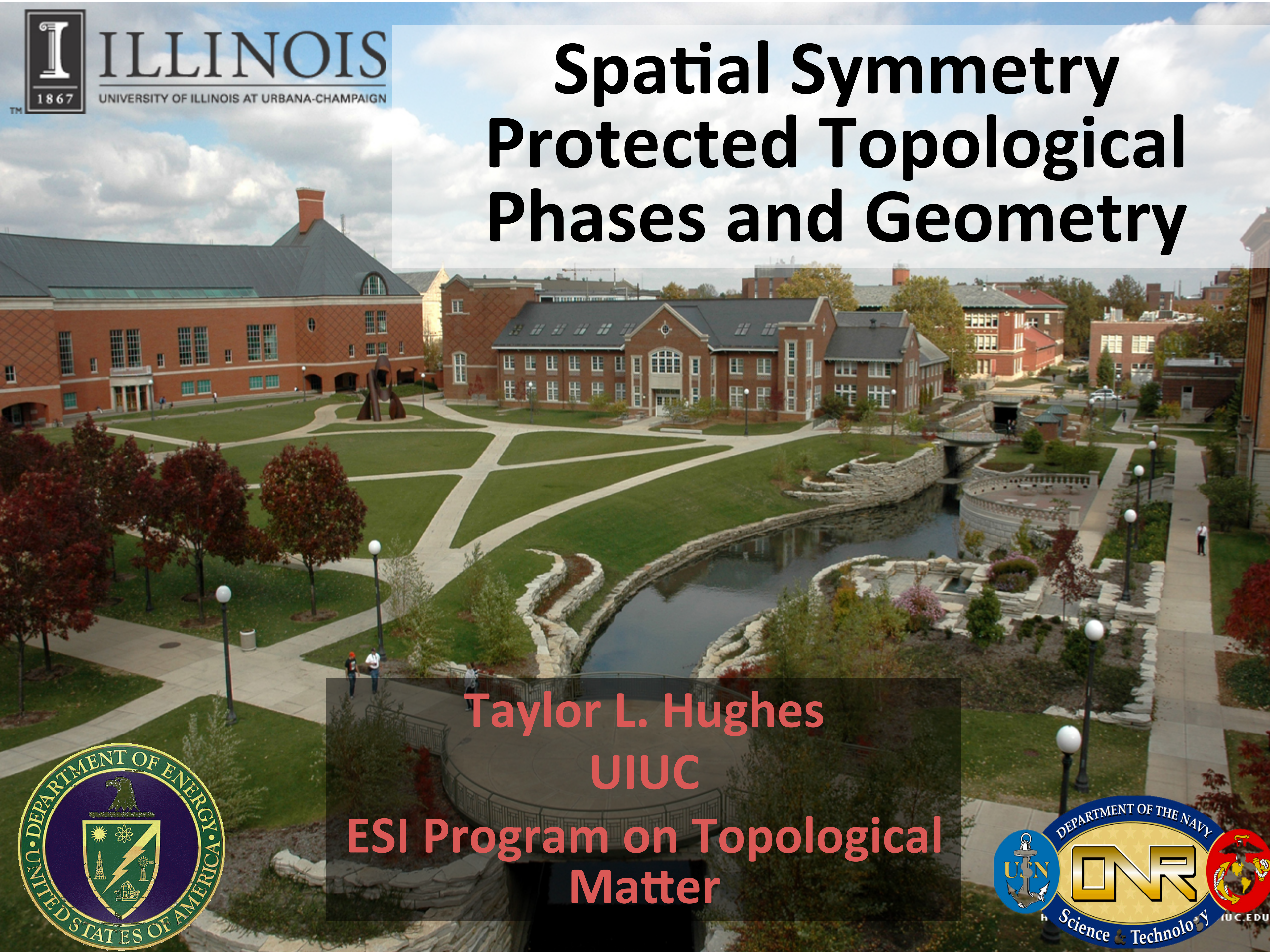
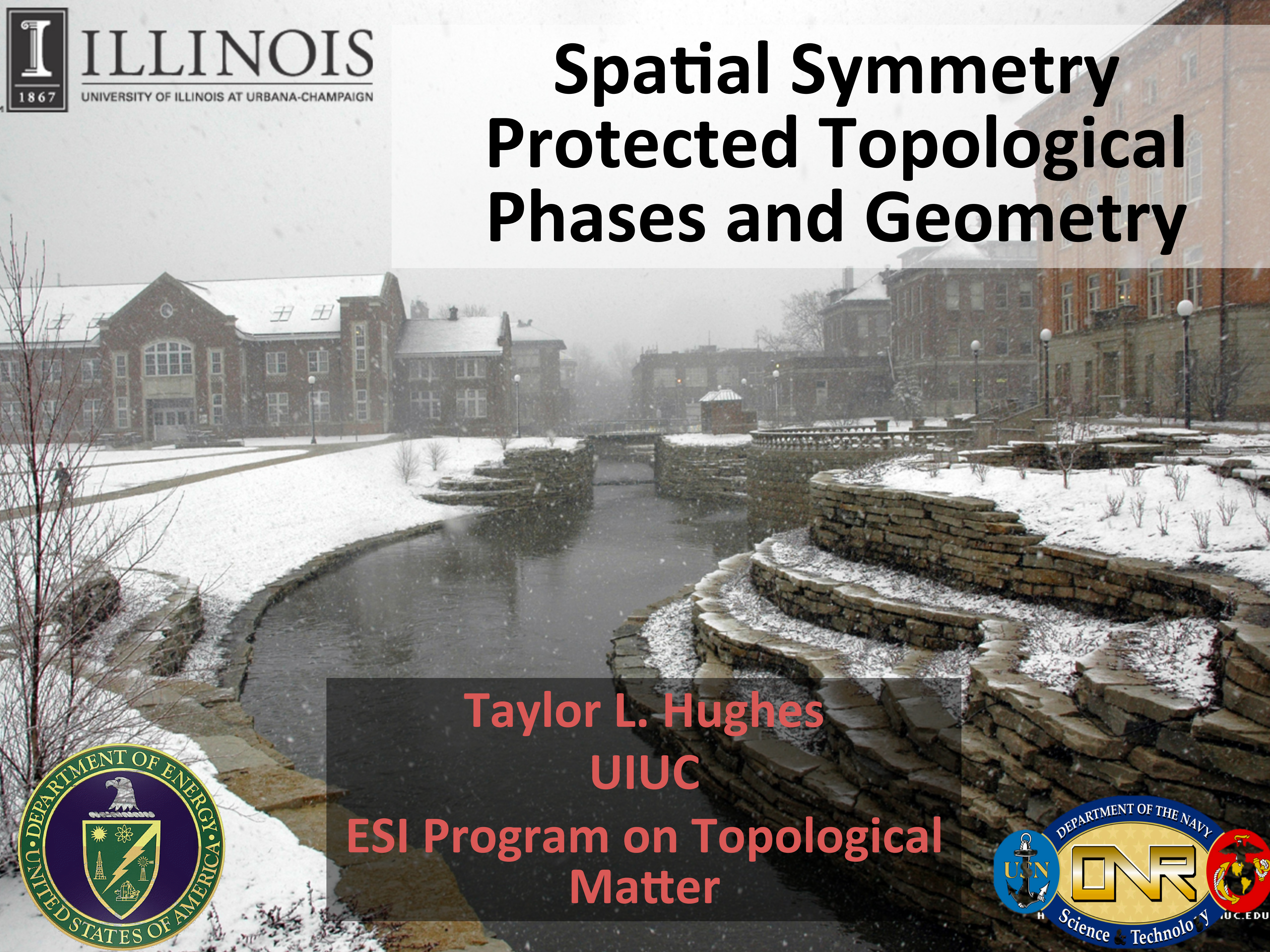
Spatial Symmetry Protected Topological Phases and Geometry

Taylor L. Hughes
UIUC

ESI Program on Topological Matter



Spatial Symmetry Protected Topological Phases and Geometry



Taylor L. Hughes

UIUC

ESI Program on Topological Matter

Outline

Part 0: Introduction

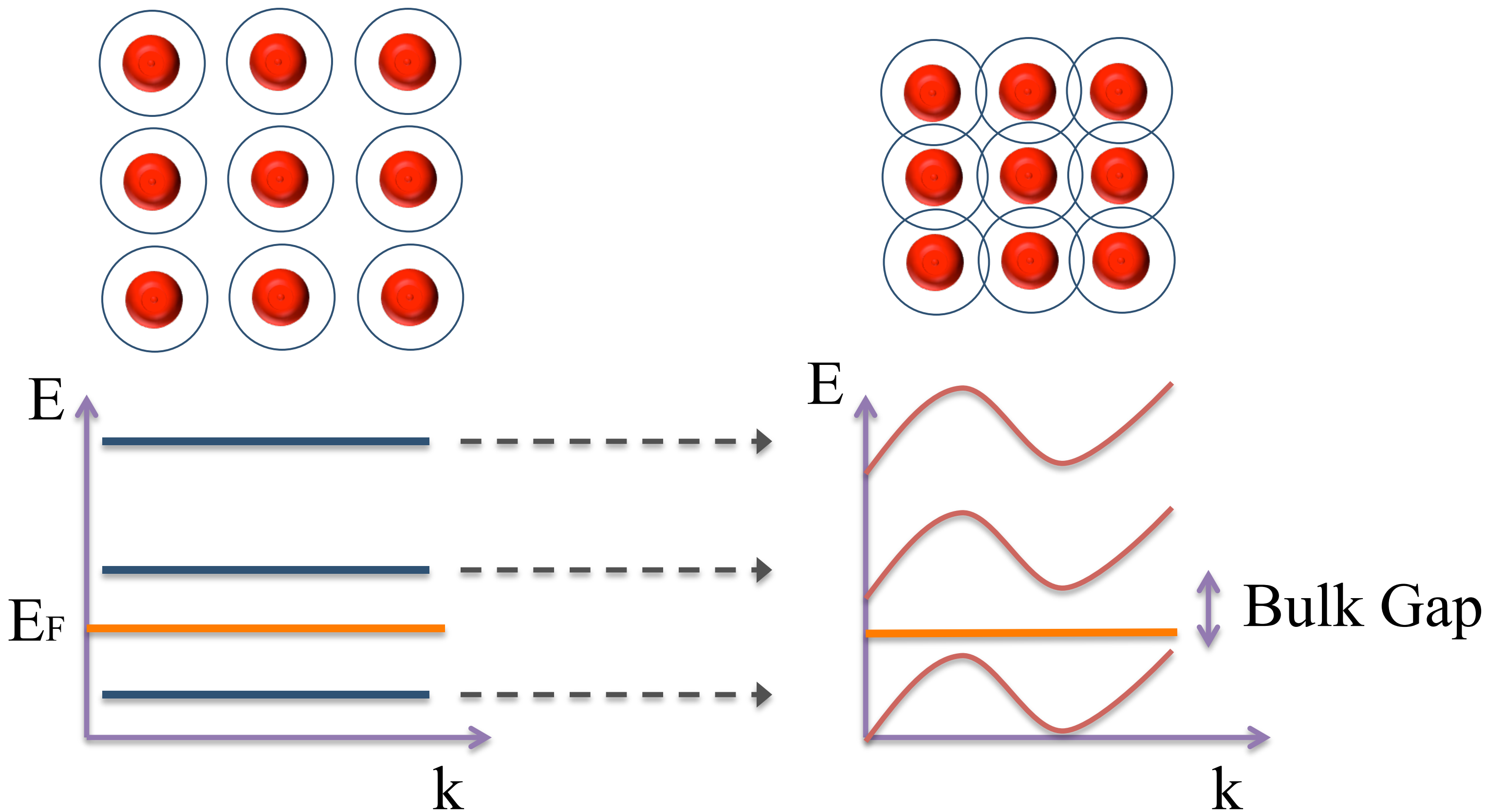
Part 1: Topological phases protected by discrete translation and rotation symmetries

Part 2: Bound states on geometric defects in point-group protected topological phases

Part 3: Interaction-Induced Topological Phases Protected by Point-Group Symmetry

Part 0: Brief Introduction To Topological Insulators

(Atomic) Band Insulators



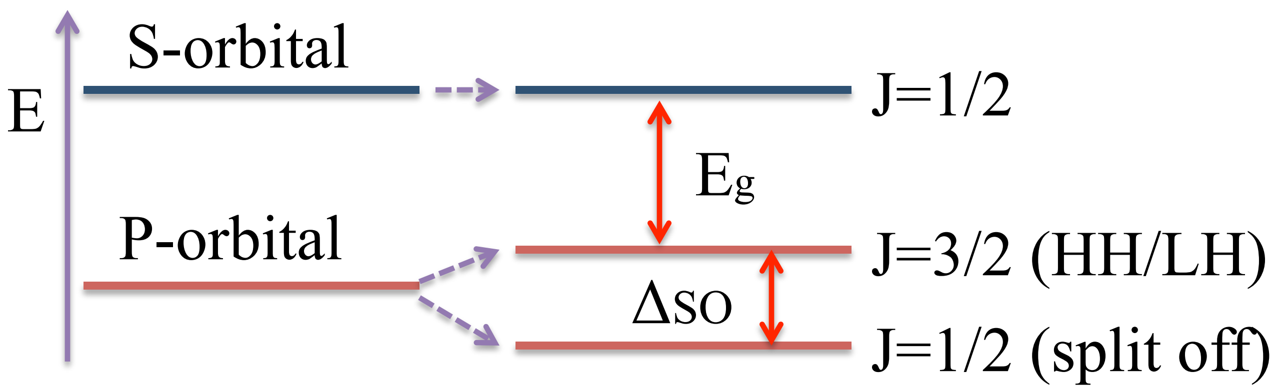
“Bands” of atomic orbitals

Dispersing bands which are adiabatically connected.

