Gauge theory of states of matter - old and new

Jürg Fröhlich
Em., ETH Zurich

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Dedicated to the memory of *Edward Nelson*

Credits

S. Bieri, A. Boyarsky, V. Cheianov, G.M. Graf, T. Kerler, I. Levkivskyi, B. Pedrini, O. Ruchayskiy, Chr. Schweigert, U. Studer, E. Sukhorukov, E. Thiran, J. Walcher, Ph. Werner, A. Zee – R. Morf

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1. Introduction

Purpose of analysis (90's): Classify

- states of non-relativistic (cond.) matter at small or zero temperatures,
- its surface states,

using ideas and concepts from gauge theory, current algebra & GR, such as

the use of: (scaling limits of) effective actions/free energies, gauge invariance, *anomaly cancellation*, power counting, "holography", …

We will illustrate this program on 2D and 3D electron gases; but it can also be applied to cold-atom gases – e.g. (rotating) Bose gases, etc. – the primordial plasma in the universe, stellar matter,…

General ideas of approach

(1) Symmetries: In the absence of external fields and pressure, etc., identify fundamental and accidental approximate internal symmetries and the corresp. approximately conserved currents.

(2) Gauging: Promote approx. global to local symmetries: "gauging of symmetries".

(3) Response: Study response of system to turning on corresp. external gauge fields and varying g_{ij}.

➔ Determine form of *effective action/free energy*
= generating functional of current Green functs.

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(1)+ id. of "order parameter" ➔ Landau Theory

(2)– (3) ➔ *"Gauge Theory of States of* Matter" (90's), "topological order"

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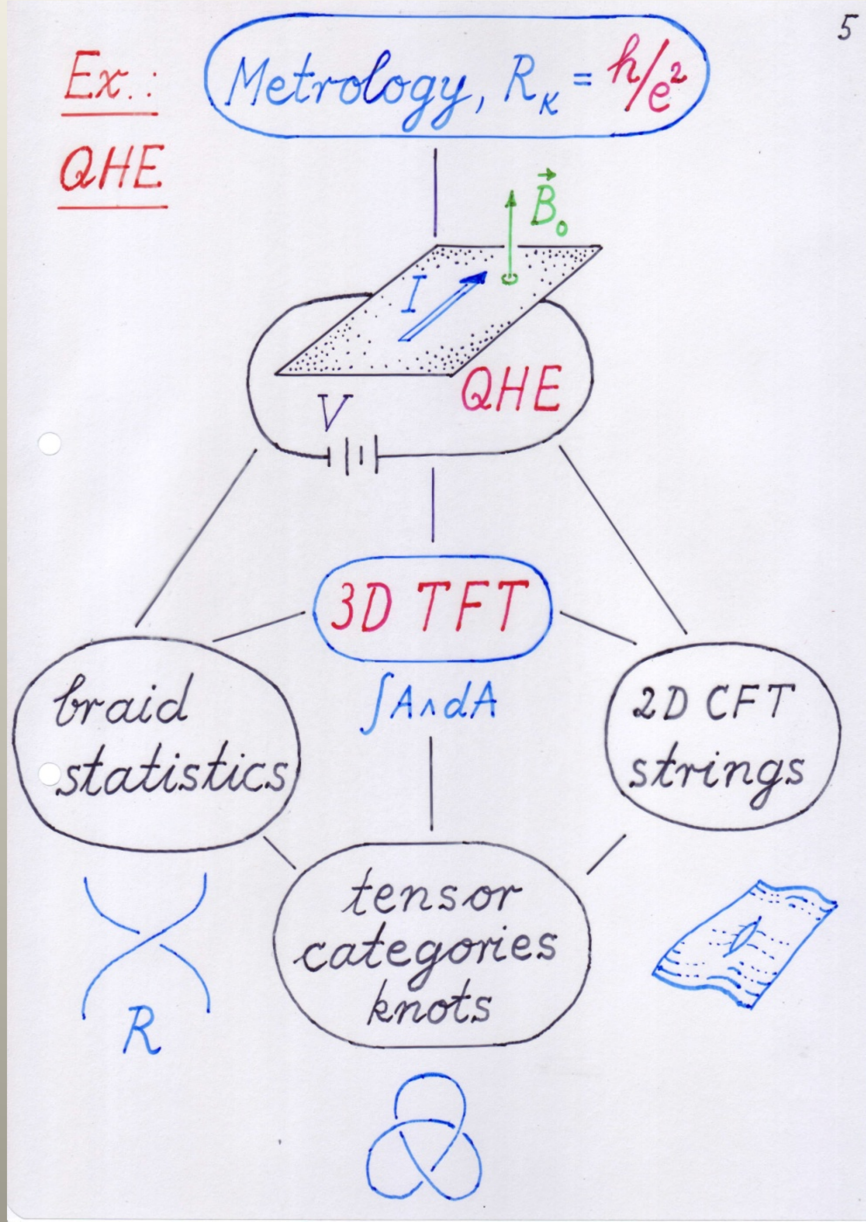
Today, I propose to illustrate this program by examining phases of *electron gases* (with a bulk mobility gap): *QHE, (2+1)D & (3+1)D topological insulators, axion QED*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Ex.:
QHE
Metrology, $R_K = h/e^2$
$\vec{B}_0$
I
QHE
V
3D TFT
$\int A \wedge dA$
braid statistics
2D CFT strings
tensor categories knots
R

