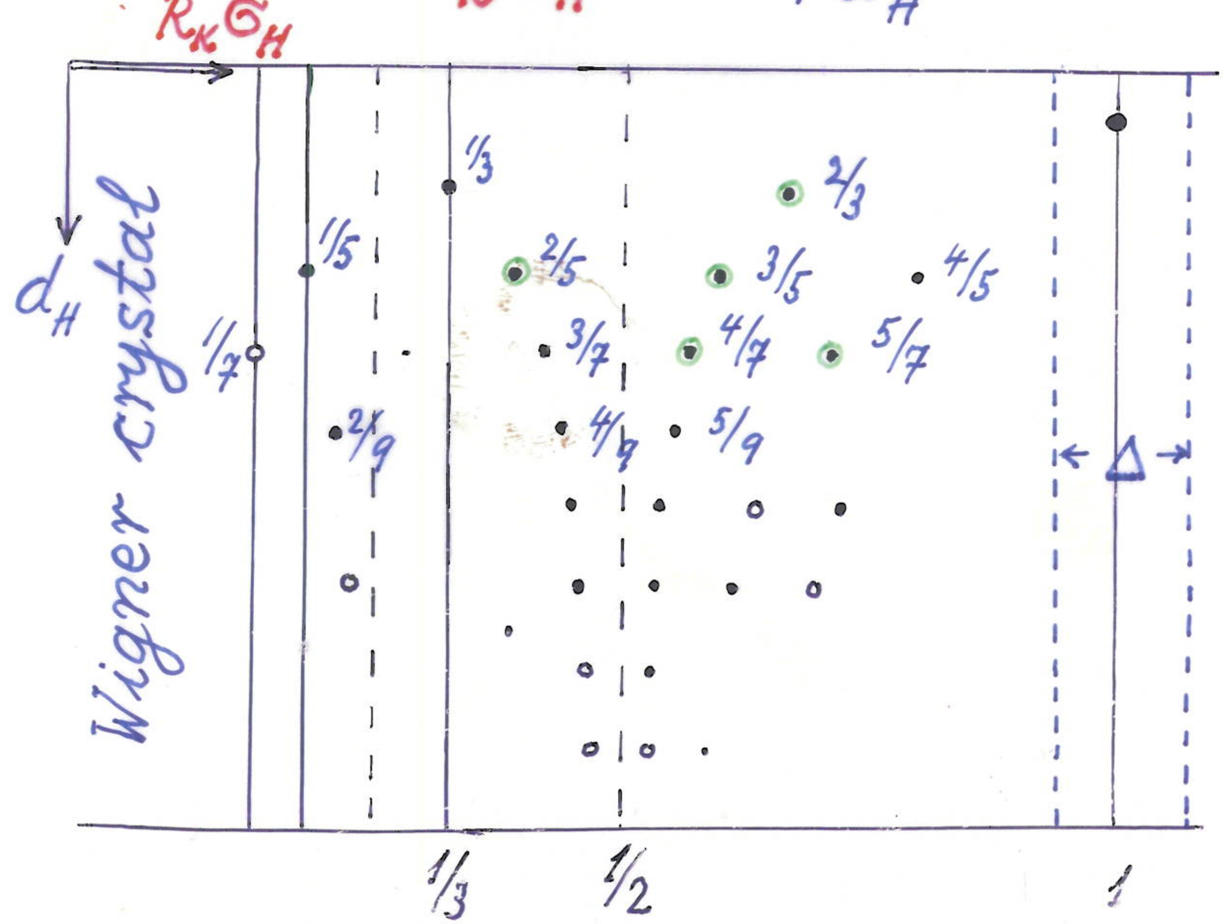


III. The cleaner the sample,

- the more plateaux. obs.
- the narrower the plat.

IV. If $R_K \sigma_H \notin \mathbb{Z}$ (FQHE) $\rightarrow$ fract. el. charges (Tsui; Glattli et al.; interferom.)

$$R_K \sigma_H = n_H / d_H \in \mathbb{Q}$$



Tasks for theorists

(1) For what values of ν is $R_L = 0$ (pos. mobility gap)? How do plateau-width, Δ, scale with disorder? …

→ Many-body th., computer

(2) Assuming that $R_L = 0$ (IHF), what can we say about:

(i) possible values of σ_H? ✓

(ii) spectrum & properties of quasi-particles? ✓

(3) Nature of transitions between neighboring IQHF's?

(4) Wigner crystal for $\nu \lesssim 1/7$;

(5) New exp. tests of theor. predictions? (e.g., interferom.)

<u>Applications</u>:

- Metrology & fund. consts.
- Novel computer memories
- q-<u>bits</u> for topological quantum computers (J.F.!) ↔ exploitation of quasi-particles w. braid stat.

before Friedman, Kitaev, ...!

3. Electrodynamics of IQHF

2DEG confined to planar domain Ω in $\vec{B}_0$; bulk mobility gap $>0 \leftrightarrow R_L = 0$.

Response of 2DEG to small pert. e.m. field, $\vec{E}, \vec{B}$, with

$$\vec{B}^{tot.} = \vec{B}_0 + \vec{B},$$

slowly time-dep.; ("ad. lim.")

Orbital dyn. of e^- dep. only on $B_3^{tot} =: B_0 + B$, $\vec{E}_{\parallel} \equiv \underline{E} = (E_1, E_2)$.

$A := (\phi, A_1, A_2)$ is vector pot. of

$$F := \begin{pmatrix} 0 & E_1 & E_2 \\ -E_1 & 0 & -B \\ -E_2 & B & 0 \end{pmatrix}$$

$\langle (\cdot) \rangle_A$: state of 2DEG

$$j^\mu(x) := \langle J^\mu(x) \rangle_A \,, \quad \mu = 0,1,2 \,. \quad (1)$$

($\underline{1}$) Hall's law ($R_L = 0$)

$$j^k(x) = \sigma_H \, \varepsilon^{kl} E_l(x) \,, \qquad (2)$$

$k, l = 1,2 \,, \quad x = (t, \underline{x}) \in \Lambda := \mathbb{R} \times \Omega$

($\underline{2}$) Charge conservation

$$\frac{\partial}{\partial t} \rho(x) + \underline{\nabla} \cdot \underline{j}(x) = 0 \qquad (3)$$

($\underline{3}$) Faraday's induction law

$$\frac{\partial}{\partial t} B_3^{tot}(x) + \underline{\nabla} \wedge \underline{E}(x) = 0 \qquad (4)$$

Laws ($\underline{1}$) – ($\underline{3}$) imply:

$$\frac{\partial}{\partial t}\rho \overset{(2)}{=} -\underline{\nabla}\cdot\underline{j} \overset{(1)}{=} -\sigma_H \underline{\nabla}\wedge\underline{E}$$

$$\overset{(3)}{=} \sigma_H \frac{\partial}{\partial t} B \qquad (5)$$

Integrate (5) in time t, with

$$j^0(x) := \rho(x) + en$$

$$B(x) = B_3^{tot}(x) - B_0$$

Then $(5) \Rightarrow$

(4) Chern-Simons Gauss law

$$j^0(x) = \sigma_H B(x) \qquad (6)$$

$$(1) \,\&\, (4) \Rightarrow \boxed{j^\mu(x) = \sigma_H \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x)} \quad (7)$$

$$(2) \Leftrightarrow \partial_\mu j^\mu = 0 \Rightarrow j^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda$$

$$(3) \Leftrightarrow \partial_{[\mu} F_{\nu\lambda]} = 0 \Rightarrow F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$$

