Bulk-edge duality for topological insulators

Gian Michele Graf
ETH Zurich

Topological Phases of Quantum Matter
Erwin Schrödinger Institute, Vienna
September 8-12, 2014

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joint work with Marcello Porta
thanks to Yosi Avron

Topological insulators: first impressions

- Insulator in the Bulk: Excitation gap
  For independent electrons: band gap at Fermi energy

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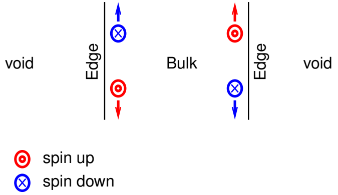

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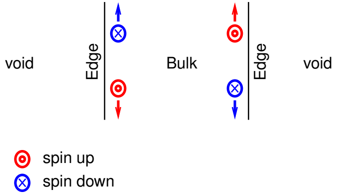
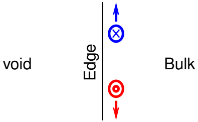


- Topology: In the space of Hamiltonians, a topological insulator can not be deformed in an ordinary one, while keeping the gap open and time-reversal invariance.

Bulk-edge correspondence



In a nutshell: Termination of bulk of a topological insulator implies edge states

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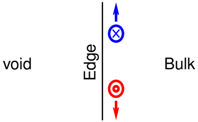

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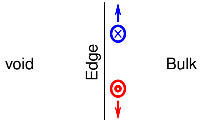
- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

Bulk-edge correspondence



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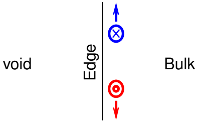

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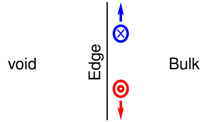
- Bulk-edge duality: Can it be shown that the two indices agree?

Bulk-edge correspondence. Done?



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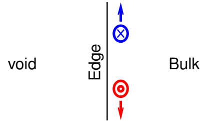
- Goal: State the (intrinsic) topological property distinguishing different classes of insulators.

More precisely:

- Express that property as an Index relating to the Bulk, resp. to the Edge. Yes, e.g. Kane and Mele.
- Bulk-edge duality: Can it be shown that the two indices agree? Schulz-Baldes et al.; Essin & Gurarie

Bulk-edge correspondence. Today



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- Express that property as an Index relating to the Bulk, resp. to the Edge. Done differently.
- Bulk-edge duality: Can it be shown that the two indices agree? Done differently.

Rueda de casino. Time $0'15''$

Rueda de casino. Time $0'25''$

Rueda de casino. Time $0'35''$

Rueda de casino. Time $0'44''$

Rueda de casino. Time $0'44.25''$

Rueda de casino. Time $0'44.50''$

Rueda de casino. Time $0'44.75''$

Rueda de casino. Time $0'45''$

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Rueda de casino. Time 0′46″

Rueda de casino. Time $0'47''$

Rueda de casino. Time $0'55''$

Rueda de casino. Time $1'16''$

Rueda de casino. Time $3'23''$

Rules of the dance

Dancers

- start in pairs, anywhere
- end in pairs, anywhere (possibly elseways & elsewhere)
- are free in between
- must never step on center of the floor

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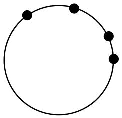
There are dances which can not be deformed into one another.

What is the index that makes the difference?

The index of a Rueda

A snapshot of the dance



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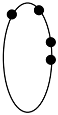
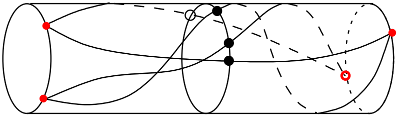


Dance D as a whole



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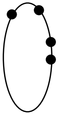

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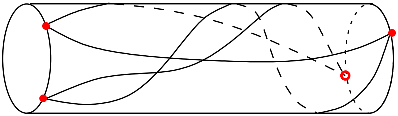

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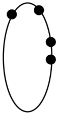

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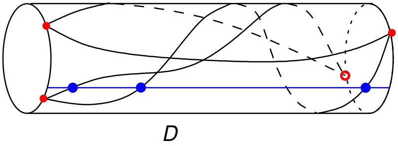