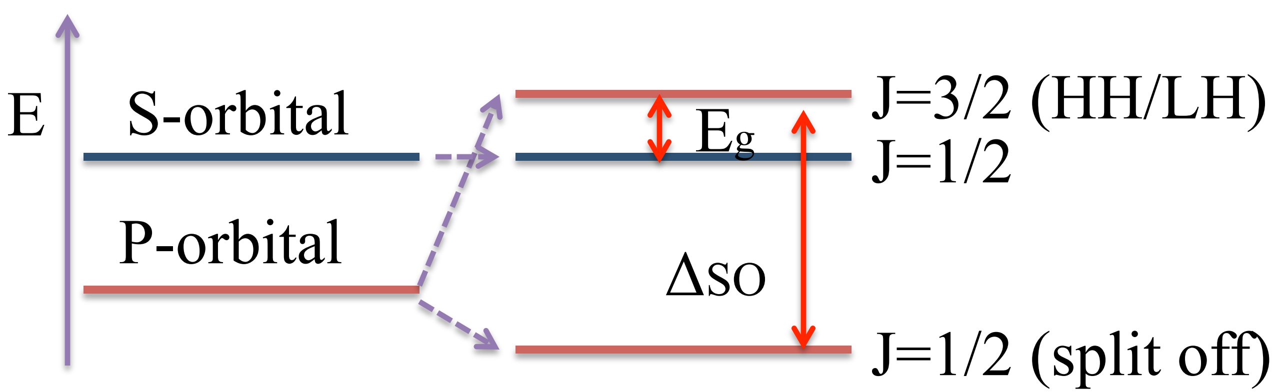
Are all (non-interacting) insulators essentially atomic insulators?

Inverted Band Order From Strong Spin-Orbit Coupling

Take GaAs: Add spin-orbit



Take HgTe:

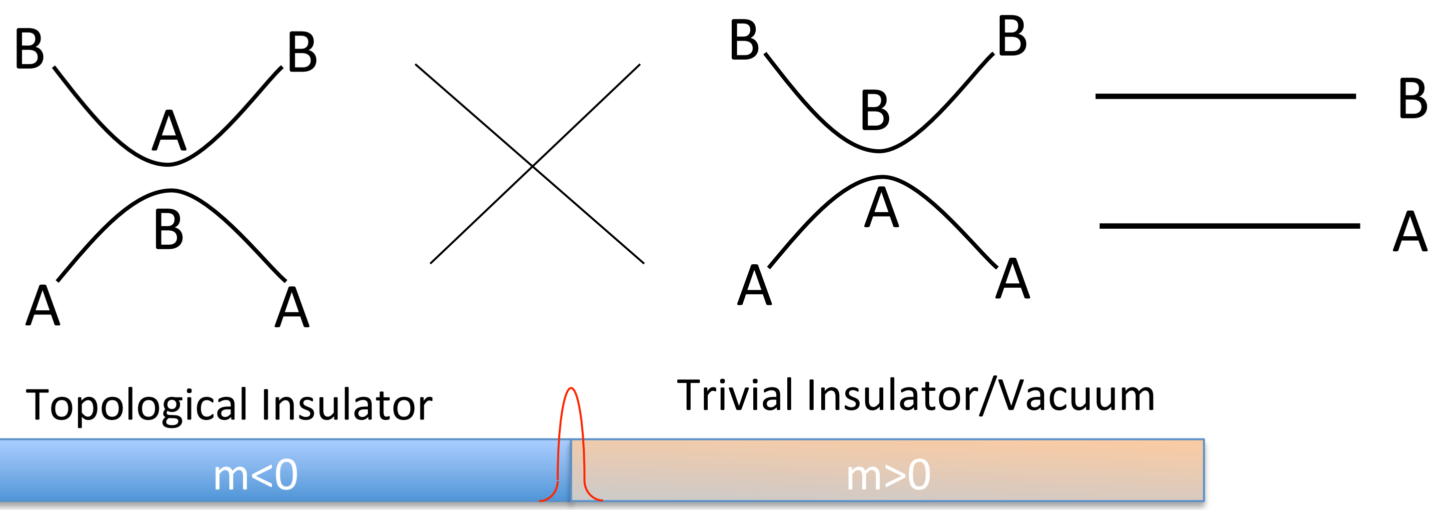


Simple Insulator with Band Inversion: 1D Dirac Model

A B

$$H = \sum_k c_k^\dagger (k\sigma^x + m\sigma^z) c_k = \sum_k c_k^\dagger \begin{pmatrix} m & k \\ k & -m \end{pmatrix} c_k$$

$$E_\pm(k) = \pm\sqrt{k^2 + m^2}$$



Part 1:Topological Phases and Response Protected by Spatial Symmetries

Inversion

Periodic Table of Free Fermion Topological Phases

Dim/Symmetry	C,Ť	C	C,T	T	T,Č	Č	Č,Ť	Ť
(0+1)d	Z_2	Z_2	0	Z	0	0	0	Z
(1+1)d	Z	Z_2	Z_2	0	Z	0	0	0
(2+1)d	0	Z	Z_2	Z_2	0	Z	0	0
(3+1)d	0	0	Z	Z_2	Z_2	0	Z	0
(4+1)d	0	0	0	Z	Z_2	Z_2	0	Z
(5+1)d	Z	0	0	0	Z	Z_2	Z_2	0
(6+1)d	0	Z	0	0	0	Z	Z_2	Z_2
(7+1)d	Z_2	0	Z	0	0	0	Z	Z_2

0	χ
Z	0
0	Z
Z	0
0	Z
Z	0
0	Z
Z	0
0	Z

The non-zero entries represent "strong" topological invariants of the bulk that distinguish gapped phases from a trivial atomic limit.

Does not include unitary symmetries. Important to consider spatial symmetries such as translation, reflection, (discrete) rotation.

Schnyder,Ryu,Furusaki,Ludwig: PRB (2008)
Kitaev: Adv. in Theoretical Phys. 2009
Qi, Hughes, Zhang: PRB(2008)

Spatial Symmetries and Topology

Precursor of Spatial-Symmetry Protected Topological Phases:

- Zak, J. "Berry's phase for energy bands in solids." *Physical review letters* **62**, 2747 (1989).

-Wannier center locations are quantized in inversion symmetric crystals, i.e., polarization is quantized.

Modern Inception of Field:

- Fu, L., Kane, C. L., & Mele, E. J. (2007). Topological insulators in three dimensions. *Physical review letters*, *98*(10), 106803.
- Moore, J. E., and Leon Balents. "Topological invariants of time-reversal-invariant band structures." *Physical Review B* 75.12 (2007): 121306.
- Roy, R. "Topological phases and the quantum spin Hall effect in three dimensions." *Physical Review B*, 79, 195322 (2009).

-Introduction of weak topological insulators protected by time-reversal and translation symmetry

- Fu, Liang, and Charles L. Kane. "Topological insulators with inversion symmetry." *Physical Review B* **76,** 045302 (2007).

-TIs with time-reversal and inversion symmetry are classified in 2D and 3D. First discrete eigenvalue formula.

- Teo, Jeffrey CY, Liang Fu, and C. L. Kane. "Surface states and topological invariants in three-dimensional topological insulators: Application to Bi_ {1− x} Sb_ {x}." *Physical Review B,* **78**, 045426 (2008).

-Introduction of mirror Chern number in 3D materials. Call for a complete topological band theory including all point-group symmetries.

Recent Work

Resulting Classification:

- Fu, Liang. "Topological crystalline insulators." *Physical Review Letters* 106.10 (2011): 106802.
- Hughes, Taylor L., Emil Prodan, and B. Andrei Bernevig. "Inversion-symmetric topological insulators." *Physical Review B* 83.24 (2011): 245132.
- Turner, Ari M., et al. "Quantized response and topology of magnetic insulators with inversion symmetry." *Physical Review B* 85.16 (2012): 165120.
- Fang, Chen, Matthew J. Gilbert, and B. Andrei Bernevig. "Bulk topological invariants in noninteracting point group symmetric insulators." *Physical Review B* 86.11 (2012): 115112.
- Jadaun, Priyamvada, et al. "Topological classification of crystalline insulators with space group symmetry." *Physical Review B* 88.8 (2013): 085110.
- Slager, Robert-Jan, et al. "The space group classification of topological band-insulators." *Nature Physics* 9.2 (2012): 98-102.
- Teo, Jeffrey CY, and Taylor L. Hughes. "Existence of Majorana-Fermion Bound States on Disclinations and the Classification of Topological Crystalline Superconductors in Two Dimensions." *Physical review letters* 111.4 (2013): 047006.
- Chiu, Ching-Kai, Hong Yao, and Shinsei Ryu. "Classification of topological insulators and superconductors in the presence of reflection symmetry." *Phys. Rev. B* **88**, 075142 (2013).
- Zhang, Fan, C. L. Kane, and E. J. Mele. "Topological Mirror Superconductivity." *Phys. Rev. Lett.* **111**, 056403 (2013).

Material Prediction and Experimental Confirmations

- Hsieh, Timothy H., et al. "Topological crystalline insulators in the SnTe material class." *Nat. Comm.* **3**, 982 (2012).
- Tanaka, Y., et al. "Experimental realization of a topological crystalline insulator in SnTe." *Nat. Phys.* **8**, 800 (2012).
- Dziawa, P., et al. "Topological crystalline insulator states in Pb1− xSnxSe." *Nat. Mat.* **11**, 1023 (2012).
- Xu, Su-Yang, et al. "Observation of a topological crystalline insulator phase and topological phase transition in Pb1− xSnxTe." *Nat. Com.* **3**, 1192 (2012).

Example: Su-Schrieffer-Heeger model in 1D

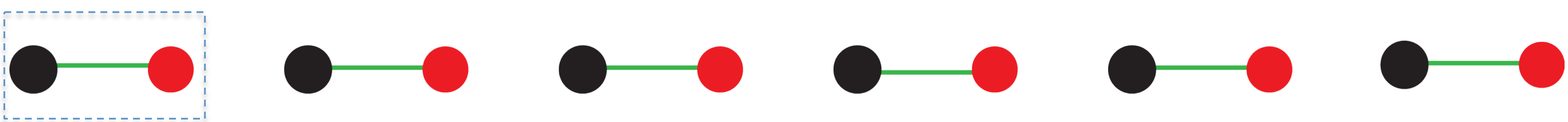
Class D insulator in 1+1-d with (fine-tuned) particle-hole symmetry. Strong invariant: Z_2.