Then (7) $\Rightarrow$

$$j^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu b_\lambda = \sigma_H\, \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda ,$$

or

$$\boxed{db = \sigma_H\, dA} \qquad (8)$$

But wherever $\sigma_H \neq const.$, e.g. at $\partial\Omega$, Eq. (8) inconsistent:

$$0 = \partial_\mu j^\mu = \varepsilon^{\mu\nu\lambda}(\partial_\mu \sigma_H) F_{\nu\lambda} \neq 0 \quad (8')$$

on $\Sigma :=$ support $(\text{grad}\, \sigma_H)$! $\rightarrow$

$$j^\mu = j^\mu_{bulk} \neq j^\mu_{tot} = j^\mu_{bulk} + j^\mu_{edge},$$

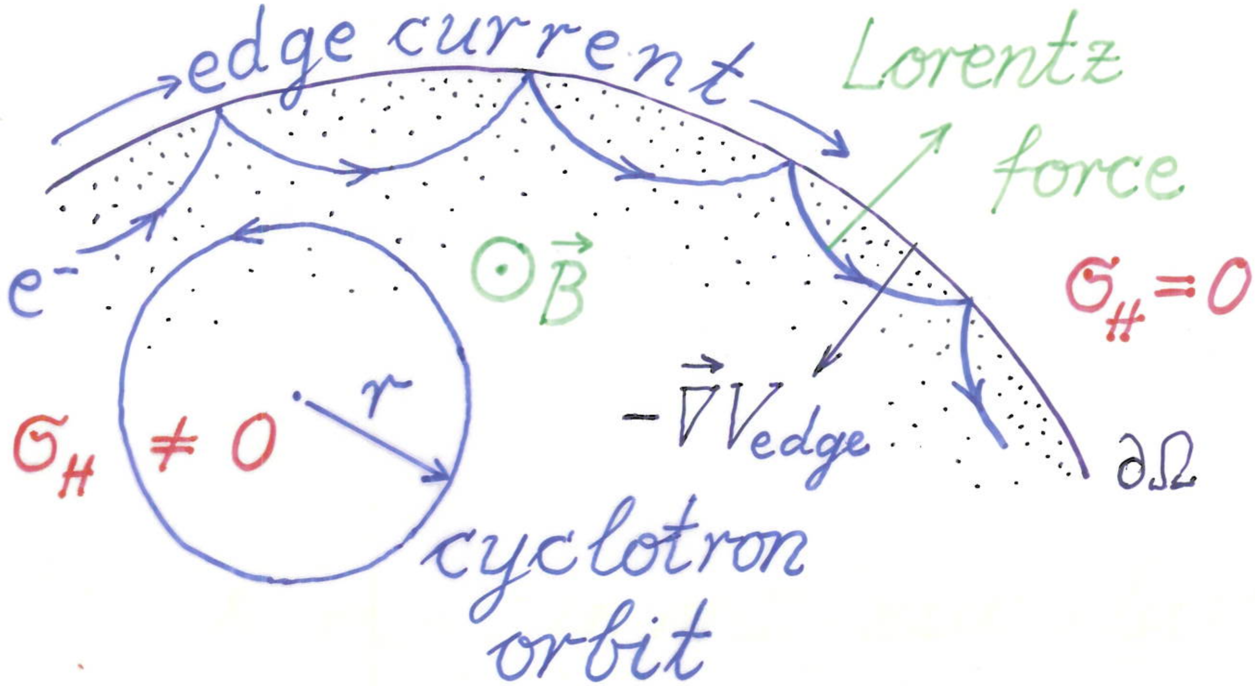
$$\partial_\mu j^\mu_{tot} = 0, \quad \text{supp}\, j^\mu_{edge} = \Sigma,$$

$\underline{j}_{edge} \perp \underline{\nabla}\, \sigma_H$. Then (7) $\Rightarrow$

$$\boxed{\partial_\mu j^\mu_{edge} = -\partial_\mu j^\mu_{bulk} = -\sigma_H\, E_{\parallel}\big|_{\Sigma}} \qquad (9)$$

Chiral anomaly in (1+1)D

Edge current, j^μ_{edge}, is anomalos chiral current in (1+1)D:



At edge:

$$e\,\frac{\vec{v}}{c}\wedge\vec{B} = -\vec{\nabla}V_{edge} \rightarrow \vec{v}$$

Analogous phen. in class. physics: <u>Hurricanes</u>!

$\vec{B}\rightarrow\vec{\omega}_{earth}$, Lorentz $\rightarrow$ Coriolis

$-\vec{\nabla}V_{edge} \longrightarrow -\vec{\nabla}$pressure

Chiral anomaly in (1+1)D:

$$\partial_\mu j^\mu_{edge} = \frac{e^2}{h}\Big(\sum_{species} Q_i^2\Big) E_{\parallel}\Big|_\Sigma ,$$

where $eQ_1, \dots, eQ_n$ are el. charges = c.c.'s to "quasi-part. contributing to j^μ_{edge}. Thus

$$R_K \sigma_H = \sum_{species} Q_i^2 \qquad (10)$$

If $R_K \sigma_H \notin \mathbb{Z} \Rightarrow$ some Q_i's fractional!

<u>IQHF</u>: Each filled "Landau level" contributes one spec. of e^- to j^μ_{edge} $\Rightarrow$

$R_K \sigma_H$ = # filled Landau lev.

4. The bulk of an IHF

$\Sigma = \partial\Omega$, $\Lambda = \mathbb{R} \times \Omega$; $(R_K = 1)$.

$\mathcal{J}^\mu(x)$: q.m. current density

$\langle(\cdot)\rangle_A$: state of IHF in ext. field $(\underline{E} = \dot{A}, B = \nabla \wedge A)$

$S_\Lambda(A)$: eff. action of IHF.

$$j^\mu_{bulk}(x) = \langle \mathcal{J}^\mu_{bulk}(x) \rangle_A = \delta S_\Lambda(A) / \delta A_\mu(x)$$

$$\stackrel{!}{=} \sigma_H \, \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}(x), \quad (x \notin \partial\Lambda)$$

$$\Rightarrow S_\Lambda(A) = \frac{\sigma_H}{2} \int_\Lambda \varepsilon^{\mu\nu\lambda} A_\mu(x) F_{\nu\lambda}(x) dx + \Gamma_{\partial\Lambda}(a)$$

$$= \underbrace{\frac{\sigma_H}{2} \int_\Lambda A \wedge dA}_{\text{from incompr. + P-breaking}} + \Gamma_{\partial\Lambda}(a) \quad (11)$$

where $a := A|_{\partial\Lambda}$, $\Gamma_{\partial\Lambda}(a)$: gen. fu. of Green fns of edge curr.