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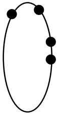

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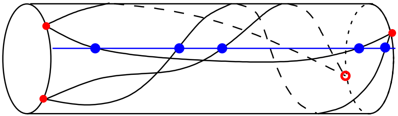

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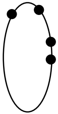

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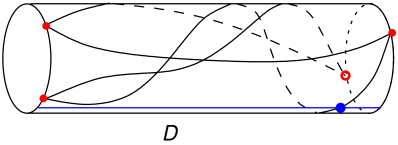

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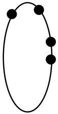

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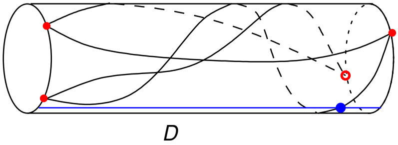
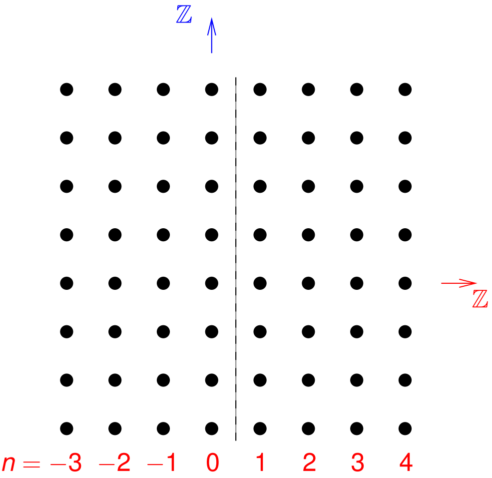


$\mathcal{I}(D) =$ parity of number of crossings of fiducial line

Bulk Hamiltonian

Hamiltonian on the lattice $\mathbb{Z} \times \mathbb{Z}$ (plane)



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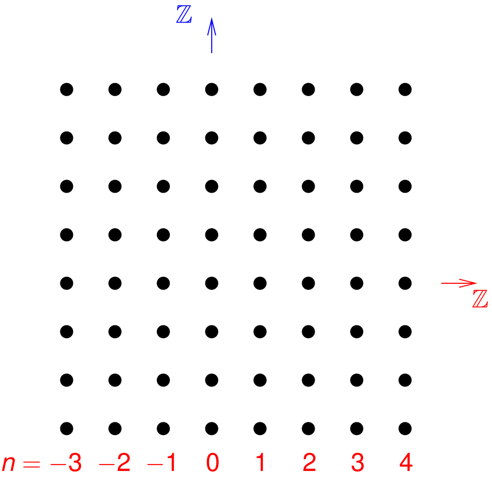

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End up with wave-functions $\psi = (\psi_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z};\mathbb{C}^N)$ and Bulk Hamiltonian

$$\big(H(k)\psi\big)_n = A(k)\psi_{n-1} + A(k)^*\psi_{n+1} + V_n(k)\psi_n$$

with
$V_n(k) = V_n(k)^* \in M_N(\mathbb{C})$ (potential)
$A(k) \in \mathrm{GL}(N)$ (hopping)

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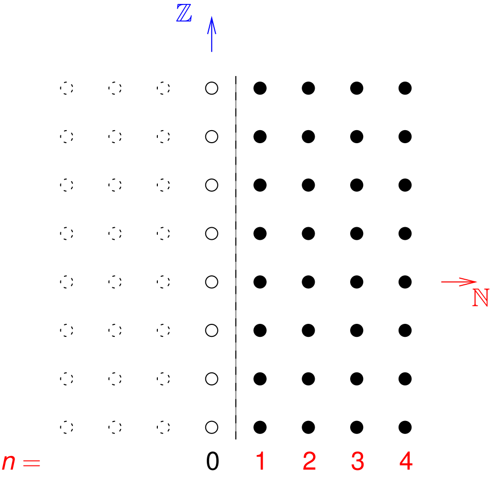
Hamiltonian on the lattice $\mathbb{N} \times \mathbb{Z}$ (half-plane) with $\mathbb{N} = \{1, 2, \ldots\}$