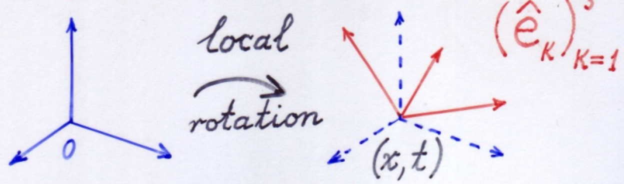
2. Electron Gases in 2D & 3D

QM of a single electron (e.g., in the periodic potential of a crystal) is governed by the *Pauli Equation for a 2-component spinor*, which – in the absence of external fields – has the following global symmetries:

(1) Symmetries of Pauli Equation:

- Global phase rotation of spinor: *U(1)* (fund.!), conserved current is usual *electric current*
- Rotations in spin space: *SU(2)* ➔ *spin current*

(2) Gauge these symmetries:



space-time dep. frames

$$\frac{\hbar}{i}\frac{\partial}{\partial x^j} \longmapsto \frac{\hbar}{i}D_j := \frac{\hbar}{i}\partial_j + a_j + w_j$$

$$i\hbar\frac{\partial}{\partial t} \longmapsto i\hbar D_0 := i\hbar\partial_t + a_0 + w_0$$

U(1) SU(2)

$$w_\mu = w_\mu^\kappa \sigma_\kappa$$

covariant derivatives

$$(\delta^{ij}) \longmapsto (g^{ij}(x))$$

Covariant Pauli Eq.:

$$i\hbar D_0\Psi_t = -\frac{\hbar^2}{2m}\frac{1}{\sqrt{g}} D_k\left(\sqrt{g}\, g^{kl}\right)D_l\Psi_t$$

(CPE)

Action:

$$S(\Psi^*,\Psi;g,a,w) :=$$

$$\int dt\int\sqrt{g}\, d^2x \Big[\Psi_t^*\cdot i\hbar D_0\Psi_t - \frac{\hbar^2}{2m}(D_k\Psi_t)^*\cdot g^{kl}D_l\Psi_t - \lambda|\Psi_t|^2 * \Phi\, |\Psi_t|^2\Big]$$

For $\lambda = 0$,

$$\frac{\delta S}{\delta\Psi^*} = 0 \iff \text{(CPE)}$$

$\lambda \neq 0$: 2-body interactions with potential $\lambda\Phi$

Many-body th.:

$(\Psi^*,\Psi) \longmapsto (\bar{\Psi},\Psi)$ Grassm. V.

(3) Berezin integral:

$$Z(g,a,w) := \int \mathcal{D}\overline{\Psi}\, \mathcal{D}\Psi\, e^{\frac{i}{\hbar} S(\overline{\Psi},\Psi;g,a,w)}$$

$$\boxed{S_{eff}(g,a,w) := \frac{\hbar}{i} \ln Z(g,a,w)}$$

"effective action"

At $T = (k_B \beta)^{-1} > 0$, for g, a, w time-independent, may study thermal equilibr.:

$$S_{eff} \longmapsto F_{eff}(\beta; g, a, w)$$

"eff. free energy"

$$\lim_{\beta\to\infty} F_{eff}(\beta; g,a,w) = \Delta E_0(g,a,w)$$

g.s. energy

Properties of S_{eff}/F_{eff}

(i) $$\frac{\delta S_{eff}}{\delta g^{ij}(x)} = \langle \underbrace{T_{ij}(x)}_{\text{stress tensor}} \rangle_{g,a,w}$$

$$\frac{\delta S_{eff}}{\delta a_\mu(x)} = \langle \underbrace{j^\mu(x)}_{\text{em current density}} \rangle_{g,a,w}$$

$$\frac{\delta S_{eff}}{\delta w_\mu^\kappa(x)} = \langle \underbrace{S_\kappa^\mu(x)}_{\text{spin current -"-}} \rangle_{g,a,w}$$

Higher derivatives: Connected current Green functions of $n \geq 2$ current densities.