$\int_\Lambda A \wedge dA$ not gauge-inv. under $A \to A + d\chi$, $\chi|_{\partial\Lambda} \neq 0$; cured by $\Gamma_{\partial\Lambda}(a) \Rightarrow$

J^μ_{edge} is $U(1)$ Kac-Moody curr.

$\partial_\mu J^\mu = 0 \Rightarrow J^\mu = \sqrt{\sigma_H}\, \varepsilon^{\mu\nu\lambda} \partial_\nu B_\lambda$

$S_\Lambda(B, A)$: action of B coupled to vector pot. A.

(11) $\Longleftrightarrow$

$$S_\Lambda(B,A) = \frac{1}{2}\int_\Lambda B \wedge dB + \int_\Lambda J^\mu A_\mu \tag{12}$$

$$+ \underbrace{\text{bd. term}}_{\text{action for } U(1) \text{ Kac-Moody c.}}$$

$S_\Lambda(\mathcal{B}, A)$: action of topol. $U(1)$-Chern-Simons th.

Charge operator:

$$Q_{\mathcal{O}} := \int_{\mathcal{O}} d^2x \, J^0(t, \underline{x})$$

$$= \sqrt{\sigma_H} \int_{\partial\mathcal{O}} \mathcal{B} \qquad \text{(Stokes)}$$

$\Rightarrow e^{iQ_{\mathcal{O}}} = $ Wilson loop$[\partial\mathcal{O}]$

Curvature ($\propto J^\mu$) of $\mathcal{B}$-field conc. in <u>loc. static sources</u>, $|(q,\lambda), z\rangle$, with

$$Q_{\mathcal{O}} |(q,\lambda), z\rangle = \sqrt{\sigma_H}\, q\, |(q,\lambda), z\rangle, \quad z \in \mathcal{O} \tag{13}$$

q: flux of $\mathcal{B}$-field

λ: "internal" quantum #

Mobility gap in bulk $> 0 \Rightarrow$

Bulk of IHF described by

3D TFT

$\sim$ {family of "sectors" $[(q,\lambda)]$} $=: \mathcal{C}$

$\mathcal{C}$ equipped w. composition rule, $\otimes$, and quantum statistics given by braiding, ε; $(S, T, \cdots)$.

$[(q,\lambda), z]$: Space of quasi-particle states w. el. charge $\sqrt{\sigma_H}\, q$ & internal quantum # λ localized near $z \in \mathbb{C} \simeq \mathbb{R}^2$

Spin of quasi-particles:

$$U(Rot_{2\pi})\restriction_{[(q,\lambda)]} = e^{2\pi i\, s(q,\lambda)}\, \mathbb{1}\restriction_{[(q,\lambda)]}$$

$$s(q,\lambda) = q^2/2 + h_\lambda \quad \Big(\underset{i.g.}{\notin} \tfrac{1}{2}\mathbb{Z}\,!\Big) \quad (16)$$

(Fusion rules finite)

$$Ass.^* \Rightarrow h_\lambda \in \mathbb{Q} \quad (\text{Vafa's thm.})$$

If (q^*,λ^*) = quantum # of e^-
then

$$(13) \Rightarrow \sqrt{\sigma_H}\, q^* \overset{!}{=} -1$$

$$(16) \Rightarrow s(q^*,\lambda^*) \overset{!}{=} \ell + \frac{1}{2}, \quad \ell \in \mathbb{Z}.$$

Thus $\frac{1}{2\sigma_H} + h_{\lambda^*} = \ell + \frac{1}{2}$

$$\overset{Vafa}{\Longrightarrow} \quad \sigma_H =: \frac{n_H}{d_H} \in \mathbb{Q}$$

min. el. charge $= e\frac{1}{k\,d_H}$, $k = 1,2,3,\cdots$

Digression on QHE

Classification of incompr. (gapped) bulk theories in scaling limit: 3D TFT's

$\simeq$ quasi-rat., braided, modular tensor cats., $\mathcal{C}$,

w. ab. charge, Q_{em}, simple current, J_e, $[Q_{em}, J_e] = -J_e$, describing electrons.

$$\mathcal{C} = \{\mathcal{O}, (N_{\alpha\beta}^{\gamma}), B, F, Q_{em}, J_e\}$$

$$\ell + 1/2 = \frac{1}{2\sigma_H} + \Delta_e, \quad \Delta_e \in \mathbb{Q}$$

$$\Longrightarrow \quad \sigma_H \in \mathbb{Q}\,!$$

"Holography"

$\exists$ QFT of gapless edge degs. of freedom, QFT_{edge}, with:

(i) chiral Kac-Moody current, j_{em}, descr. diam. em edge currents;

(ii) superselection sectors described by $\mathcal{C}$.

In general, QFT_{edge} not conformal, because diff. edge modes have different propagation speeds, $\{v_i\}$.

Example. If $\mathcal{C}$ has abelian braid statistics

$$\mathcal{C} \leftrightarrow \{\Gamma, Q \in \Gamma^* \text{ visible}\},$$

$$\Gamma \ni q_e, \text{ w. } Q \cdot q_e = -1,$$

$$\sigma_H = \langle Q, Q \rangle, \langle q, q \rangle = Q \cdot q \mod. 1.$$

$$\Gamma \supset \Gamma_{Kneser} \oplus \underbrace{\Gamma_{Witt}}_{A, D, E_6, E_7 \text{ root lattices}}$$

$$Q \cdot \Gamma_W = 0, \ (\text{for } \sigma_H \leq 1)$$

$$N = \operatorname{rank} \Gamma$$

Edge degrees of freedom

$\exists$ N gapless, chiral scalar Bose fields, $\varphi_1, \dots, \varphi_N$, with propagation speeds $v_1, \dots, v_N$ (possibly all different); vacuum Ω, with

(i) $j_{em} = \sum_{i=1}^{N} Q_i \cdot \partial\varphi_i$, $Q = (Q_1, \dots, Q_N)$

(ii) Phys. states obt. by appl.

- polyn. in $\{\partial\varphi_1, \dots, \partial\varphi_N\}$
- vertex operators

$$:\exp i \sum_{j=1}^{N} q_j^\alpha \varphi_j:,$$

to Ω,

$$q^\alpha = \begin{pmatrix} q_1^\alpha \\ \vdots \\ q_N^\alpha \end{pmatrix} \in \Gamma$$

(q_j^α) analogous to CKM.

Theory has "approx." Kac-

Moody symm. at level 1 corresp. to Γ_{Witt}. But if $v_1, \dots, v_N$ not all equal symm. not exact, QFT_{edge} not conformal.

Observable consequences.

"Visibility" in interference experiments (M-Z, ↗ L & S) involving tunneling of quasi-particles across quantum point contacts (drawing!)