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Note: $\sigma_{\text{ess}}(H^\sharp(k)) \subset \sigma_{\text{ess}}(H(k))$, but typically $\sigma_{\text{disc}}(H^\sharp(k)) \not\subset \sigma_{\text{disc}}(H(k))$

General assumptions

- Gap assumption: Fermi energy μ lies in a gap for all $k \in S^1$:

$$\mu \notin \sigma(H(k))$$

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- Fermionic time-reversal symmetry: $\Theta : \mathbb{C}^N \to \mathbb{C}^N$
  - Θ is anti-unitary and $\Theta^2 = -1$;
  - Θ induces map on $\ell^2(\mathbb{Z}; \mathbb{C}^N)$, pointwise in $n \in \mathbb{Z}$;
  - For all $k \in S^1$,

$$H(-k) = \Theta H(k) \Theta^{-1}$$

    Likewise for $H^\sharp(k)$

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  $$H\psi = E\psi \implies H(\Theta\psi) = E(\Theta\psi)$$

  and $\Theta\psi = \lambda\psi$, $(\lambda \in \mathbb{C})$ is impossible:

  $$-\psi = \Theta^2\psi = \bar{\lambda}\Theta\psi = \bar{\lambda}\lambda\psi \qquad (\Rightarrow\Leftarrow)$$

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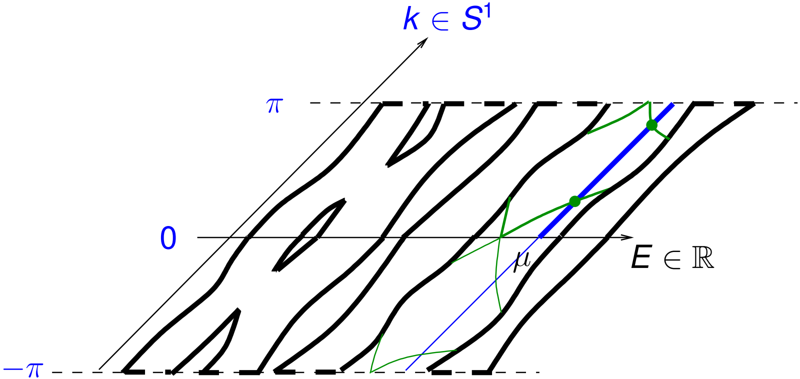
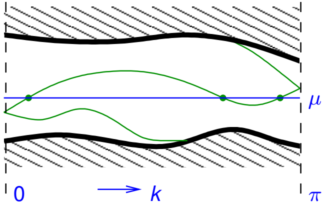


Bands, Fermi line (one half **fat**), edge states

The edge index

The spectrum of $H^\sharp(k)$

symmetric on $-\pi \le k \le 0$



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Definition: Edge Index

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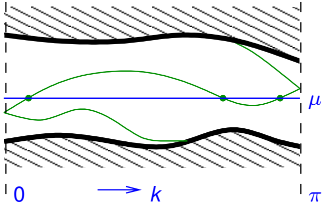

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Collapse upper/lower band to a line and fold to a cylinder: Get rueda and its index.

Towards the bulk index

Let $z \in \mathbb{C}$. The Schrödinger equation

$$(H(k) - z)\psi = 0$$

(as a 2nd order difference equation) has $2N$ solutions $\psi = (\psi_n)_{n\in\mathbb{Z}}$, $\psi_n \in \mathbb{C}^N$.

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Let $z \notin \sigma(H(k))$. Then

$$E_{z,k} = \{\psi \mid \psi \text{ solution}, \psi_n \to 0, \ (n \to +\infty)\}$$

has

- $\dim E_{z,k} = N$.