(ii) Gauge invariance

$$S_{eff}(g, a_\mu + \partial_\mu \chi, U w_\mu U^{-1} + U \partial_\mu U^{-1})$$
$$= S_{eff}(g, a_\mu, w_\mu)$$

χ real-valued, U $SU(2)$-valued

$\Leftrightarrow$ j^μ cons., s^μ_k covar. cons.

(ii') General covariance

(iii) Bulk mobility gap

$\Downarrow$

Cl. props. of current Green fns.

$\Downarrow$

In scaling lim,

$S_{eff} = \sum_n \int$ "gauge-inv." local poly. + bd. terms

scaling dim n $(= -1, 0, 1, 2, \cdots)$

Retain only "most relevant" terms $(n = -1, 0, 1)$. Ex.: $D = 2, w = 0$

$$S_{eff}(a) = \frac{\sigma_H}{2} \int_\Lambda \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \, d^3x \quad \text{QHE}$$
$$+ \text{bd. term}$$

3. Phys. Interpretation of a & w

$$a_0 = e\varphi - \rho^{-1} p \quad \cdots$$

$$a_k^{tot} = eA_k^{(0)} + \underbrace{eA_k + mV_k}_{\equiv\, a_k}$$

curl $A^{(0)}$: time-indep., per. or hom.

V: velocity field of moving background – gauge inv., usually div $V = 0$.

p: pressure …

$$w_0^{tot} = \left(\frac{g\mu_B}{2}\vec{B} + \frac{\hbar}{4}\,\mathrm{curl}\vec{V}\right)\cdot\vec{\sigma}$$

↖ Zeeman

$$+ \vec{W}_0\cdot\vec{\sigma}$$ ← Weiss exch. field

$$+ \vec{\omega}_0\cdot\vec{\sigma}$$ ← spin conn.

$$w_k^{tot} = \left(\frac{g\mu_B}{2} + \frac{e}{4mc}\right)\left(\vec{\sigma}\wedge(\vec{E}+\dot{\vec{V}})\right)_k$$

↖ spin-orbit

$$+ \vec{W}_k\cdot\vec{\sigma} \quad + \vec{\omega}_k\cdot\vec{\sigma}$$

↖ Weiss ↖ spin conn.

Only spin connection transf. inhomogeneously under SU(2)-gauge trsf.; other contribs. to w_μ^{tot} transf. homogeneously.

4 Effective Actions/Free Energies – Examples

Electron gas in ext. gauge fields $(a_\mu^{tot}, w_\mu^{tot})$ in 2+1 or 3+1 D.

U(1) $a_\mu^{tot} = a_\mu^{(0)} + a_\mu = eA_\mu^{(0)} + eA_\mu + mV_\mu$ ← gauge-inv

SU(2) $w_\mu^{tot} = w_\mu^{(0)} + w_\mu$ ← trsf. homog.

$a^{(0)}, w^{(0)}$: diff. geom. of sample backgd., static em field

↪ explicit breaking of P, T.

For simplicity, g^{ij} fixed $(g^{ij} = \delta^{ij})$. Apply (i)÷(iii):

A short digression: *“Emergent gauge fields”*

Consider an electron gas in a crystalline background with the property that – *as long as electron interactions are neglected* – there are N bands, $\varepsilon_i(k)$, $i=1,\dots,N$, where the crystal momentum k belongs to the Brillouin zone.

Suppose that n bands are filled in the ground state of the non-interacting gas.

Let us study the effect of turning on weak electron-electron interactions for a system with rather *flat* bands (large effective masses). Using a Born-Oppenheimer type approximation, we find an effective theory for this system with a $U(n)$- effective gauge field, w, (sometimes called Berry connection)! This gauge field can be modulated with the help of external fields.

Other instances of “emergent gauge fields” are encountered in the analysis of the fractional quantum Hall effect; (the gauge fields are then the abelian vector potentials of conserved currents). See, e.g., my Les Houches Lectures of 1994.