Given: $$H(k) \ni CH(k)C^{-1} = -H^T(-k)$$

Construct: $$A^{mn}(k) = -i\langle u_m(k)|\partial_k|u_n(k)\rangle$$

Calculate: $$\theta = \int_{-\pi/a}^{\pi/a} dk \, \mathrm{Tr}\,[A(k)]$$

Θ=0



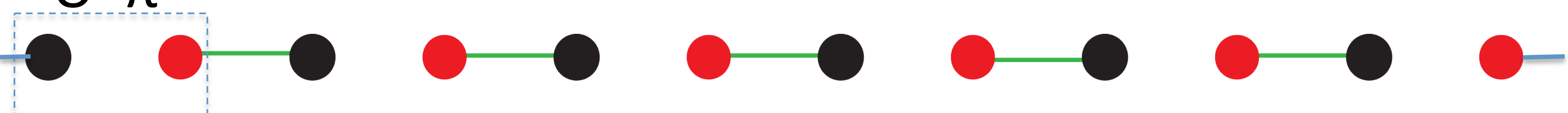
Θ=π

Example: Su-Schrieffer-Heeger model

Θ=0

$$H(k) = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix} \qquad A(k) = 0$$

Θ=π



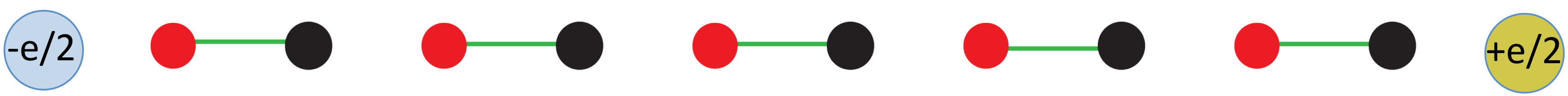
$$H(k) = \begin{pmatrix} 0 & -te^{ika} \\ -te^{-ika} & 0 \end{pmatrix} \qquad A(k) = \frac{1}{2}$$

$$A^{mn}(k) = -i\langle u_m(k)|\partial_k|u_n(k)\rangle \qquad \theta = \int_{-\pi/a}^{\pi/a} dk \, \mathrm{Tr}\,[A(k)]$$

Electromagnetic Response in 1D

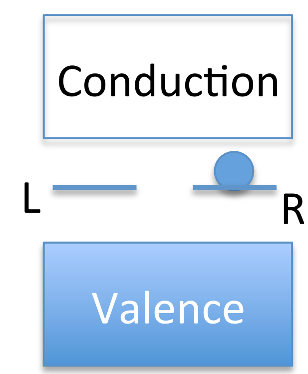
Θ=0

Θ=π



Connection between strong topological invariant and EM response– the charge polarization.

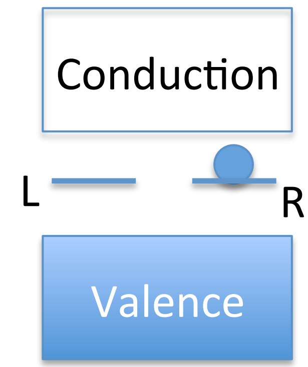
$$P_1 = \frac{e\theta}{2\pi} \mod Ze$$



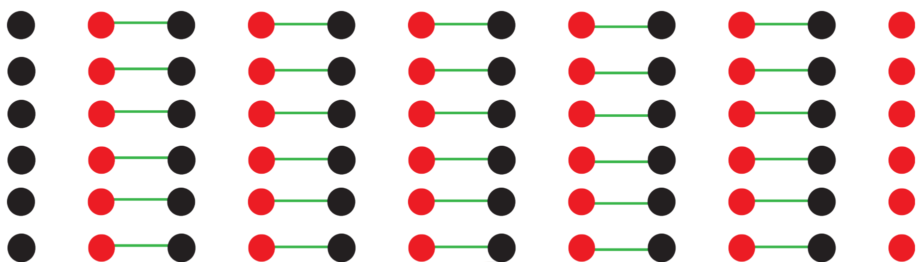
Electromagnetic Response Actions

1D

$$S_1[A_\mu] = \frac{e}{4\pi}\int dx dt \; \theta \epsilon^{\mu\nu} F_{\mu\nu} = \int dx dt \; P_1 E$$

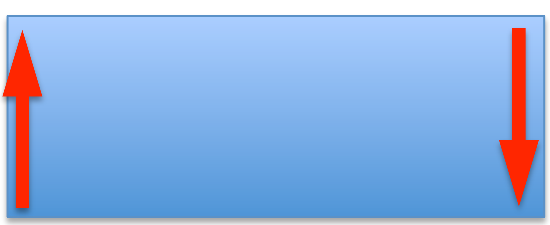


2D (weak)



$$S_2[A_\mu] = \frac{e}{8\pi^2}\int d^2x dt \; G_i \epsilon^{i\mu\nu} F_{\mu\nu} = \int d^2x dt \; \mathbf{P}_1 \cdot \mathbf{E}$$

2D (strong)



$$S_{CS}[A_\mu] = \frac{e^2}{4h}\int d^2x dt \; A_\mu \epsilon^{\mu\nu\rho} F_{\nu\rho}$$

Quantization of Z_2 Electromagnetic Response

$$S_1[A_\mu] = \frac{e}{4\pi}\int dxdt\ \theta\epsilon^{\mu\nu}F_{\mu\nu} = \int dxdt\ P_1 E$$

This type of quantized response appears in all even spacetime dimensions

- Under C symmetry P_1 transforms to $-P_1$ (odd). This constrains P_1=-P_1.

Z_2 Quantization of P...

(Zak

als P_1 is periodic i.e. P_1= P_1+ ne e/2

$$S_3[A_\mu] = \int d^3xdt\ P_3\mathbf{E}\cdot\mathbf{B}$$ (odd under T, T^2=-1)

$$S_5[A_\mu] = \int d^5xdt\ P_5E_{01}B_{23}B_{45}$$ (odd under C, C^2=-1)

$$S_7[A_\mu] = \int d^7xdt\ P_7E_{01}B_{23}B_{45}B_{67}$$ (odd under T, T^2=+1)

Interestingly, *every* action has an **E**-field, thus also odd under inversion!

Turner, Zhang, Vishwanath (2010)
TLH, Prodan, Bernevig (2011)
Turner, Zhang, Mong, Vishwanath (2011)

Inversion Protected Topological Phases

Stabilize topology with inversion instead of C or T symmetry. Leads to new material possibilities, e.g., insulating magnets.

Also, allows efficient calculation of bulk topological invariants:

Example: $PH(k)P^{-1} = H(-k)$

$$P_1 = \frac{e}{2\pi i}\text{Log}\left(\frac{\det B(\pi/a)}{\det B(0)}\right) = \frac{e}{2\pi i}\text{Log}\left(\prod_{a\in occ.} \zeta(\pi/a)\zeta(0)\right)$$

$$\text{Tr}[A(-k)] = -\text{Tr}[A(k)] - i\nabla_k \text{Log}[\det B(k)]$$

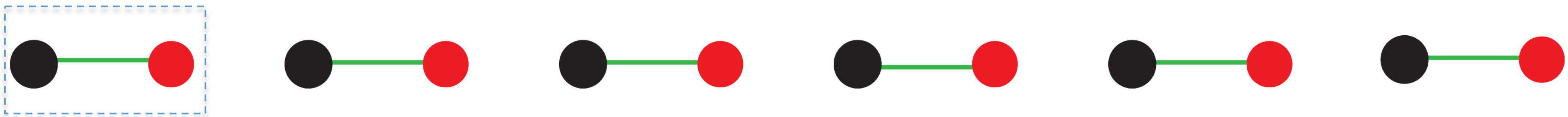
$$B_{mn}(k) \equiv \langle u_m(-k)|P|u_n(k)\rangle$$

If we know the inversion eigenvalues of the occupied bands we can determine polarization. Continuous integral -> discrete data.

Inversion Eigenvalue Example

Θ=0

$$P = \sigma^x$$



$$H(k) = \begin{pmatrix} 0 & -t \\ -t & 0 \end{pmatrix} \qquad A(k) = 0$$

$$\zeta(k=0) = \zeta(k=\pi/a) = +1$$

Θ=π

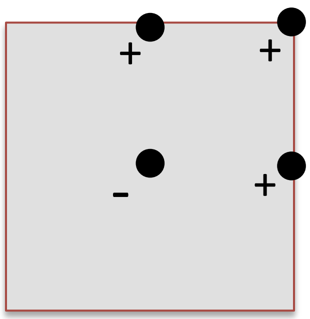
$$H(k) = \begin{pmatrix} 0 & -te^{ika} \\ -te^{-ika} & 0 \end{pmatrix} \qquad A(k) = \frac{1}{2}$$

$$\zeta(k=0) = -\zeta(k=\pi/a) = +1$$

Higher Dimensional Cases with Inversion

2D:

Chern Number



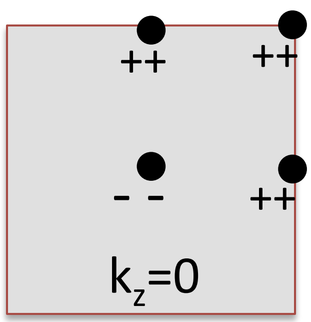
$$(-1)^{C_1} = \prod_{\Lambda,\alpha \in occ.} \zeta_\alpha(k=\Lambda)$$

Inversion(C_2) determines Chern number mod 2 (Hughes et al., Turner et al.)

C_n rotation determines Chern number mod n (Fang et al.)

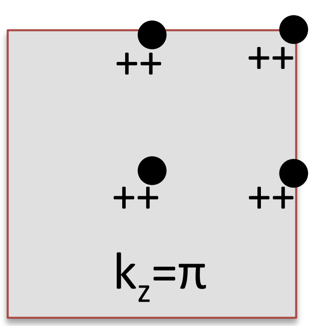
3D:

Magneto-electric polarization



$$S_3[A_\mu] = \int d^3x dt \; P_3 \mathbf{E} \cdot \mathbf{B}$$

With T and P we can use the Fu-Kane formula:



$$P_3 = \prod_{\Lambda,\alpha \in occ./2} \zeta_\alpha(k=\Lambda)$$

Eigenvalues come in Kramers' pairs with T & P.

But if we break T, how do we choose half the occupied states?

Part 1:Topological Phases and Response Protected by Spatial Symmetries

Translation

Weak Invariants Protected by Translation Symmetry

Preserving translation invariance introduces a new series of invariants generically called "weak" topological invariants.

Dim/Symmetry
(0+1)d
(1+1)d
(2+1)d
(3+1)d
…

C	C & Translation	Invariants
Z_2	Z_2	G_0
Z_2	Z_2+Z_2	$G+G_0$
Z	$Z+2Z_2+Z_2$	$G+G_a+G_0$
0	$0+3Z+3Z_2+Z_2$	$0+G_a+G_{ab}+G_0$
…	…	…

While strong invariants are isotropic, the weak invariants are anisotropic.