⟶

5. Summary

Physics of IQHF ($\sigma_H, q^{el.}_{min}, \ldots$)

encoded in

3D TFT ($\supset$ CS th. for B)

$\sim$ quasi-rat. braided $\otimes$ cat.

holography $\sim$ chiral 2D QFT

$\uparrow$

descr. edge degs. of freedom

(anomalous chir. edge curr.)

$\rightarrow$ quasi-parts. w. quantum #

(q, λ), exh. braid statistics.

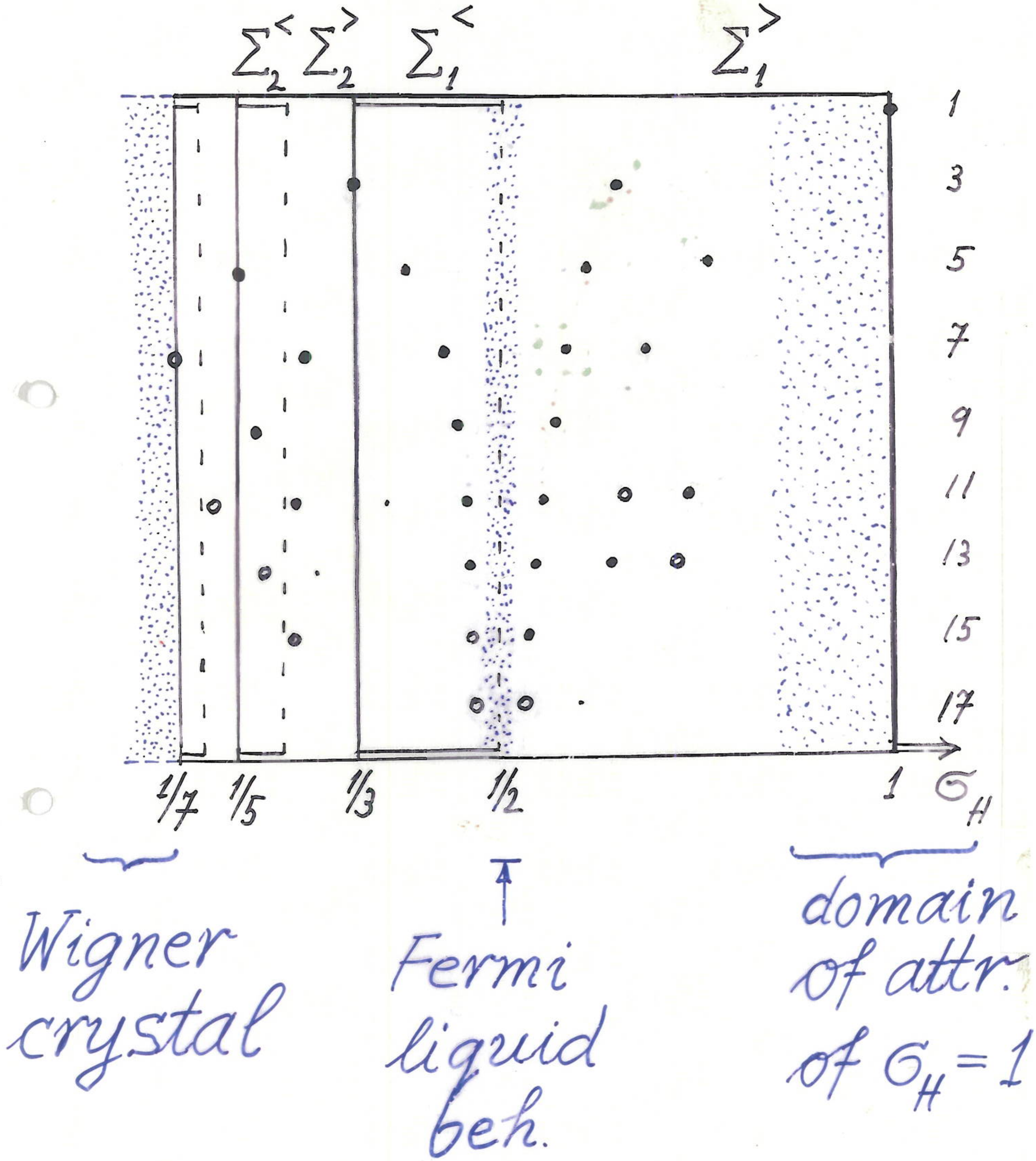
For abelian braid statistics:

IQHF $\leftrightarrow$ (Γ, $\underline{Q} \in \Gamma^*$, "CKM")