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- $E_{\bar{z},-k} = \Theta E_{z,k}$

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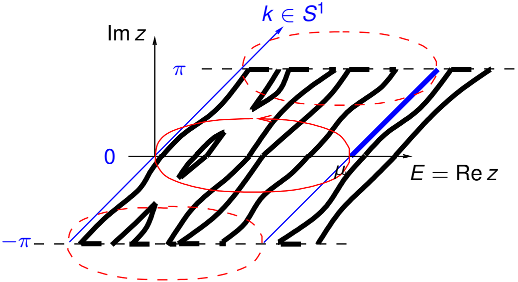
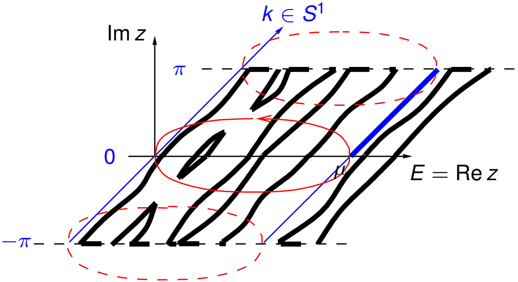


Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and $\Theta^2 = -1$.

The bulk index



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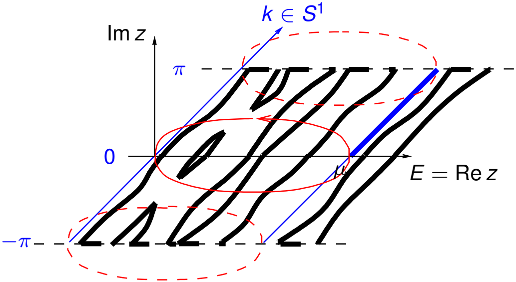
Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

The bulk index



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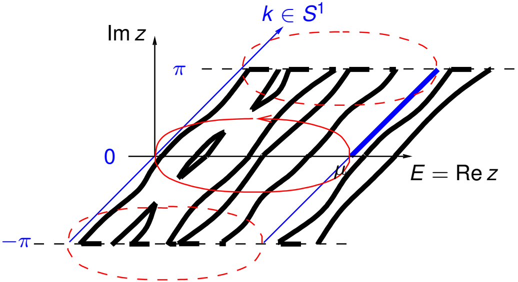
The bulk index



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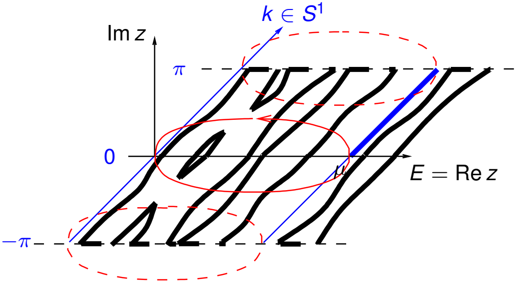
What's behind the theorem? How is $\mathcal{I}(E)$ defined?

The bulk index



Loop γ and torus $\mathbb{T} = \gamma \times S^1$

Vector bundle E with base $\mathbb{T} \ni (z, k)$, fibers $E_{z,k}$, and $\Theta^2 = -1$.

Theorem In general, vector bundles $(E, \mathbb{T}, \Theta)$ can be classified by an index $\mathcal{I}(E) = \pm 1$ (besides of $N = \dim E$)

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What's behind the theorem? How is $\mathcal{I}(E)$ defined? Aside . . . a rueda . . .

Time-reversal invariant bundles on the torus

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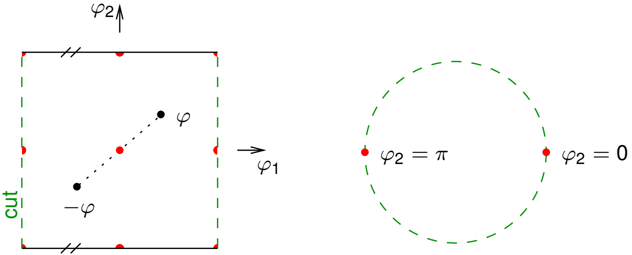

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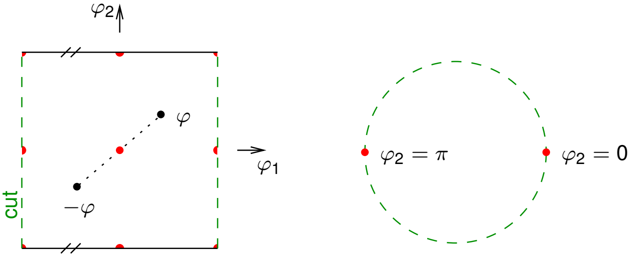