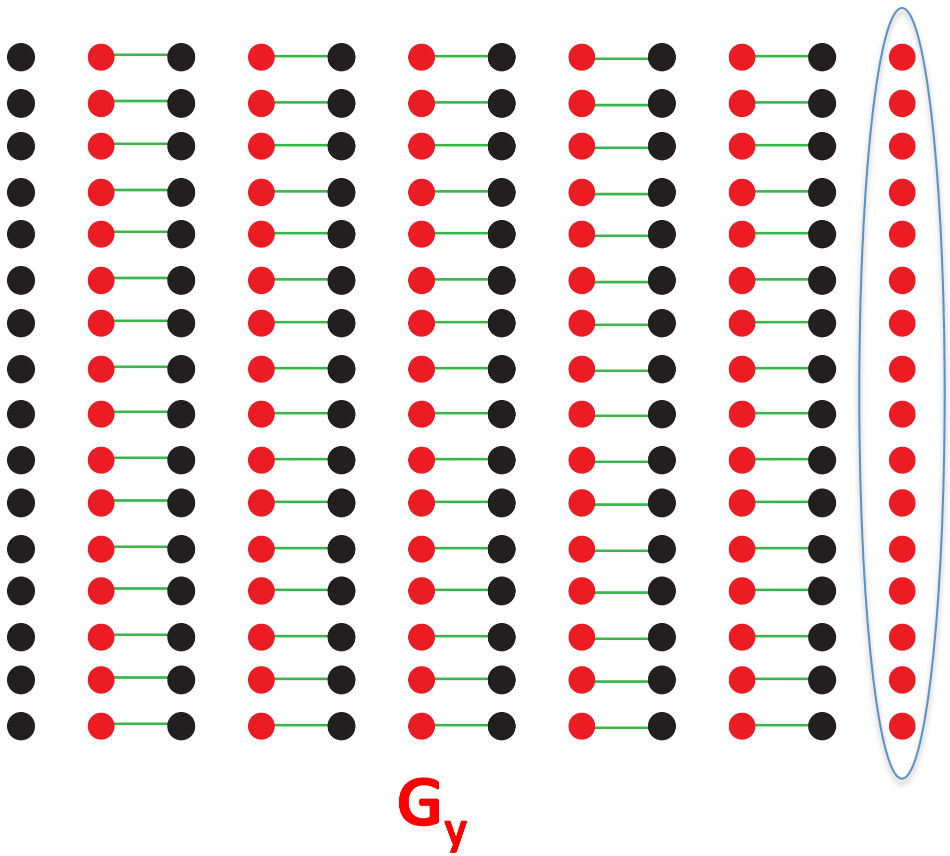
(Fu-Kane-Mele 2007, Moore-Balents 2007, Roy 2009)

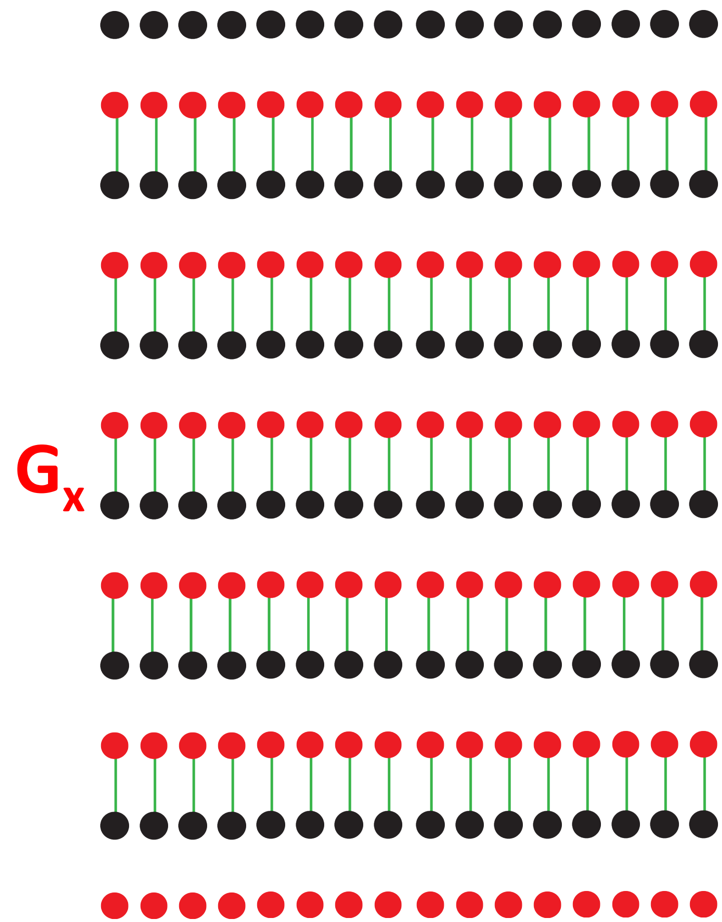
K-theory classification on torus instead of sphere (Kitaev 2009)

Strong+**Weak**+**Secondary Weak**+**Global**

Example: Weak Invariants from SSH

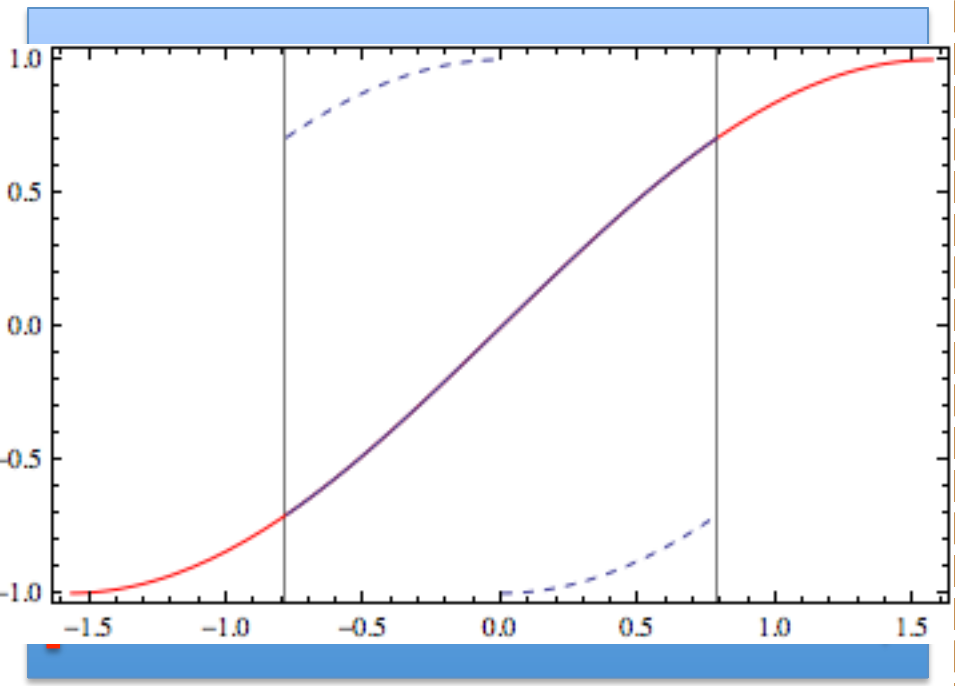
Class D in 2d: Z+**2Z$_2$**

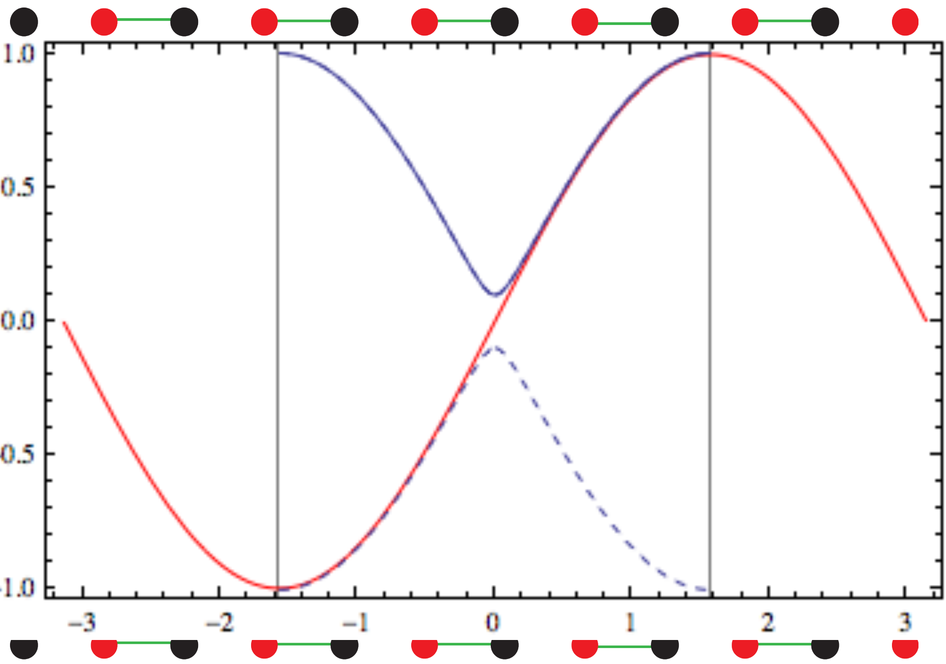




Weak vs. Strong in 2D

Class D in 2d: $Z+2Z_2$





First Chern Number: C_1

G_x

If only the weak invariant is non-zero, breaking translation symmetry (even just on the edge) allows us to gap the system!

Part 1:Topological Phases and Response Protected by Spatial Symmetries

Rotation

Classification of C4 Invariant 2D Superconductors

Description of Mean-Field Superconductors with rotation symmetry

- BdG Hamiltonian in class D (T-breaking)

$$\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$$

- C4 rotation symmetry (square lattice)

$$\hat{r} H_{BdG}(\mathbf{k})\hat{r}^{\dagger} = H_{BdG}(r \cdot \mathbf{k})$$

$$\Xi \hat{r} \Xi^{-1} = \hat{r}$$

$$\hat{r}^4 = -1$$

Teo , TLH; PRL 2013

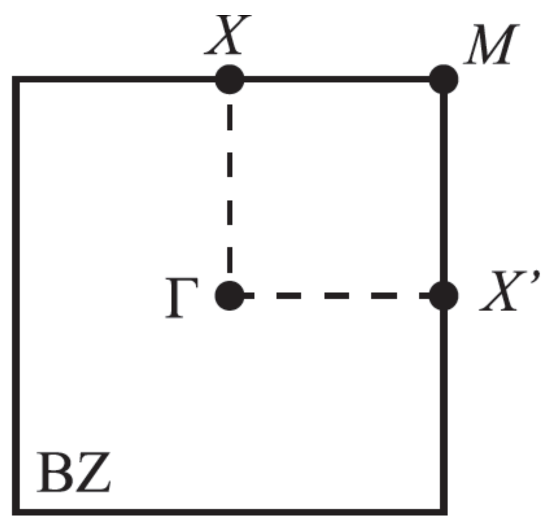
Classification of C4 Symmetric Superconductors

- Topological invariants (all T-breaking)

  (i) First Chern Number $$ch = \frac{i}{2\pi}\int_{BZ} \mathrm{Tr}(d\mathcal{A})$$

  (ii) Rotation invariants

  3 integers defined from rotation eigenvalues at special points in the BZ



Full Classification

$$\mathbb{Z}^4 = \{(ch; n_4, n_6, n_7)\}$$

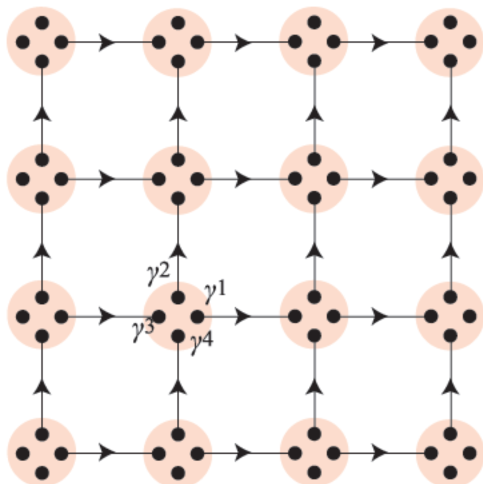
Note: Adding T-symmetry restricts all invariants to vanish!

Teo , TLH; PRL 2013

Some Model Hamiltonians

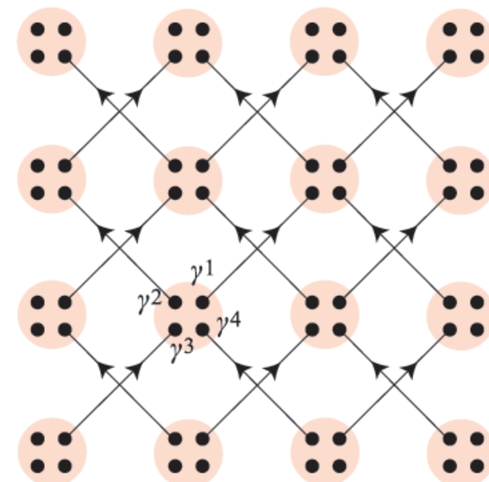
Arrays of Kitaev p-wave wires that preserve C4 symmetry

H_b



Sr_2RuO_4?