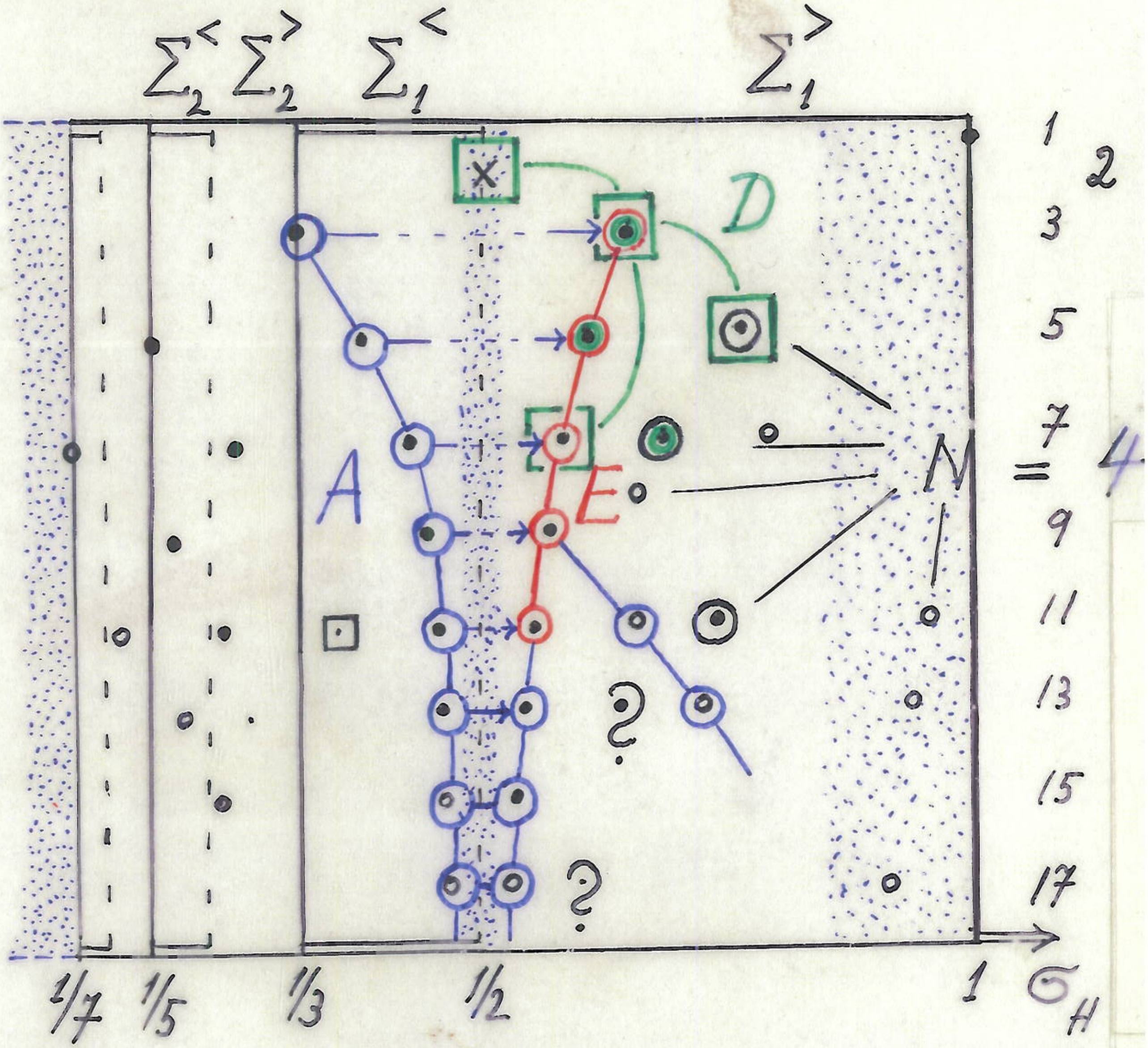
Γ: odd-int. lattice

Q: visible vect. in Γ^*: $\sigma_H = Q \cdot Q \in \mathbb{Q}$

Exp. data.


$\Sigma_2^<$ $\Sigma_2^>$ $\Sigma_1^<$ $\Sigma_1^>$
1
3
5
7
9
11
13
15
17
1/7
1/5
1/3
1/2
1
σ_H
Wigner crystal
Fermi liquid beh.
domain of attr. of $\sigma_H = 1$

Exp. data.



Wigner crystal

Fermi liquid beh. (marginal liquid)

domain of attr. of $\sigma_H = 1$

FROM THE QHE TO "TOPOLOGICAL INSULATORS"

A UNIFIED PERSPECTIVE

Jürg Fröhlich

em., ETH Zürich

Bieri, Boyarsky, Cheianov, Graf, Kerler, Levkivsky, Pedrini, Ruchayskiy, Schweigert, Studer, Sukhorukov, Thiran, Walcher, Werner, Zee – R. Morf

1. Introduction

Purpose of analysis (90's)

Classify

- states of NR bulk (cond.) matter at $T \geq 0$; &
- its surface states —

using ideas & concepts from gauge theory & GR: (scaling lim of) effective actions, gauge invariance, anomaly cancellation, "holography".

Illustrate program on 2D & 3D electron gases; ($\cdots\rightarrow$ atom gases, BEC, ..., primordial plasma, stellar matter, ... : N.t.)

General ideas of approach:

(1) Find all (fund. & accid.) global (int.) symmetries & corresp. conserved currents.

(2) Promote global to local symmetries – "gauging".

(3) Study response of syst. to turning on "tiny" external gauge fields & varying g_{ij}.

→ determine (form of) effective action (free energy) = gener. funct. of <u>current</u> Green fns.

(1) + "order param." → Landau theory

(1) ÷ (3) → "<u>Gauge Theory of States of Matter</u>" (early 90's)

"topological <u>order</u>".

Today, will apply this to el. systems w. bulk mobility gap: QHE, (2+1)D & (3+1)D "topol. insulators", axion QED

2. Electron Gases in 2&3 D

QM of <u>single el.</u> governed by **Pauli Equation**:

$$i\hbar \frac{\partial}{\partial t} \Psi_t = \frac{1}{2m}\left(\frac{\hbar}{i}\vec{\nabla}\right)^2 \Psi_t \,, \quad \text{(PE)}$$

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow(x) \end{pmatrix} \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$$

(1) Symmetries of (PE)

- Global phase rot. : $U(1)$

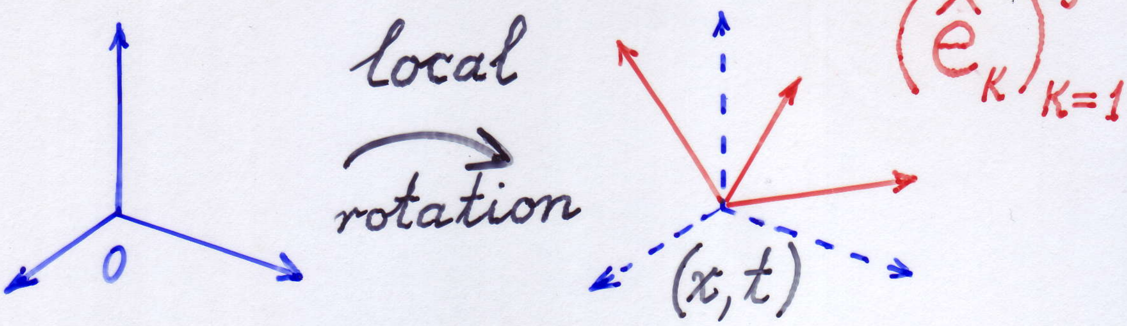
Conserved current: $j^0 = \Psi^* \cdot \Psi$,

$$\vec{j} = \frac{i\hbar}{2m}\left\{ \Psi^* \cdot \vec{\nabla}\Psi - (\vec{\nabla}\Psi^*)\cdot \Psi \right\}$$

- Rotations in spin space: $SU(2)$

$\longrightarrow$ spin current: s^0_κ, $\vec{s}_\kappa \in su(2)$.

(2) Gauge these symmetries:



space-time dep. frames

$$\frac{\hbar}{i}\frac{\partial}{\partial x^j} \longmapsto \frac{\hbar}{i} D_j := \frac{\hbar}{i}\partial_j + a_j + w_j$$

$$i\hbar\frac{\partial}{\partial t} \longmapsto i\hbar D_0 := i\hbar\partial_t + a_0 + w_0$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad U(1) \quad SU(2)$$

$$w_\mu = w_\mu^\kappa \sigma_\kappa$$

covariant derivatives

$$(\delta^{ij}) \longmapsto (g^{ij}(x))$$

Covariant Pauli Eq.:

$$i\hbar D_0 \Psi_t = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}} D_k\left(\sqrt{g}\, g^{kl}\right) D_l \Psi_t$$

(CPE)

Action:

$$S(\Psi^*, \Psi; g, a, w) :=$$

$$\int dt \int \sqrt{g}\, d^p x \Big[\Psi_t^* \cdot i\hbar D_0 \Psi_t - \frac{\hbar^2}{2m} (D_k \Psi_t^*) \cdot g^{kl} D_l \Psi_t - \lambda |\Psi_t|^2 * \Phi\, |\Psi_t|^2 \Big]$$

For $\lambda = 0$,

$$\frac{\delta S}{\delta \Psi^*} = 0 \iff \text{(CPE)}$$

$\lambda \neq 0$: 2-body interactions with potential $\lambda \Phi$

Many-body th.:

$$(\Psi^*, \Psi) \longmapsto (\bar{\Psi}, \Psi) \quad \text{Grassm. V.}$$

(3) Berezin integral:

$$Z(g,a,w) := \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\, e^{\frac{i}{\hbar} S(\bar{\Psi},\Psi; g,a,w)}$$

$$\boxed{S_{eff}(g,a,w) := \frac{\hbar}{i} \ln Z(g,a,w)}$$

"effective action"