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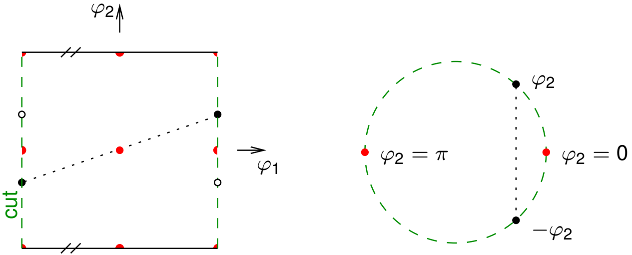

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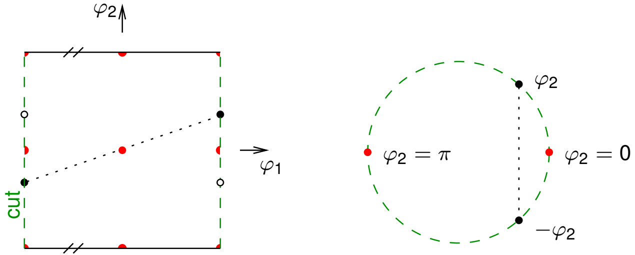

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$$\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0 \,, \qquad (\varphi_2 \in S^1)$$

with $\Theta_0 : \mathbb{C}^N \to \mathbb{C}^N$ antilinear, $\Theta_0^2 = -1$

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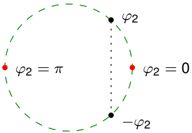

- $\Theta_0 T(\varphi_2) = T^{-1}(-\varphi_2)\Theta_0$
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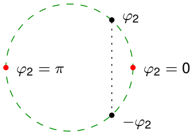

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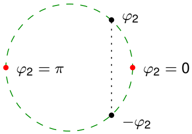

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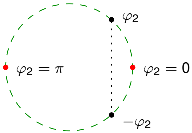

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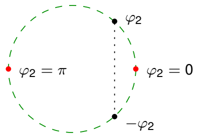

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Definition (Index): $\mathcal{I}(E) := \mathcal{I}(D)$

Remark: $\mathcal{I}(E)$ agrees (in value) with the Pfaffian index of Kane and Mele.

. . . aside ends here.

Main result

Theorem Bulk and edge indices agree:

$$\mathcal{I} = \mathcal{I}^\sharp$$

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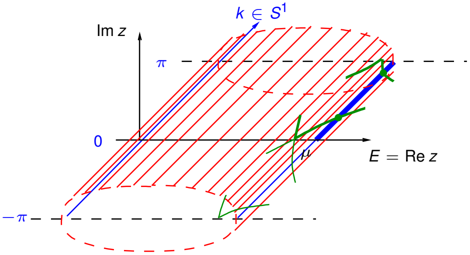
$\mathcal{I} = +1$: ordinary insulator
$\mathcal{I} = -1$: topological insulator

Idea of proof of $\mathcal{I} = \mathcal{I}^\sharp$



Fermi line (one half **fat)**
edge states
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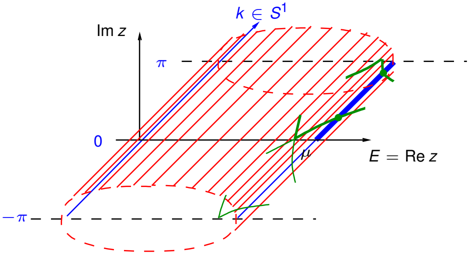

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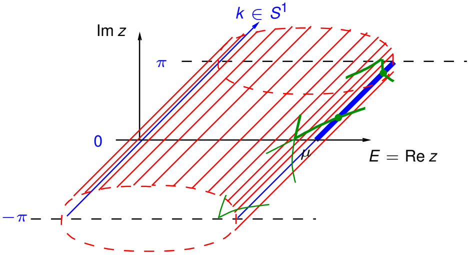