• **2+1 D**, $w_\mu = 0$, $a_\mu \neq 0$.

$$S_{eff}(a) = (2\lambda^2)^{-1}\int_\Lambda (a^T)^2 d^3x \qquad I$$
$$+ \frac{\sigma_H}{2}\Big[\int_\Lambda a\wedge da + \Gamma(a|_{\partial\Lambda})\Big] \qquad II$$
$$+ \frac{1}{2}\int_\Lambda (\varepsilon\,\underline{\mathcal{E}}^2 + \mu^{-1}\mathcal{B}^2)\,d^3x + \cdots, \qquad III$$

$$\underline{\mathcal{E}} = -e\underline{\nabla}\varphi + e\dot{\underline{A}} + \underline{\nabla}(\rho^{-1}p) + m\dot{\underline{V}}$$
$$\mathcal{B} = \text{curl}(e\underline{A} + m\underline{V})$$

I) Supercond., London eq. ("relevant")

II) Hall effect ("marginal")

III) Maxwell term → diel. cst. ε, magn. perm. μ ("irrelevant" → neglect)

Set $\lambda^{-2} = 0$, neglect III.

Under a gauge trsf., $a \mapsto a + d\chi$,

$$\int_\Lambda a\wedge da \mapsto \int_\Lambda a\wedge da + \int_{\partial\Lambda} \chi\, da$$

$\Rightarrow \Gamma(a|_{\partial\Lambda})$ is anomalous chir. action = gen. fu. of Green fu. of chiral edge currents.

Response Eqs. – see (i):

$$j^0 = \sigma_H\Big(B + \frac{m}{e}\,\text{curl}\,\underline{V} + \varkappa K + \cdots\Big)$$
$$j^k = \sigma_H \varepsilon^{kl}\underbrace{\Big(E_l + \frac{m}{e}\dot{V}_l\Big)}_{\mathcal{E}_l} + \cdots \quad \text{(HE)}$$
$$\frac{\delta\Gamma(a|_{\partial\Lambda})}{\delta a|_{\partial\Lambda}} = j_{edge}$$
$$\partial_\mu j^\mu_{edge} = -\sigma_H\, \mathcal{E}|_{\partial\Lambda} \quad \text{(Chir. An.)}$$

- $2+1\,D$, $a_\mu = 0$, $w_\mu \neq 0$

$$S_{eff}(w) = \chi \int_\Lambda tr(w_0^{\,2})\, d^3x$$
$$+ \tilde{\chi} \int_\Lambda tr(\underline{w}^2)\, d^3x$$
$$+ \frac{k}{4\pi}\Big[\int_\Lambda tr\big(w^{tot} \wedge dw^{tot} + \frac{2}{3}(w^{tot})^{\wedge 3}\big) + \Gamma_{WZW}\big(w^{tot}\big|_{\partial\Lambda}\big)\Big]$$
$$+ \cdots \quad , \; k \in \mathbb{Z}$$

$$\boxed{k = 0, \pm 1}$$ "$\mathbb{Z}_2$"

spin (-current) Hall effect

Γ_{WZW} : gen. fu. of Green fus. of chiral edge spin currents at level $k = 2 \times$ spin of quasi-pt.

→ *Chiral spin liquids, edge spin currents, spin QHE*

- $3+1\,D$, $w_\mu = 0$, $a_\mu \neq 0$

$$S_{eff}(a) = S_{Maxwell}(a) + S_\theta(a)$$
$$S_{Maxwell}(a) = \frac{1}{2\alpha}\int [\varepsilon \vec{\mathcal{E}}^2 + \mu^{-1}\vec{\mathcal{B}}^2]\, d^4x$$
$$S_\theta(a) = \theta \frac{1}{4\pi^2}\int \vec{\mathcal{E}} \cdot \vec{\mathcal{B}}\, d^4x$$

∞ ext. sample, $\vec{\mathcal{E}}, \vec{\mathcal{B}}$ vanish at ∞ $\Rightarrow$

$$\frac{1}{4\pi^2}\int \vec{\mathcal{E}} \cdot \vec{\mathcal{B}}\, d^4x = n \in \mathbb{Z}$$

$\Rightarrow$ Bulk of system inv. under P & T iff

$$\boxed{\theta = 0, \pi}$$

"topological insulator" in 3 dim.