At $T = (k_B \beta)^{-1} > 0$, for g, a, w time-independent, may study thermal equilibr.:

$$S_{eff} \longmapsto F_{eff}(\beta; g,a,w)$$

"eff. free energy"

$$\lim_{\beta\to\infty} F_{eff}(\beta; g,a,w) = \Delta E_0(g,a,w)$$

g.s. energy

Properties of S_{eff}/F_{eff}

(i) $$\frac{\delta S_{eff}}{\delta g^{ij}(x)} = \langle \underbrace{T_{ij}(x)}_{\text{stress tensor}} \rangle_{g,a,w}$$

$$\frac{\delta S_{eff}}{\delta a_\mu(x)} = \langle \underbrace{j^\mu(x)}_{\text{em current density}} \rangle_{g,a,w}$$

$$\frac{\delta S_{eff}}{\delta w_\mu^{\kappa}(x)} = \langle \underbrace{S^\mu_\kappa(x)}_{\text{spin current --"--}} \rangle_{g,a,w}$$

Higher derivatives: Connected current Green functions of $n \geq 2$ current densities.

(ii) Gauge invariance

$$S_{eff}(g, a_\mu + \partial_\mu \chi, U w_\mu U^{-1} + U \partial_\mu U^{-1})$$

$$= S_{eff}(g, a_\mu, w_\mu)$$

χ real-valued, U $SU(2)$-valued

$\Leftrightarrow$ j^μ <u>cons.</u>, s^μ_κ <u>covar. cons.</u>

(ii′) <u>General covariance</u>

(iii) Bulk mobility gap

$\Downarrow$

Al. props. of current Green fus.

$\Downarrow$

In scaling lim,

$$S_{eff} = \sum_n \underbrace{\int \text{"gauge-inv." local poly.}}_{\text{scaling dim } n \,(= -1, 0, 1, 2, \dots)} + \text{bd. terms}$$

Retain only "most relevant" terms $(n=-1,0,1)$. Ex.: $D=2, \omega=0$

$$S_{eff}(a) = \frac{\sigma_H}{2} \int_\Lambda \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \, d^3x$$

$$+ \text{bd. term}$$

QHE

3. Phys. Interpretation of a & ω

$$a_0 = e\varphi - \rho^{-1} p \quad \cdots$$

$$a_k^{tot} = eA_k^{(0)} + \underbrace{eA_k + mV_k}_{\equiv a_k}$$

curl $A^{(0)}$: time-indep., per. or hom.

V: velocity field of moving background – gauge inv., usually div $V = 0$.

p: pressure ...

$$w_0^{tot} = \left(\frac{g\mu_B}{2}\vec{B} + \frac{\hbar}{4}\,curl\vec{V}\right)\cdot\vec{\sigma}$$

Zeeman

$$+\ \vec{W}_0\cdot\vec{\sigma}$$

Weiss exch. field

$$+\ \vec{\omega}_0\cdot\vec{\sigma}$$

spin conn.

$$w_k^{tot} = \left(\frac{g\mu_B}{2} + \frac{e}{4mc}\right)\left(\vec{\sigma}\wedge\left(\vec{E}+\dot{\vec{V}}\right)\right)_k$$

spin-orbit

$$+\ \vec{W}_k\cdot\vec{\sigma} \quad + \ \vec{\omega}_k\cdot\vec{\sigma}$$

Weiss   spin conn.

Only spin connection transf. inhomogeneously under SU(2)-gauge trsf.; other contribs. to w_μ^{tot} transf. homogeneously.

4 Effective Actions / Free Energies – Examples

Electron gas in ext. gauge fields $(a_\mu^{tot}, w_\mu^{tot})$ in 2+1 or 3+1 D.

U(1) $a_\mu^{tot} = a_\mu^{(0)} + a_\mu = eA_\mu^{(0)} + eA_\mu + mV_\mu$

gauge-inv

SU(2) $w_\mu^{tot} = w_\mu^{(0)} + w_\mu$

trsf. homog.

$a^{(0)}, w^{(0)}$: diff. geom. of sample backgd., static em field

$\hookrightarrow$ explicit breaking of P, T.

For simplicity, g^{ij} fixed ($g^{ij} = \delta^{ij}$). Apply (i)÷(iii):

- $\underline{2+1\,D}$, $w_\mu = 0$, $a_\mu \neq 0$.

$$S_{eff}(a) = (2\lambda^2)^{-1} \int_\Lambda (a^T)^2 d^3x \qquad \mathrm{I}$$

$$+ \frac{\sigma_H}{2} \Big[\int_\Lambda a \wedge da + \Gamma(a|_{\partial\Lambda}) \Big] \qquad \underline{\mathrm{II}}$$

$$+ \frac{1}{2} \int_\Lambda (\varepsilon\, \underline{\mathcal{E}}^2 + \mu^{-1} \mathcal{B}^2)\, d^3x + \cdots, \qquad \underline{\mathrm{III}}$$

$$\underline{\mathcal{E}} = -e\underline{\nabla}\varphi + e\dot{\underline{A}} + \underline{\nabla}(\rho^{-1}p) + m\dot{\underline{V}}$$

$$\mathcal{B} = \mathrm{curl}\,(e\underline{A} + m\underline{V})$$

I) Supercond., London eq. ("relevant")

$\underline{\mathrm{II}}$) Hall effect ("marginal")

$\underline{\mathrm{III}}$) Maxwell term → diel. cst. ε, magn. perm. μ ("irrelevant" → neglect)

Set $\lambda^{-2} = 0$, neglect $\underline{\text{III}}$.

Under a gauge trsf., $a \mapsto a + d\chi$,

$$\int_\Lambda a \wedge da \mapsto \int_\Lambda a \wedge da + \int_{\partial\Lambda} \chi \, da$$

$\Rightarrow \Gamma(a|_{\partial\Lambda})$ is anomalous chir. action = gen. fu. of Green fu. of <u>chiral edge currents</u>.

<u>Response Eqs.</u> – see (i):

$$j^0 = \sigma_H \left(B + \frac{m}{e} \operatorname{curl} \underline{V} + \varkappa K + \cdots\right)$$

$$j^k = \sigma_H \varepsilon^{kl} \underbrace{\left(E_l + \frac{m}{e} \dot{V}_l\right)}_{\mathcal{E}_l} + \cdots \quad (HE)$$

$$\frac{\delta \Gamma(a|_{\partial\Lambda})}{\delta a|_{\partial\Lambda}} = j_{edge}$$

$$\partial_\mu j^\mu_{edge} = -\sigma_H \mathcal{E}|_{\partial\Lambda} \quad (\text{Chir. An.})$$

- $\underline{2+1\,D}$, $a_\mu = 0$, $w_\mu \neq 0$

$$S_{eff}(w) = \chi \int_\Lambda tr(w_0^2)\, d^3x$$
$$+ \tilde{\chi} \int_\Lambda tr(\underline{w}^2)\, d^3x$$
$$+ \frac{k}{4\pi}\Big[\int_\Lambda tr\Big(w^{tot} \wedge dw^{tot} + \frac{2}{3}(w^{tot})^{\wedge 3}\Big) + \Gamma_{WZW}\big(w^{tot}\big|_{\partial\Lambda}\big)\Big]$$
$$+ \cdots, \quad k \in \mathbb{Z}$$

$$\boxed{k = 0, \pm 1} \qquad \text{"}\mathbb{Z}_2\text{"}$$

spin (-current) Hall effect

Γ_{WZW}: gen. fu. of Green fus. of chiral edge spin currents at level $k = 2 \times$ spin of quasi-pt.