H_c



Raghu, Kapitulnik, Kivelson, 2010

TB model	ch	n_4	n_6	n_7
H_b	0	1	−1	1
H_c	0	2	0	0

Teo , TLH; PRL 2013

Part 2: Bound States on Topological Defects in Spatial Symmetry Protected Phases

Boundstate Production Mechanisms

For free fermion models the Dirac domain wall/vortex is the generic mechanism for topological boundstates. However, this does not apply for more complicated interacting systems.

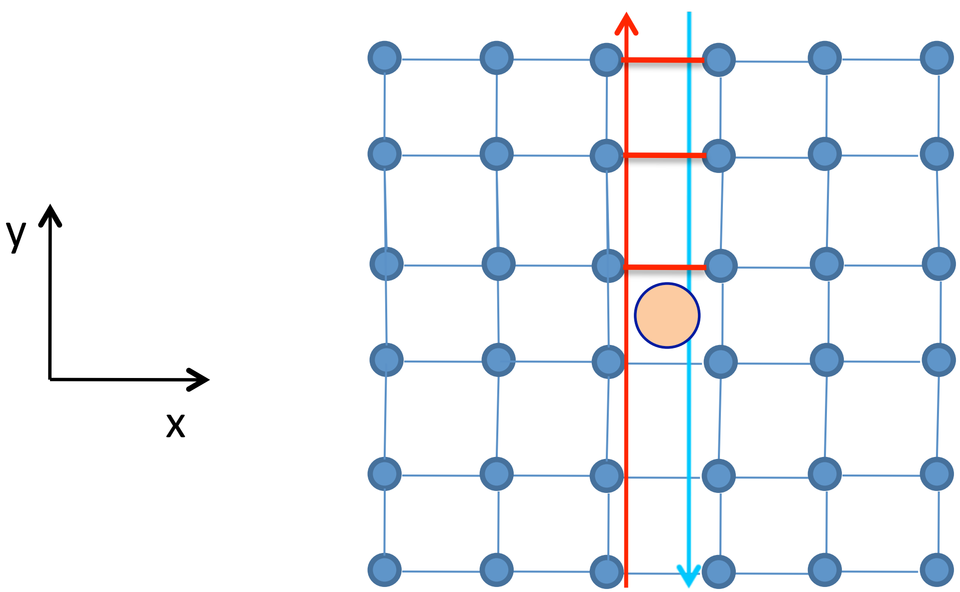
Another mechanism which can be used even with interactions are considering "gauge fluxes" of a global symmetry.

Symmetry	Flux
U(1) Global Charge Conservation	Magnetic flux
Translation Symmetry	Dislocation
Rotation Symmetry	Disclination
Anyonic Symmetry	Twist Defect

In the case of free fermions the mechanisms coincide.

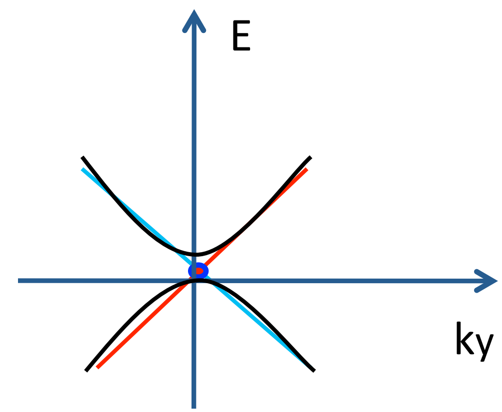
Bound States on a flux in the QAHE/Chern Insulator

Topological Phase Protected by Global U(1) symmetry: global charge conservation



$$m(y) = m$$

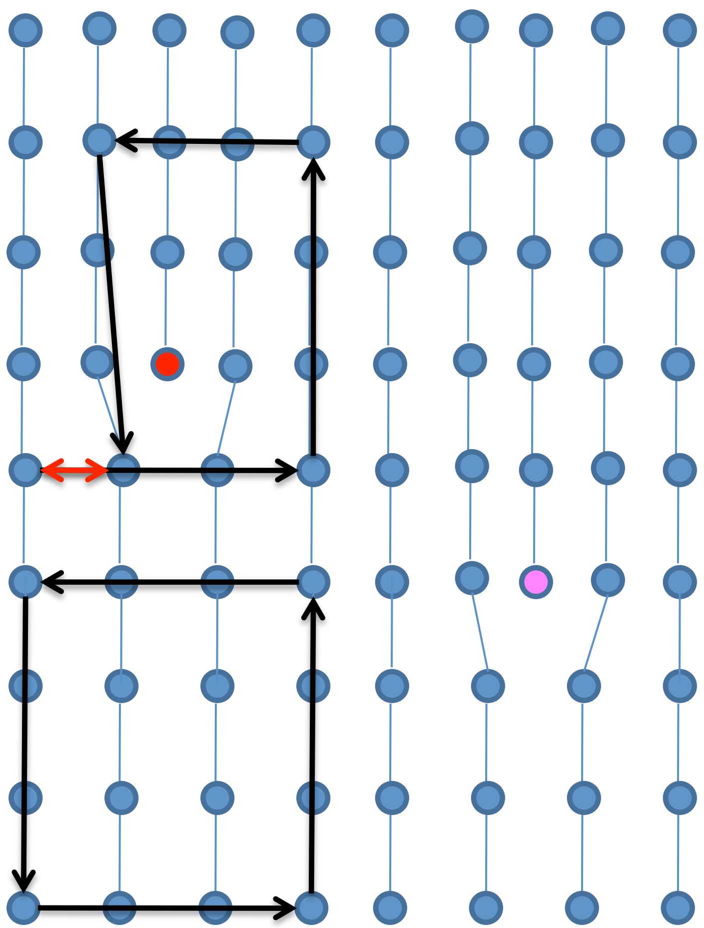
Gapless fermion spectrum on cut



$$H = \begin{pmatrix} p_y & m(y) \\ m(y) & -p_y \end{pmatrix}$$

Lee, Zhang, Xiang PRL (2007)

Crystal Dislocations: Translation Defects



Let’s take a path in the lattice
3 steps right
3 steps up
3 steps left
3 steps down
This path is closed in the reference state.

The amount of translation is the Burgers vector and it is a vector of topological charges. It doesn’t change if you continuously deform the dislocation.

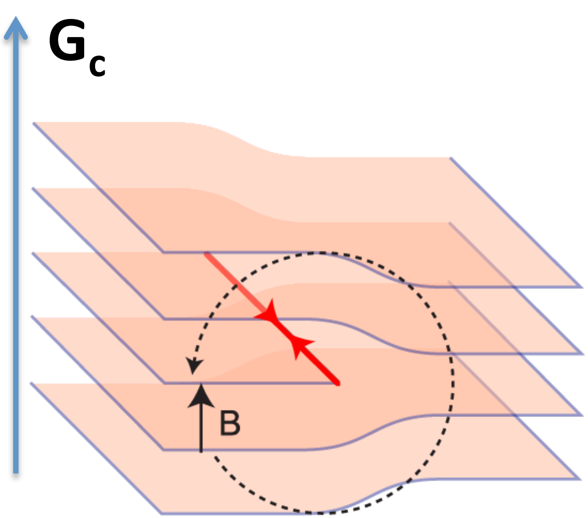
Dislocation Bound States in Translation Protected Topological States

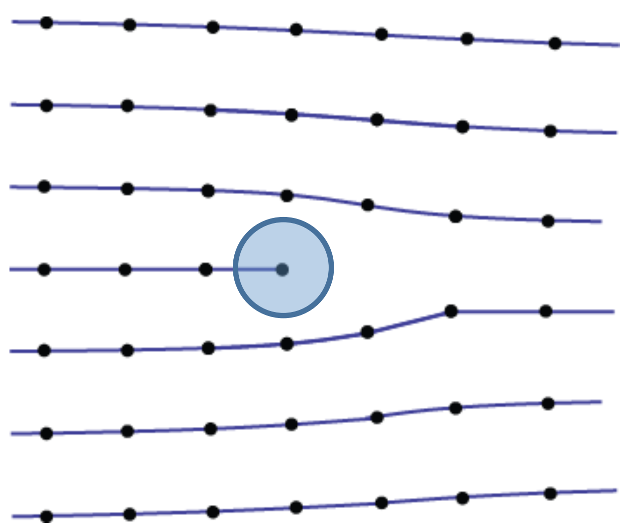
Topological insulators/ superconductors (class D) with weak indices $(G_1, G_2, G_3)=\mathbf{G_c}$

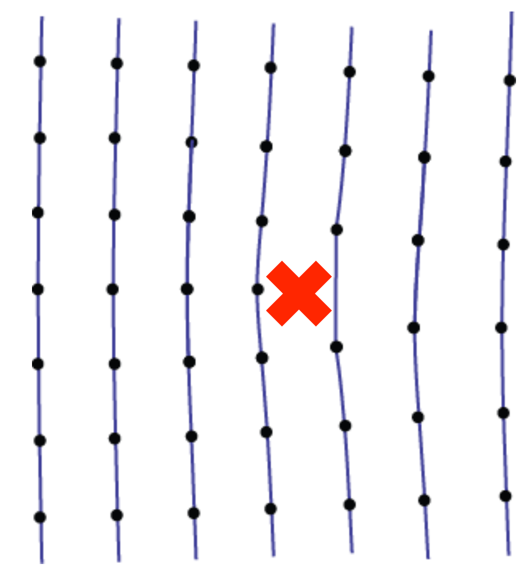
$$n = \frac{1}{2\pi}\mathbf{G}_c \cdot \mathbf{B}$$

Burger's vector characterizing dislocation

Ran, Zhang, Vishwanath, 2009



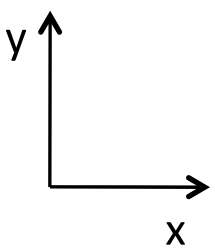


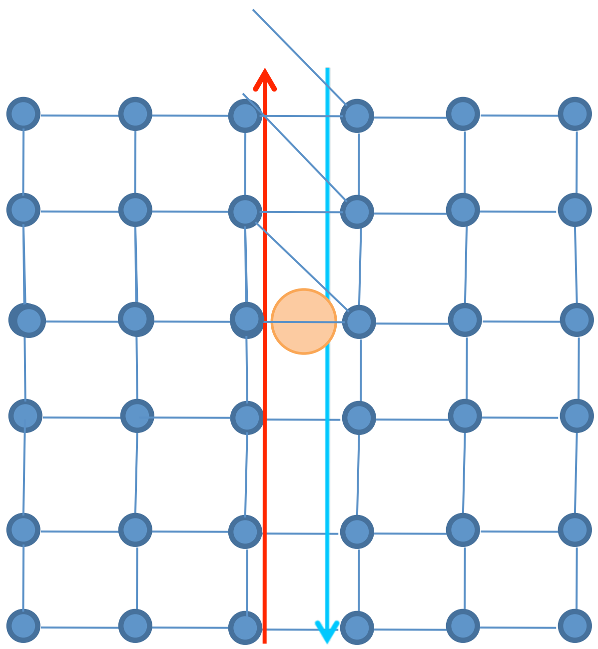


Teo, Kane, 2010, Ran 2010, Asahi, Nagaosa, 2012 Juricic, et al., 2012 TLH, Yao, Qi, 2013

Bound States on Dislocations

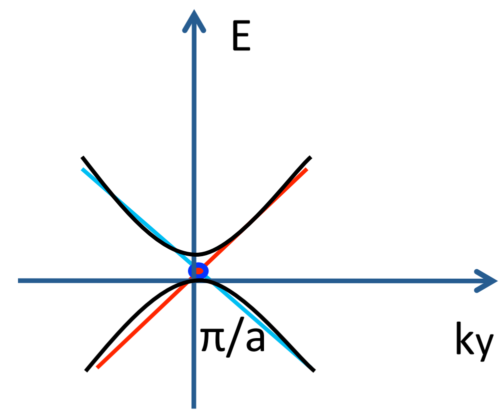
$$m(y) = me^{i\mathbf{b}\cdot\mathbf{K}}$$





$$m(y) = m$$

Gapless fermion spectrum on cut



$$H = \begin{pmatrix} p_y & m(y) \\ m(y) & -p_y \end{pmatrix}$$

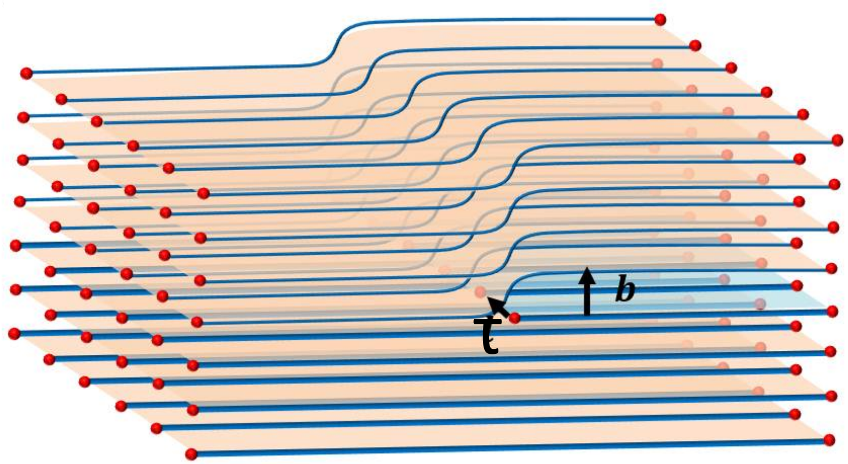
Ran, Zhang, Vishwanath Nat. Phys. (2009).