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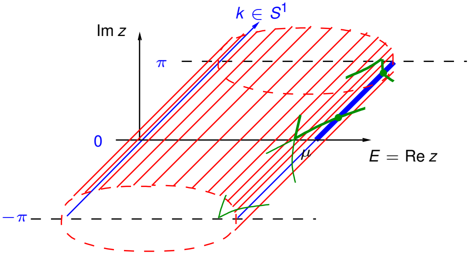

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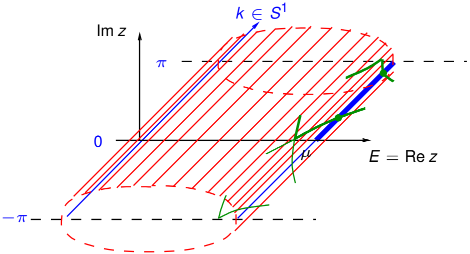

Fermi line (one half **fat**)
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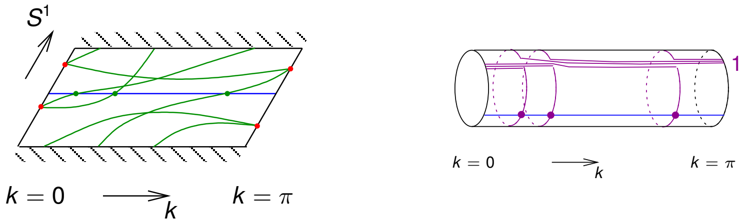
Edge states define a rueda
Bulk torus cut along Fermi line; frame bundle admits a section, which is “aware of the edge”; transition matrix defines rueda.

Proof of Theorem: Dual ruedas



Edge rueda: edge eigenvalues; Bulk rueda: eigenvalues of $T(k)$

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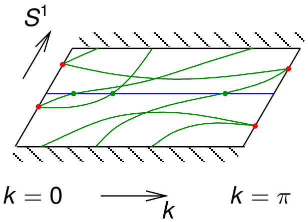

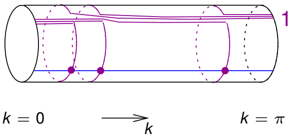

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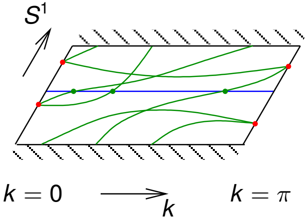

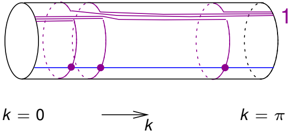

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The two ruedas share intersection points. Hence indices are equal □

A simpler result in a simpler setting:

Quantum Hall effect in doubly periodic lattices

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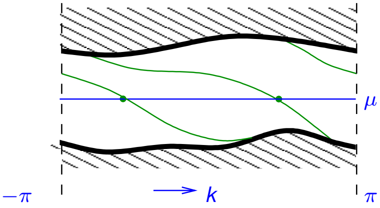

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Chern number $\mathrm{ch}(E_j)$ is computed by • cutting torus to cylinder • taking global section of E_j on cylinder • defines transition matrix T (phase factor) along circle • $\mathrm{ch}(E_j) =$ winding number of T.

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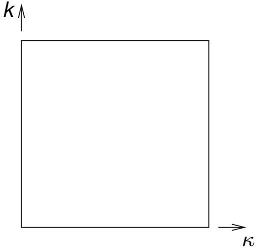
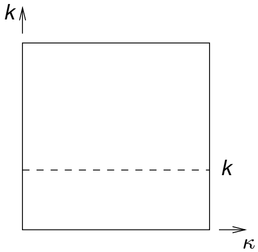
(cf. Hatsugai) Here via scattering and Levinson's theorem.