2+1 D "topological insulator"

(J.F. et al. 1993)

• <u>3+1 D</u>, $w_\mu = 0$, $a_\mu \neq 0$

$$S_{eff}(a) = S_{Maxwell}(a) + S_\theta(a)$$

$$S_{Maxwell}(a) = \frac{1}{2\alpha}\int[\varepsilon\vec{\mathcal{E}}^2 + \mu^{-1}\vec{\mathcal{B}}^2]\,d^4x$$

$$S_\theta(a) = \Theta\frac{1}{4\pi^2}\int\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\,d^4x$$

∞ ext. sample, $\vec{\mathcal{E}}, \vec{\mathcal{B}}$ vanish at ∞ $\Rightarrow$

$$\frac{1}{4\pi^2}\int\vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\,d^4x = n \in \mathbb{Z}$$

$\Rightarrow$ Bulk of system inv. under P & T iff

$$\boxed{\Theta = 0, \pi}$$

"topological insulator" in 3 dim.

Consider finite sample confined to Λ w. bd. $\partial\Lambda$.

$$S_\theta^\Lambda(a) \underset{\uparrow}{=} \frac{\theta}{4\pi^2} \int_{\partial\Lambda} a \wedge da$$

Stokes

For $\theta = \pi$, $S_{\theta=\pi}^\Lambda$ is effective action of (2+1)D 2-component relativistic Dirac fermions (F-M-S '76; D-J-T, Redl. '80s)

= surface modes of (3+1)D "topological insulator"

Promote Θ to dyn. field φ

= "axion field"

$$S(\varphi, a) = \frac{1}{4\pi^2}\int \varphi\, F_a \wedge F_a \qquad \text{(AxTI)}$$

diml. reduction of (4+1)D Hall effect (F-P[-W] '99)

(AxTI) may arise in insulators with 2 filled bands (bonding, anti-bd.). If els. couple to P-, T- breaking background (e.g., anti-ferro order, chiral structure)

$\rightarrow$ (AxTI), φ : backgd. flucts.

Add axion action

$$\frac{J}{2}\int[\dot{\varphi}^2-(v\vec{\nabla}\varphi)^2-U(\varphi)]\,d^4x$$

→ Possibility of axion domain walls → surface modes ~ 2-comp. Dirac fs.

- Instabilities (F-P '99, ...)

Simplest example:

$\vec{\mathcal{E}}=0$, $\vec{\mathcal{B}}$ time-indep.

$\dot{\varphi}=:\Delta\mu$ — " —

$$F_{eff}(\vec{a})=cst\int\vec{\mathcal{B}}^2d^3x+\frac{\Delta\mu}{4\pi^2}\int\vec{a}\cdot\vec{\mathcal{B}}\,d^3x$$

Minimize F_{eff} → magnetic instability for $|\vec{k}|\leqslant cst.\,\Delta\mu$.

The 4D/5D QHE and Cosmic Magnetic Fields

Jürg Fröhlich
ETH Zurich & IAS, Princeton
August 2014

Credits:

Work carried out with

Bill Pedrini (1999) (& Philipp Werner)

and continued with

Alexey Boyarsky and Oleg Ruchayskiy

(2009-present)

Summary

It is argued that the evolution of magnetic fields in a primordial plasma at $T > 10$ MeV is affected by the *chiral anomaly*. An asymmetry between left- and right-chiral leptons, reflected in a non-zero difference between left- and right chemical potentials, develops in the presence of strong magnetic fields. This results in a transfer of magnetic helicity from short to long distance scales. The asymmetry between left and right chiral leptons strongly affects many processes in the early universe.

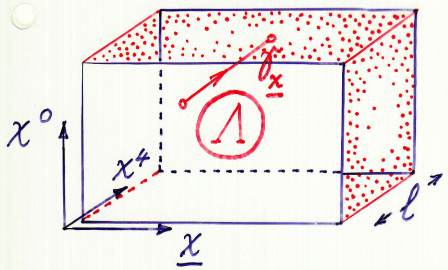
The difference between (space-time dependent) *left- and right chemical potentials* is interpreted as the time derivative of an *axion field.* Equations of motion for axion – electrodynamics are derived, and an instability leading to the growth of magnetic fields in the early universe is exhibited.

A related instability appears in the theory of certain TI's.

This story can be viewed as a 5D analogue of the QHE.

A higher-diml. cousin of the QHE

"Space-time", Λ = slab of width $\ell \subset M^5$; $\partial\Lambda$: two $\parallel$ 3-branes ("visible world")



5D vector pot.

$\hat{A} = (A, A_4)$

$A := \hat{A}_{\parallel}\big|_{\partial\Lambda}$

Bulk degrees of freedom: Very heavy particles (4-comp. Dirac fermions) coupled to $\hat{A}$, w. breaking of P & T.

→ ~ massless, chiral surface

(left-handed waves *(L)* at $x^4 = 0$, right-handed ones *(R)* at $x^4 = l$)

waves (masses from tunneling between branes): observed light particles ~ edge degrees of freedom in Hall fluid.

⇓

5D analogue of Hall's Law

$$j^\mu = \frac{1}{4}\sigma_H \varepsilon^{\mu\nu\lambda\rho\sigma} \hat{F}_{\nu\lambda} \hat{F}_{\rho\sigma} \qquad (13)$$

or

$$\boxed{J = \sigma_H \hat{F} \wedge \hat{F}}$$

valid except where σ_H jumps

$j^\mu_{tot} = j^\mu_{bulk} + j^\mu_{brane}$: conserved

$$j^\mu_{brane} = j^\mu_L \, \delta_{\partial_-\Lambda} + j^\mu_R \, \delta_{\partial_+\Lambda}$$

$$(13) \Rightarrow \boxed{\partial_\mu j^\mu_{L/R} = \sigma_H \, \underline{E} \cdot \underline{B}} \qquad (14)$$

Chiral anomaly in 3+1 D!

$$\sigma_H = \sum_{\substack{\text{fermion} \\ \text{species}}} \frac{Q_i^3}{4\pi^2} \qquad (15)$$

Axion

$$\varphi(\underline{x}, t) := \int_{\gamma_{\underline{x}}} \hat{A}_4(\underline{x}, x^4, t)\, dx^4$$