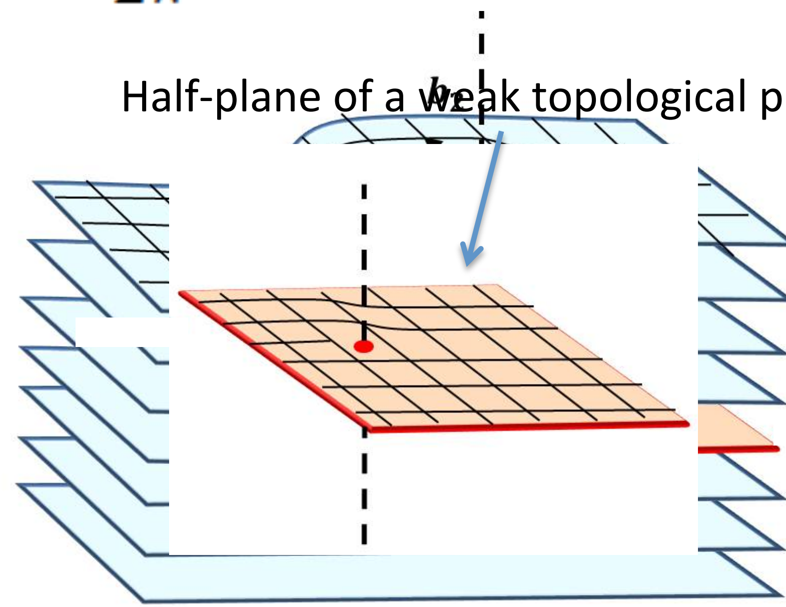
Bound States with Secondary Weak Invariants

In class D in 3d we have an antisymmetric tensor $\mathbf{G}_{ab}$

$$n = \frac{1}{2\pi} G_{ab} B^a \tau^b$$

$$n = \frac{1}{2\pi} G_{ab} B_1^a B_2^b$$





Requires translation symmetry along dislocation.
A weak invariant for the dislocation itself!

Bound state on linked dislocations does not require symmetry along dislocation.
Possible appearance in Raghu, Kapitulnik, Kivelson state of Sr_2RuO_4 where $G_{ab} \neq 0$.

TLH, Yao, Qi, 2013

Summary of Boundstate Index Theorems in Topological Superconductors

Strong Invariant (no symmetry)

$$\Theta_{vortex} = \frac{1}{2\pi}\frac{\Phi}{\phi_0}Ch \qquad \left(\phi_0 = \frac{h}{2e}\right)$$

Primary Weak Invariant (Translation)

$$\Theta_{dislocation} = \frac{1}{2\pi}\mathbf{B}\cdot\mathbf{G}_\nu$$

Secondary Weak Invariant (Translation)

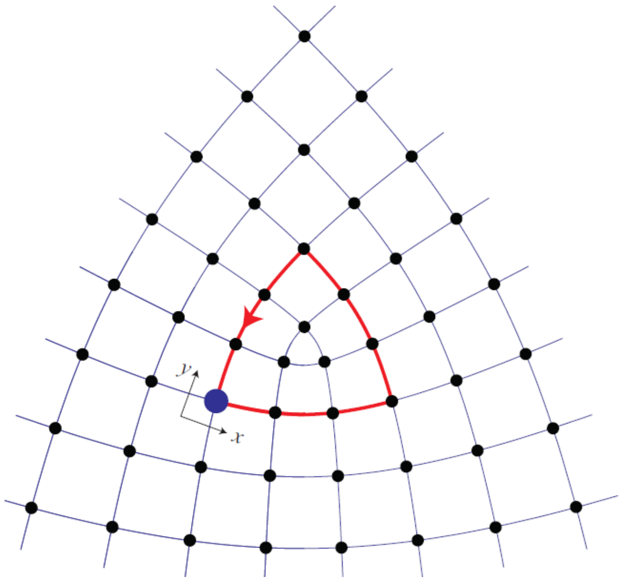
$$\Theta_{dislocation^2} = \frac{1}{2\pi}(B_1^a B_2^b)G_{ab}$$

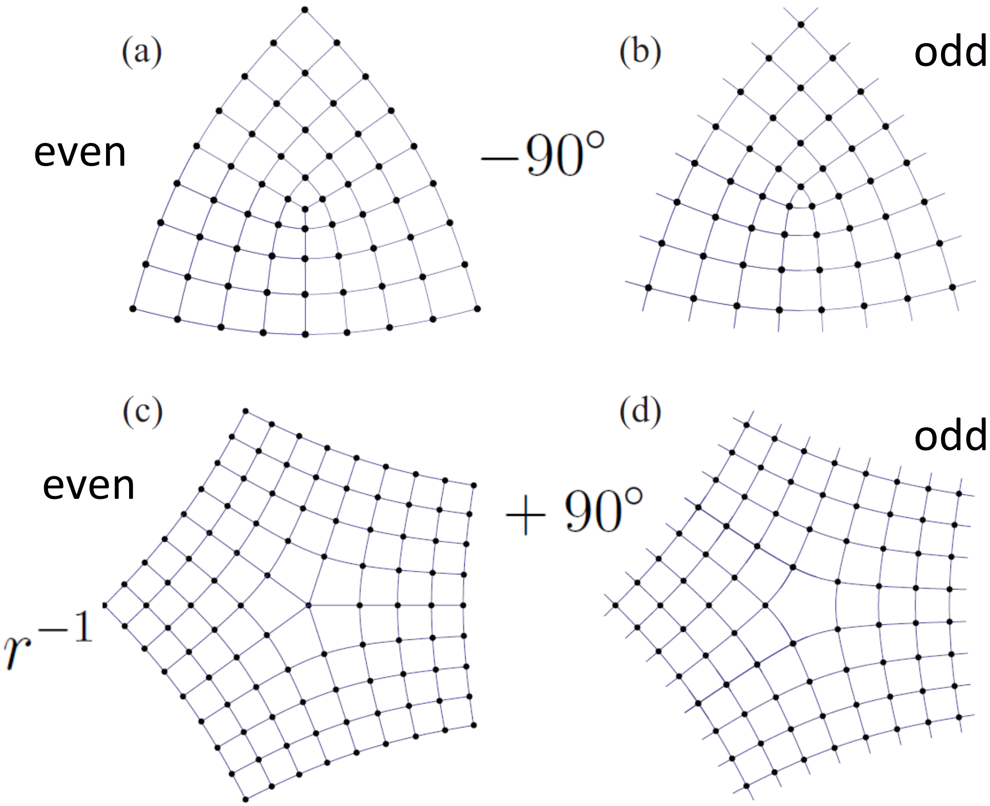
Total Index (= 0 even number of Majorana Boundstates, =1 odd number)

$$\Theta = \Theta_{vortex} + \Theta_{dislocation} + \Theta_{dislocation^2} \mod 2$$

Teo, TLH 2013, TLH, Yao, Qi, 2013

Disclinations in the Square Lattice





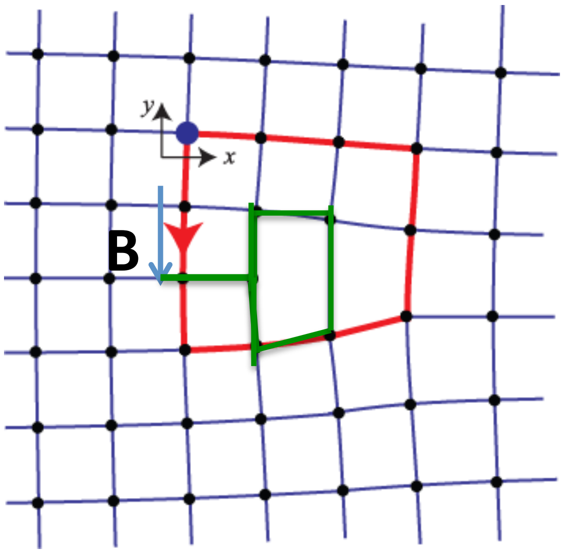
$$r(\mathbf{e_x})^3 r(\mathbf{e_x})^3 r(\mathbf{e_x})^3 = (-3\mathbf{e}_x) r^{-1}$$

Classification: $C_4 \times \mathbb{Z}_2$ Frank Angle x Translation Parity

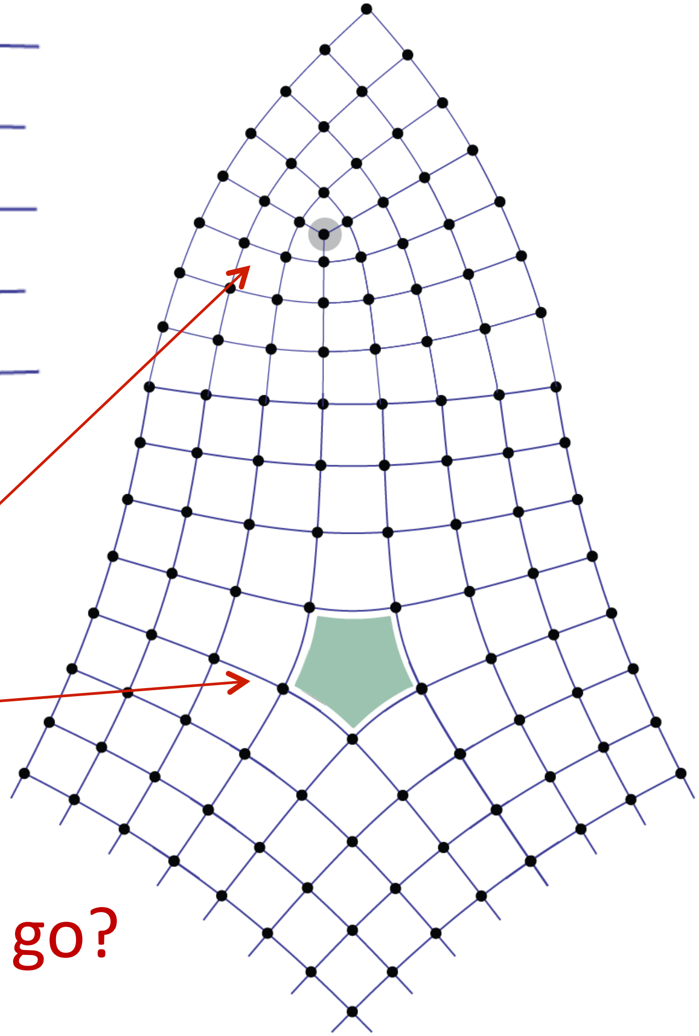
Eveness / oddness of number of translations.
Equal to number of distinct rotation centers.

Teo , TLH; PRL 2013

Dislocation = Disclination Dipole



Overall odd dislocation



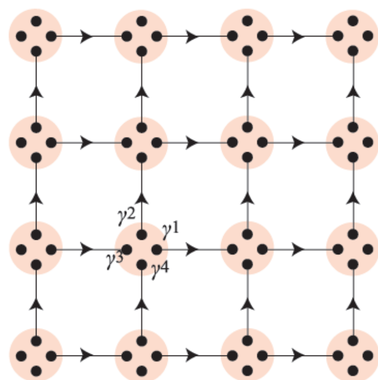
How does Majorana mode decide where to go?