Duality via scattering

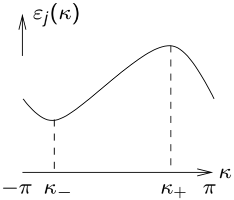


Brillouin zone $\ni (\kappa, k)$
Energy band $\varepsilon_j(\kappa, k)$

Duality via scattering



Minima $\kappa_-(k)$ and maxima $\kappa_+(k)$ of energy band $\varepsilon_j(\kappa, k)$ in κ at fixed k



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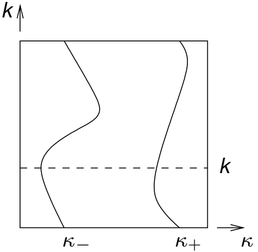

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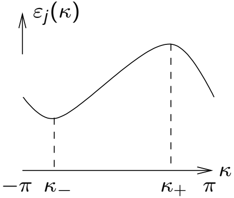

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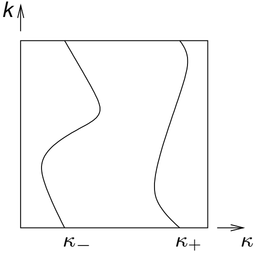

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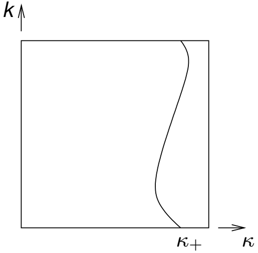

Maxima $\kappa_+(k)$

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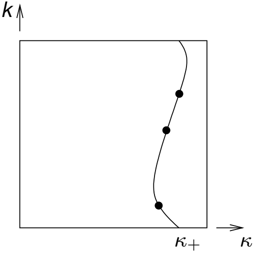
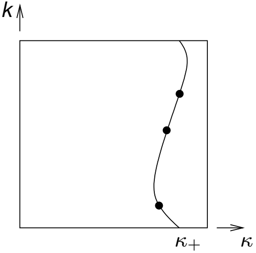


Maxima $\kappa_+(k)$ with semi-bound states (to be explained)

Duality via scattering



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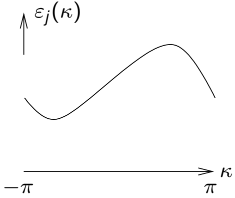
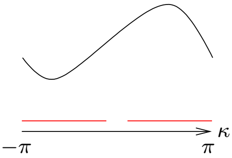


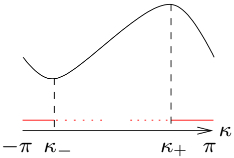
At fixed k: Energy band $\varepsilon_j(\kappa, k)$ and the line bundle E_j of Bloch states

Duality via scattering



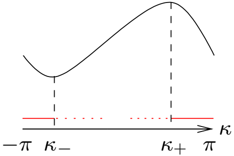
Line indicates choice of a section $|\kappa\rangle$ of Bloch states (from the given band). No global section in $\kappa \in \mathbb{R}/2\pi\mathbb{Z}$ is possible, as a rule.

Duality via scattering

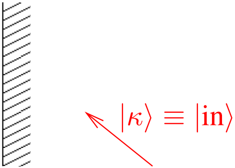


States $|\kappa\rangle$ above the solid line are left movers ($\varepsilon_j'(\kappa) < 0$)

Duality via scattering



They are incoming asymptotic (bulk) states for scattering at edge (from inside)



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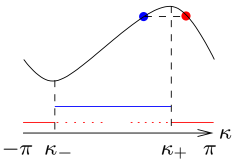
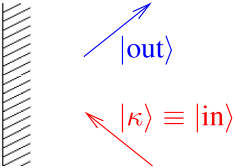
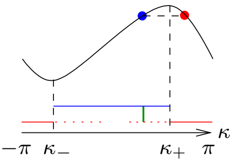
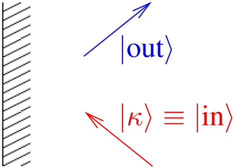


Scattering determines section $|\mathrm{out}\rangle$ of right movers above line



Duality via scattering



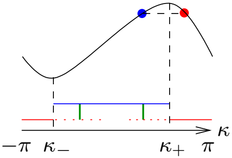


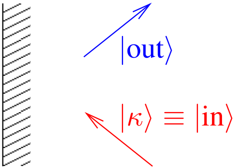
Scattering matrix

$$|\text{out}\rangle = S_+ |\kappa\rangle$$

as relative phase between two sections of the same fiber (near κ_+)

Duality via scattering





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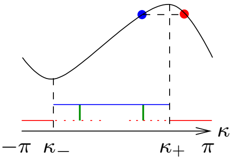
Likewise S_- near κ_-.