$\hat{A}_\mu = A_\mu$, $\mu = 0,1,2,3$, indep. of x^4.

$$\dot{\varphi}(\underline{x}, t) = \int_{\gamma_{\underline{x}}} \hat{E}_4(\underline{x}, x^4, t)\, dx^4$$

$$= V(\underline{x}, t) = \mu_R - \mu_L \qquad (16)$$

Then (13) becomes:

$$j^0 = \frac{\sigma_H}{6} \, \underline{\nabla} \cdot (\varphi \underline{B})$$

$$\underline{j} = \frac{\sigma_H}{6} \left\{ (\varphi \underline{B})^{\cdot} + \underline{\nabla} \wedge (\varphi \underline{E}) \right\} \qquad (17)$$

For $\varphi = (\mu_R - \mu_L)\, t$,

$$\boxed{\underline{j} = \frac{\sigma_H}{6} (\mu_R - \mu_L) \underline{B}} \qquad (18)$$

Plug (17) into Maxwell's Eqs.

$$\underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \wedge \underline{E} + \dot{\underline{B}} = 0 \qquad (19)$$

$$\underline{\nabla} \cdot \underline{E} = \frac{\sigma_H}{6} \, \underline{\nabla} \varphi \cdot \underline{B}$$

$$\underline{\nabla} \wedge \underline{B} - \dot{\underline{E}} = \frac{\sigma_H}{6} \left\{ \dot{\varphi} \underline{B} + \underline{\nabla} \varphi \wedge \underline{E} \right\} \qquad (20)$$

5D Maxwell $\vee$ (13) $\cdots \Rightarrow$

$$\Box \varphi = - \frac{\sigma_H}{6} \, \underline{E} \cdot \underline{B} - U'(\varphi) \qquad (21)$$

More plausibly, the axion field satisfies a non-linear diffusion equation.

U': periodic fu. of φ.

System of NL hyperbolic PDE's $\to$ Toy for mathematician

Special solution:

$\underline{E} = \underline{B} = 0$; $\varphi = \varphi(t)$, ind. of $\underline{x}$,

solution of

$$\ddot{\varphi} = -U'(\varphi) \quad \text{(pendulum)}$$

Linearization of (19)–(21)

around this solution:

Parametric resonance $\to$

unstable Fourier modes,

$\underline{\tilde{E}}_{\underline{k}}$, $\underline{\tilde{B}}_{\underline{k}}$ with $\underline{\tilde{E}}_{\underline{k}} \cdot \underline{\tilde{B}}_{\underline{k}} \neq 0$.

A simple special case:

$$\underline{j} = \sigma_T \underline{B} + \sigma_\Omega \underline{E} \leftarrow \text{Ohm}$$

$$\sigma_T = \sum_i \frac{q_i^2}{4\pi h} (\mu_r^i - \mu_\ell^i)$$

$\sigma_\Omega \gg \sigma_T > 0$: primordial plasma

Maxwell's eqs.

$$\underline{\nabla} \cdot \underline{B} = 0, \quad \underline{\nabla} \cdot \underline{E} = 0.$$

$\Rightarrow$ $\underline{E}$, $\underline{B}$ transv. pol.

$$\underline{\nabla} \wedge \underline{E} + \dot{\underline{B}} = 0, \quad \underline{\nabla} \wedge \underline{B} - \dot{\underline{E}} = \sigma_T \underline{B} + \sigma_\Omega \underline{E}$$

Fix wave vector $\underline{k} = k\underline{e}_3$, $k > 0$; $\underline{X}^T$: comp. of $\underline{X} \perp \underline{k}$

Then

$$\begin{pmatrix} \underline{\dot{E}}^T \\ \underline{\dot{B}}^T \end{pmatrix} = K(k) \begin{pmatrix} \underline{E}^T \\ \underline{B}^T \end{pmatrix}, \quad \text{with}$$

$$K(k) = \left(\begin{array}{cc|cc} -\sigma_\Omega & 0 & -\sigma_T & -ik \\ 0 & -\sigma_\Omega & ik & -\sigma_T \\ \hline 0 & ik & 0 & 0 \\ -ik & 0 & 0 & 0 \end{array}\right)$$

Eigenvalues of $K(k)$: $i\omega_\alpha(k)$, $\alpha = 1, \dots, 4$ (circ. frequ. of normal modes).

If $i\omega_{\alpha_0}(k) > 0 \Rightarrow$ expon. growing normal mode!

$$i\omega(k) = \frac{-\sigma_\Omega \pm \sqrt{\sigma_\Omega^2 - 4k(k \pm \sigma_T)}}{2}$$

(hand-made calculation)

$\Rightarrow$ For $0 < k < \sigma_T$, $\exists!$ one positive solution,

$$i\omega_{\alpha_0}(k) \approx \frac{k(\sigma_T - k)}{\sigma_\Omega}$$

$\hookrightarrow$ Origin of seed magn. fields in the universe?

But need "large", time-dependent initial axion configuration.

Remarks

(1) The (time derivative of the) axion field really might be a space-time dependent "chiral chemical potential" (rather than a dynamical degree of freedom). Thus, presumably, its equation of motion is a diffusion equation.

(2) 4D Electrodynamics in the presence of an axion field also appears in the theory of certain 3D topological insulators (TI) with two filled bands. Similar instabilities might then be observed when an external electric field is applied: If such a field exceeds a certain critical strength then, in the bulk of a TI, it is screened and converted into a magnetic field; (Ooguri & Oshikawa, Fröhlich & Werner)

Everything else next time!

Thank you!

My Manifesto

I propose that, at all colleges and universities of the so-called civilized world – in Europe and the Americas – *one or two days per semester* will be declared to be

Days of Reflection and of Protest

During these days, we will not teach or attend committee meetings, and there won't be any exercise classes. Instead, we will discuss some of the serious problems threatening our civilization, draft declarations and reach out to the media, with the aim to make it clear to all circles wielding power that we no longer accept:

My Manifesto, ctd.

- That internal tensions and conflicts in countries belonging to the so-called civilized world, such as the *Ukraine*, are “solved” by armed conflicts rather than by political dialogue and compromise.
- That innocent people are slaughtered in ugly civil wars and by terrorist activities, such as those in Syria and Iraq.
- That countries threaten other countries with warfare.
- That weapons are sold to (clans) in countries plagued by civil war or other forms of unrest and conflict.
- That religions are abused for purposes of power and suppression.
- That the dignity and the rights of women are abused and offended in the name of religion.

My Manifesto, ctd.

- That people are harassed or killed because of their race or faith.
- That nothing is done against the perversions of 21st Century Capitalism.
- That the resources of Planet Earth continue to be looted shamelessly.

These are but some examples of numerous problems threatening the survival of humankind in peace and dignity. –

Where is the *"Peace Movement"*, where are movements such as *"Occupy Wall Street"*, *"Survivre et Vivre"*? What is the *"Club of Rome"* doing? Why are the media silent about the activities of these and other groups?

***

Students and Academics of Europe and the Americas, raise your voices, arise!