Teo , TLH; PRL 2013

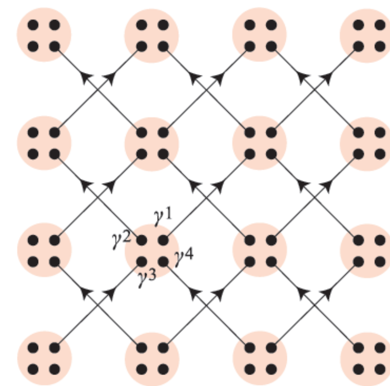
Majorana Zero Modes at Disclinations

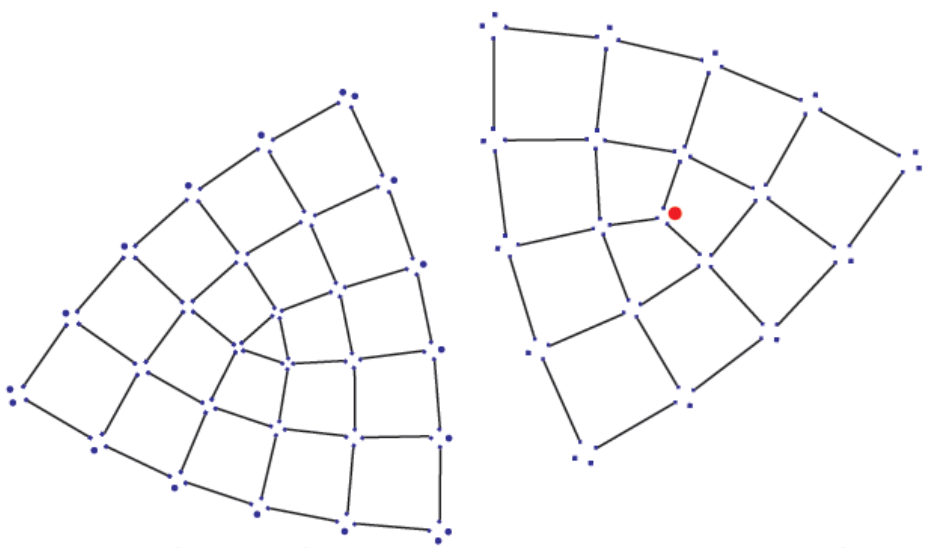
- Simple Majorana TSC Models with C4 symmetry:

$H_b =$



$H_c =$





Teo , TLH; PRL 2013

Z_2 Index for MBS on Disclinations

$$\Theta_{rot} \equiv \left[\frac{1}{2\pi} \mathbf{T} \cdot \mathbf{G}_\nu + \frac{\Omega}{2\pi} (ch + n_6 + 2n_4 + 3n_7) \right] \mod 2$$

Frank angle

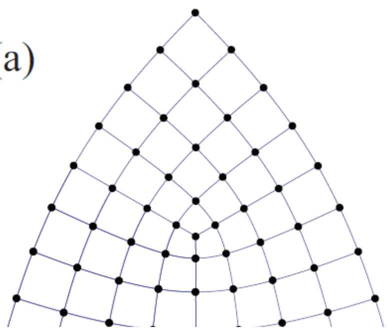
Rotation invariant from occupied bands

Weak invariant

$$\mathbf{G}_\nu = n_4 + n_6 + n_7$$

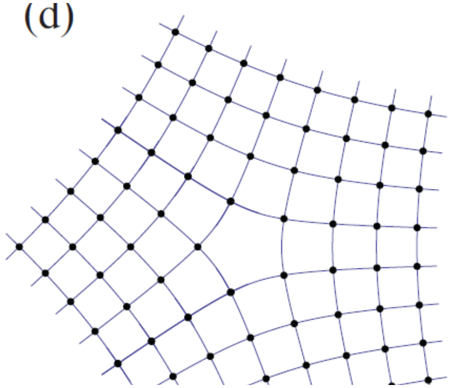
Chern invariant

(a)



$\mathbf{T} = 0, \Omega = - \pi/2$

(d)



$\mathbf{T} = 1, \Omega = +\pi/2$

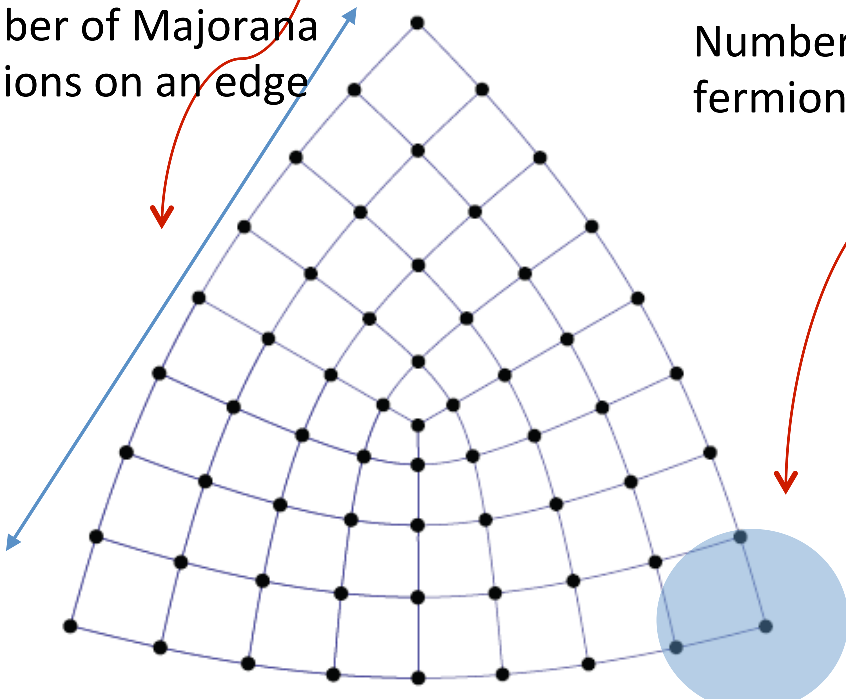
Teo , TLH; PRL 2013

Z_2 Index for MBS on Disclinations

$$\Theta_{\mathrm{rot}} \equiv \left[\frac{1}{2\pi} \mathbf{T} \cdot \mathbf{G}_\nu + \frac{\Omega}{2\pi} (ch + n_6 + 2n_4 + 3n_7) \right] \mod 2$$

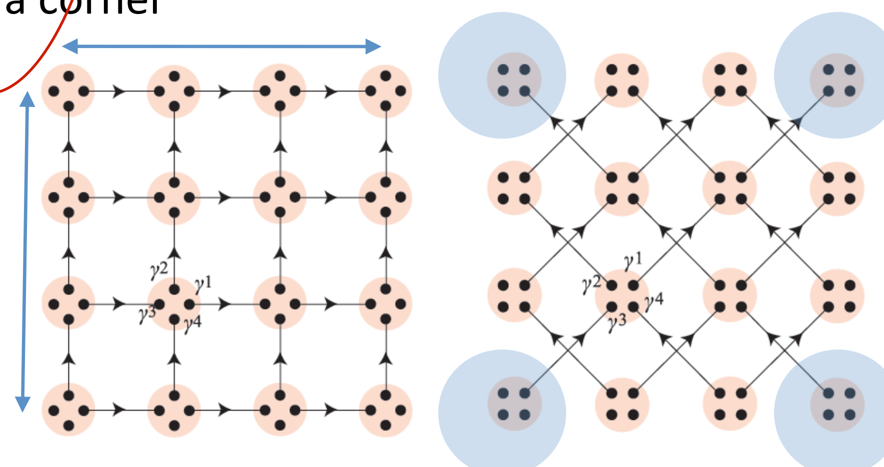
Translation piece

Number of Majorana fermions on an edge



Rotation piece

Number of Majorana fermions at a corner



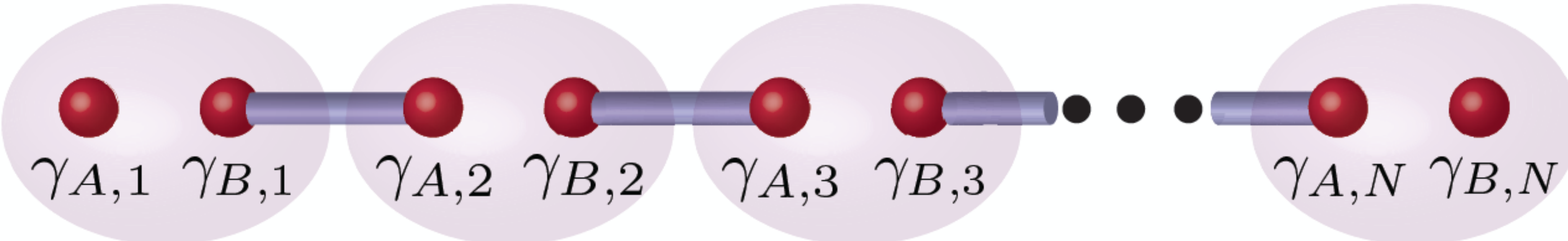
Teo , TLH; PRL 2013

Part 3: Interaction-Induced Topological Phases Protected by Point-Group Symmetry

M. F. Lapa, J. C. Y. Teo, and TLH (Submitted)

Topological Superconductors with T and P Symmetry

Take class BDI which are topological superconductors with T symmetry (T^2=+1)



$\gamma = \gamma^\dagger \quad \gamma^2 = 1$ $\qquad c_n = \gamma_{nA} + i\gamma_{nB}$

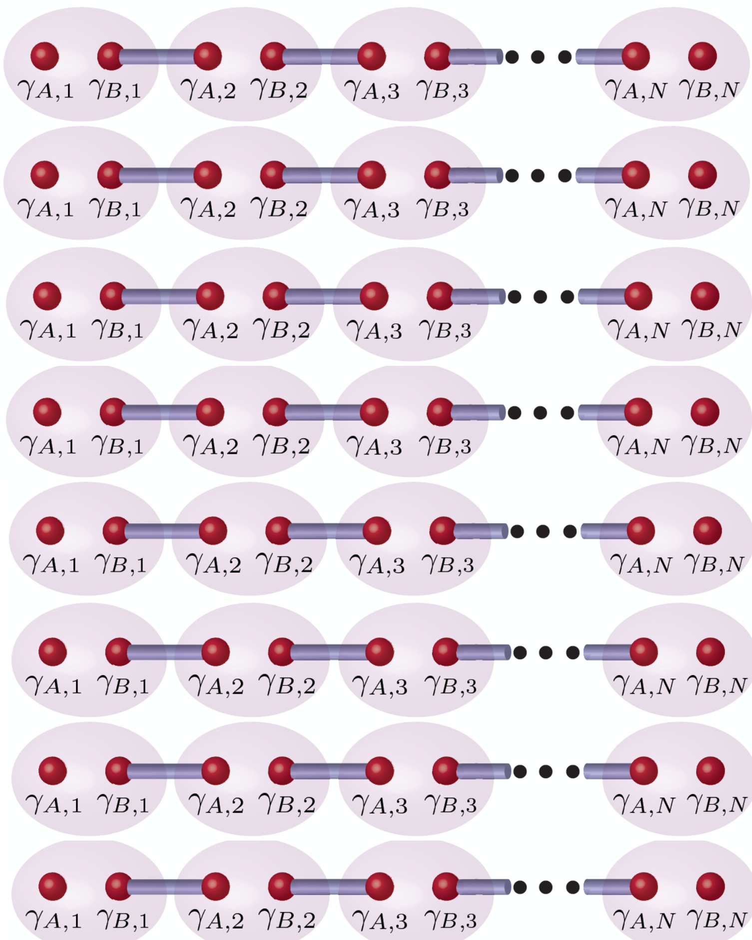
Action of T: $\quad T c_n T^{-1} = c_n$

$$T\gamma_{nA}T^{-1} = \gamma_{nA} \quad T\gamma_{nB}T^{-1} = -\gamma_{nB}$$

Topological Superconductors with T and P Symmetry

BDI Classified by an integer:

$$\nu = \#(\text{unpaired B modes}) - \#(\text{unpaired A modes})$$



$$i\gamma_{B,N,\alpha}\gamma_{B,N,\beta}$$

Now add inversion symmetry

$\nu = \{$

$$\nu \quad \rightarrow \quad -\nu \quad \text{(under inversion)}$$

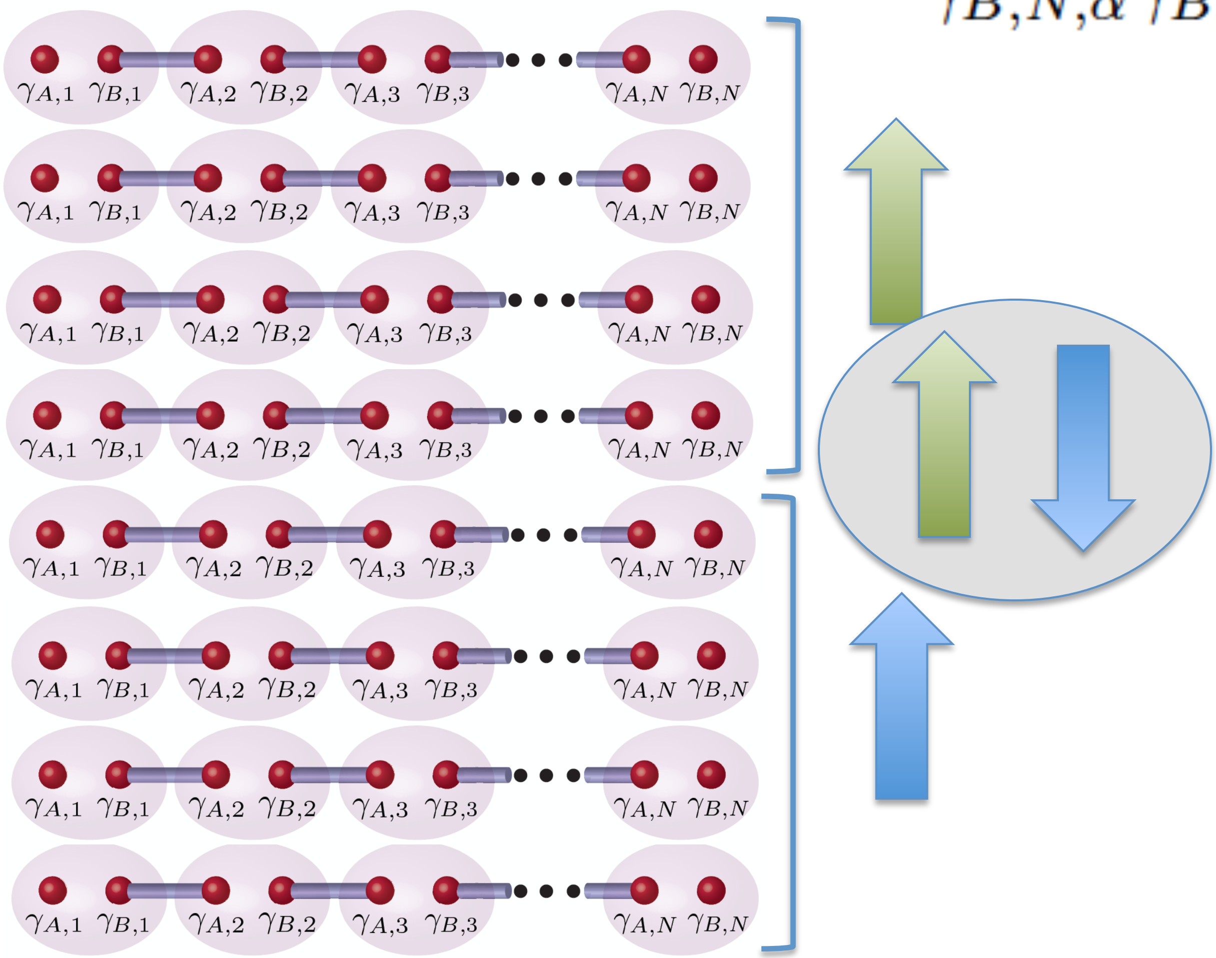
$$\implies \nu = -\nu$$

$$\implies \nu = 0 \;\; \text{since } \nu \in \mathbb{Z}$$

Topological Superconductors with T and P Symmetry

Introduce interactions (Fidkowski and Kitaev 2011):

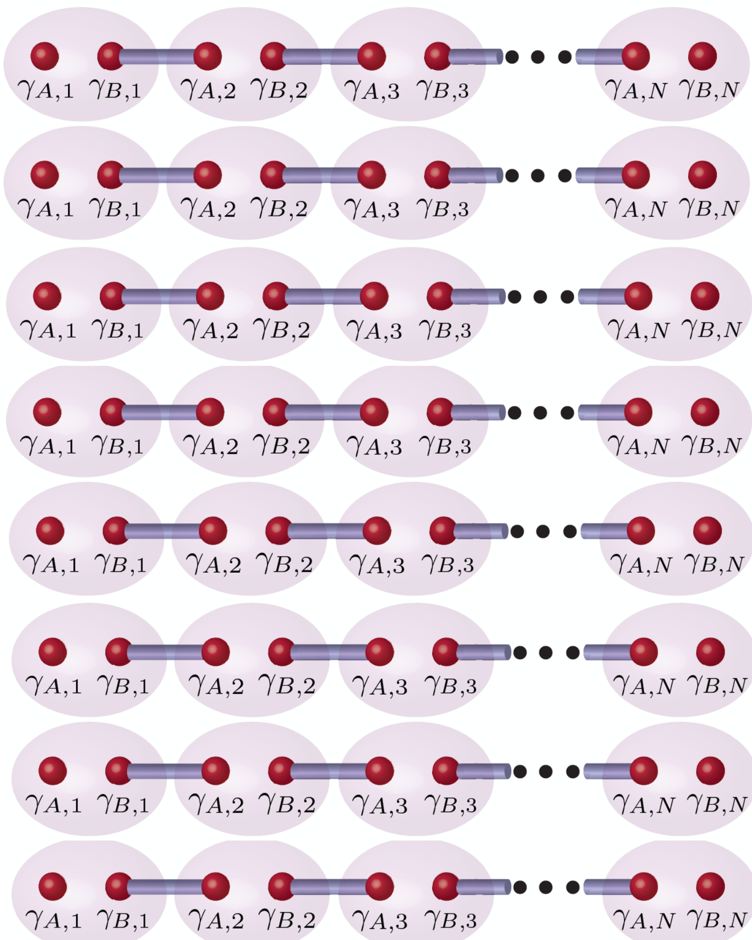
$$\gamma_{B,N,\alpha}\gamma_{B,N,\beta}\gamma_{B,N,\gamma}\gamma_{B,N,\delta}$$



$$\Longrightarrow \quad \nu = 8 \equiv \nu = 0$$

$$\Longrightarrow \quad \mathbb{Z} \to \mathbb{Z}_8$$

Topological Superconductors with T and P Symmetry



Introduce interactions (Fidkowski and Kitaev 2011):

$$\Longrightarrow \quad \nu = 8 \equiv \nu = 0$$

$$\Longrightarrow \quad \mathbb{Z} \to \mathbb{Z}_8$$

Now add inversion symmetry

$$\nu \quad \to \quad -\nu$$

$$\Longrightarrow \quad \nu = -\nu$$

$$\Longrightarrow \quad \nu = 0, 4 \ \text{ since } \nu \in \mathbb{Z}_8$$

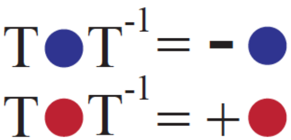
$$\boxed{0 \to \mathbb{Z}_2}$$

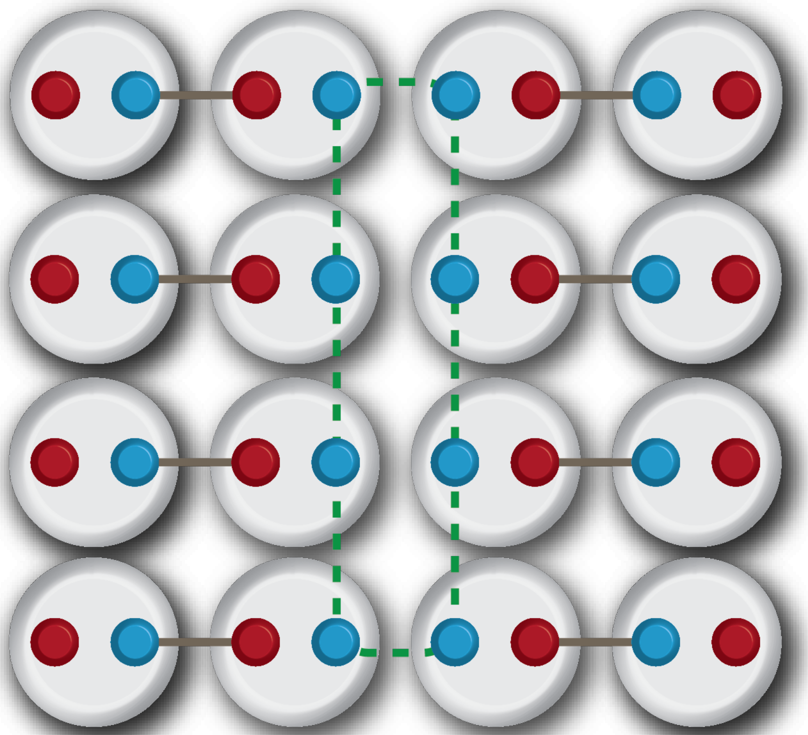
This means we have an interaction induced topological invariant that does not appear in free fermion (including mean-field) systems.

Topological Superconductors with T and P Symmetry

Now, can we find a model that represents the non-trivial phase?

Let's try to construct a simple example model



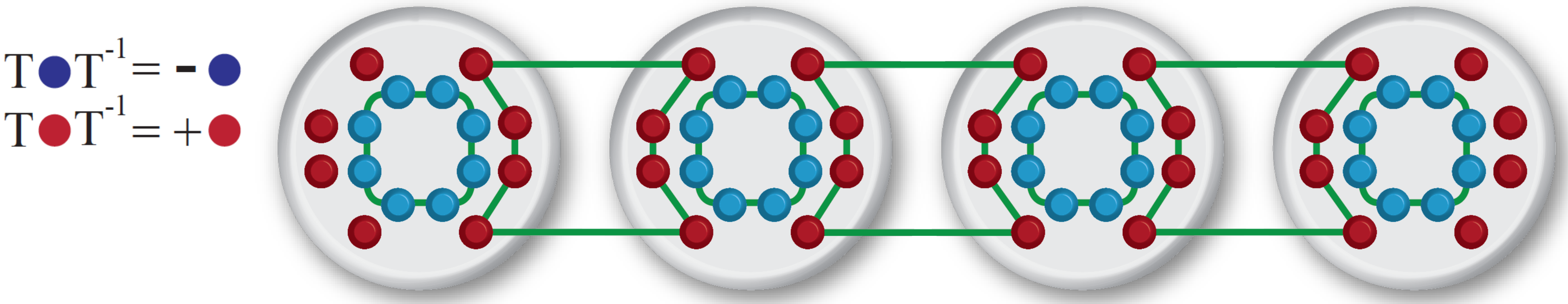


We can immediately see why strong interactions are required.

Topological Superconductors with T and P Symmetry

Now, can we find a model that represents the non-trivial phase?

Let’s do better by making it translation invariant by forming a Fidkowksi-Kitaev chain:



This model is a topological charge-4e superconductor.

Just as one can get single-electron teleportation in the Kitaev chain, we can observe teleportation of full Cooper pairs in the Kitaev-Fidkowski chain.

Acknowledgements

Thank you!

And thanks to my close collaborators

Jeffrey C. Y. Teo (UIUC Postdoc/Simons Fellow)->U. Virginia
Wladimir Benalcazar (UIUC student)
Matt Lapa (UIUC student)

Emil Prodan (Yeshiva)
Andrei Bernevig (Princeton)