Duality via scattering

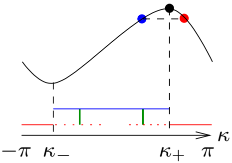


Chern number computed by sewing

$$\mathrm{ch}(E_j) = \mathcal{N}(S_+) - \mathcal{N}(S_-)$$

with $\mathcal{N}(S_\pm)$ the winding of $S_\pm = S_\pm(k)$ as $k = -\pi \dots \pi$.

Duality via scattering



As $\kappa \to \kappa_+$, whence

$$|\text{in}\rangle = |\kappa\rangle \to |\kappa_+\rangle \qquad |\text{out}\rangle = S_+|\kappa\rangle \to |\kappa_+\rangle \text{ (up to phase)}$$

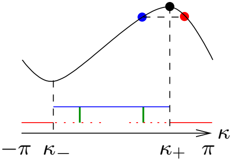
their limiting span is that of

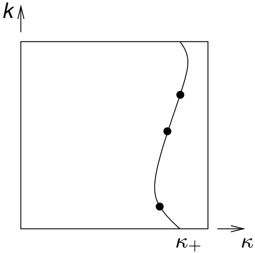
$$|\kappa_+\rangle, \quad \left.\frac{d|\kappa\rangle}{d\kappa}\right|_{\kappa_+}$$

(bounded, resp. unbounded in space). The span contains the limiting scattering state $|\psi\rangle \propto |\text{in}\rangle + |\text{out}\rangle$.

If (exceptionally) $|\psi\rangle \propto |\kappa_+\rangle$ then $|\psi\rangle$ is a semi-bound state.

Duality via scattering





As a function of k, semi-bound states occur exceptionally.

Levinson's theorem

Recall from two-body potential scattering: The scattering phase at threshold equals the number of bound states

$$\arg S\big|_{E=0+} = 2\pi N$$

Levinson's theorem

Recall from two-body potential scattering: The scattering phase at threshold equals the number of bound states

$\sigma(p^2 + V)$

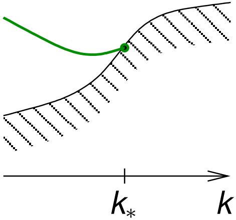
$$\arg S\big|_{E=0+} = 2\pi N$$

N changes with the potential V when bound state reaches threshold (semi-bound state $\equiv$ incipient bound state)

Levinson's theorem (relative version)

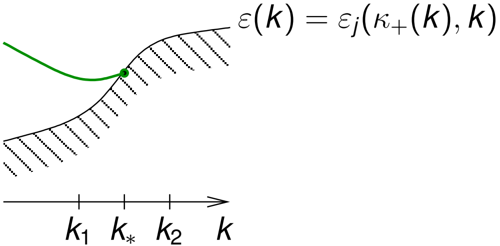
Spectrum of edge Hamiltonian

$$\varepsilon(k) = \varepsilon_j(\kappa_+(k), k)$$



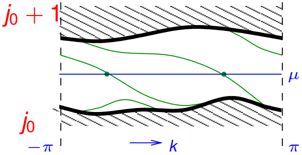
Levinson's theorem (relative version)

Spectrum of edge Hamiltonian



$$\lim_{\delta\to 0} \arg S_+(\varepsilon(k) - \delta)\Big|_{k_1}^{k_2} = \pm 2\pi$$

Proof



$$\mathcal{N}^\sharp = \mathcal{N}(S_+^{(j_0)}) \quad (= \mathcal{N}(S_-^{(j_0+1)}))$$

$$= \sum_{j=0}^{j_0} \mathcal{N}(S_+^{(j)}) - \mathcal{N}(S_-^{(j)})$$

$$= \sum_{j=0}^{j_0} \mathrm{ch}(E_j)$$

$(\mathcal{N}(S_-^{(1)}) = 0)$

Summary

Bulk = Edge

$$\mathcal{I} = \mathcal{I}^\sharp$$

- The bulk and the indices of a topological insulator (of reduced symmetry) are indices of suitable ruedas
- In case of full translational symmetry, bulk index can be defined and linked to edge in other ways