Consider finite sample confined to Λ w. bd. $\partial\Lambda$.

$$S_\theta^\Lambda(a) = \frac{\theta}{4\pi^2}\int_{\partial\Lambda} a \wedge da$$

↑ Stokes

For $\theta = \pi$, $S^\Lambda_{\theta=\pi}$ is effective action of (2+1)D 2-component relativistic Dirac fermions (F-M-S '76; D-J-T, Redl. '80s)

= surface modes of (3+1)D "topological insulator"

Promote θ to dyn. field φ = "axion field"

$$S(\varphi, a) = \frac{1}{4\pi^2}\int \varphi\, F_a \wedge F_a \quad \text{(AxTI)}$$

diml. reduction of (4+1)D Hall effect (F-P[-W] '99)

(AxTI) may arise in insulators with 2 filled bands (bonding, anti-bd.). If els. couple to P-, T- breaking background (e.g., anti-ferro order, chiral structure) → (AxTI), φ : backgd. flucts.

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Add axion action

$$\frac{J}{2}\int[\dot{\varphi}^2-(v\vec{\nabla}\varphi)^2-U(\varphi)]\,d^4x$$

→ Possibility of axion domain walls → surface modes ~ 2-comp. Dirac fs.

• Instabilities (F-P '99, ...)

Simplest example:

$\vec{\mathcal{E}}=0$, $\vec{B}$ time-indep.

$\dot{\varphi}=:\Delta\mu$ — " —

$$F_{eff}(\vec{a})=cst\int\vec{B}^2d^3x+\frac{\Delta\mu}{4\pi^2}\int\vec{a}\cdot\vec{B}\,d^3x$$

Minimize F_{eff} → magnetic instability for $|\vec{k}|\leq cst.\,\Delta\mu$.

Examples of further applications:

- Theory of hurricanes …
- Vortices in superfluids – vorticity quantization;
- Hall effect in rotating 2D (Bose) gases (!);
- Axion magneto-hydrodynamics in the primordial plasma of the early universe → growth of cosmic magnetic fields; possibly similar instabilities in cond. mat.

etc.

5. Concluding Remarks

We have seen that concepts from gauge- and gravity theory – *gauge invariance, anomaly cancellation* – combined with the use of cluster properties of current Green functions, simple scaling arguments and *power counting* lead one to a classification of *effective actions/free energies* of insulators (and superconductors), including incompressible Hall fluids and topological insulators. The effective actions/free energies determine the *response equations* of, e.g., the spin quantum Hall effect, etc. For *non-interacting 2D electron gases exhibiting the QHE*, the connection between this approach and approaches based on certain topological invariants ("index of a pair of projections") is fairly well understood.

To establish more detailed properties of 2D insulators one can make use of results on *3D TFT's*. This has led to a classification of incompressible Hall fluids and, in particular, to a list of possible (rational) values of the *Hall conductivity* and of the *spectrum of quasi-particles* (fractional charges and fractional braid statistics) encountered in incompressible Hall fluids.

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Thank you for your attention!

My Manifesto

I propose that, at all colleges and universities of the so-called civilized world – in Europe and the Americas – *one or two days per semester* will be declared to be

Days of Reflection and of Protest

During these days, we will not teach or attend committee meetings, and there won't be any exercise classes. Instead, we will discuss some of the serious problems threatening our civilization, draft declarations and reach out to the media, with the aim to make it clear to all circles wielding power that we no longer accept:

My Manifesto, ctd.

- That internal tensions and conflicts in countries belonging to the so-called civilized world, such as the *Ukraine*, are “solved” by armed conflicts rather than by political dialogue and compromise.
- That innocent people are slaughtered in ugly civil wars and by terrorist activities, such as those in Syria and Iraq.
- That countries threaten other countries with warfare.
- That weapons are sold to (clans) in countries plagued by civil war or other forms of unrest and conflict.
- That religions are abused for purposes of power and suppression.
- That the dignity and the rights of women are abused and offended in the name of religion.

My Manifesto, ctd.

- That people are harassed or killed because of their race or faith.
- That nothing is done against the perversions of 21st Century Capitalism.
- That the resources of Planet Earth continue to be looted shamelessly.

These are but some examples of numerous problems threatening the survival of humankind in peace and dignity. –

Where is the *"Peace Movement"*, where are movements such as *"Occupy Wall Street"*, *"Survivre et Vivre"*? What is the *"Club of Rome"* doing? Why are the media silent about the activities of these and other groups?

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Students and Academics of Europe and the Americas, raise your voices, arise!