The Non-Commutative Geometry of the A- and AIII- symmetry classes of TIs

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Relevant Papers

- Prodan, Hughes, Bernevig, Entanglement spectrum of a disordered Chern insulator, Phys. Rev. Lett 2010
- Prodan, Quantum transport in disordered systems under magnetic fields: A Study based on operators algebras, Appl. Math. Research Express 2013
- Prodan, Leung, Bellissard, The non-commutative n-th Chern number ($n \geq 1$), J. Phys. A 2013
- Mondragon, Song, Hughes, Prodan, Quant. Criticality in disordered chiral systems, Phys. Rev. Lett. 2014
- Prodan, Schulz-Baldes, Non-commutative odd Chern numbers and topological phases of disordered chiral systems, ArXiV 2014
- Prodan, The non-commutative geometry of the complex classes of TIs, Topo. Quant. Matter 2014

Cartan	d												
	0	1	2	3	4	5	6	7	8	9	10	11	…
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	…
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	…
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	…
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	…
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	…
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	…
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	…
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	…
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	…
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	…

For class A (d = even)

$$Ch_d(P) = \Lambda_d \int_{\mathbb{T}^d} \mathrm{Tr}\left\{ (P\, dP \wedge dP)^{\frac{d}{2}} \right\} \qquad (k\text{-space})$$

$$Ch_d(P) = \Lambda_d \sum_{s \in S_d} (-1)^s \, \tilde{\mathcal{T}} \left\{ p \prod_{i=1}^{d} \partial_{s_i} p \right\} \qquad (\text{NC form})$$

For class AIII (d = odd)

$$\tilde{Ch}_d(U) = \tilde{\Lambda}_d \int_{\mathbb{T}^d} \mathrm{Tr}\left\{ (U^{-1} dU)^d \right\} \qquad (k\text{-space})$$

$$\tilde{Ch}_d(u) = \tilde{\Lambda}_d \sum_{s \in S_d} (-1)^s \, \mathcal{T} \left\{ \prod_{i=1}^{d} u^{-1} \partial_{s_i} u \right\} \qquad (\text{NC-form})$$

Alain Connes in X-slides (X small)

Goal:

Compute the K-theory of algebras

Algorithmic and extremely general constructions

The algebraic $K_0^A(\mathcal{A})$ of a $*$-algebra

$\mathcal{P}_n(\mathcal{A})$ = the set of idempotents in $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$

$$\mathcal{P}_\infty(\mathcal{A}) = \bigcup_{n=1}^{\infty} \mathcal{P}_n(\mathcal{A})$$

$$\mathcal{P}_n(\mathcal{A}) \ni p \sim q \in \mathcal{P}_m(\mathcal{A}) \text{ if}$$

$$p = u \cdot v \quad \text{and} \quad q = v \cdot u$$

for some

$$u \in \mathcal{A} \otimes \mathcal{M}_{n \times m}(\mathbb{C})$$

$$v \in \mathcal{A} \otimes \mathcal{M}_{m \times n}(\mathbb{C})$$

Addition:

$$[p] + [q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$$

The algebraic $K_1^A(\mathcal{A})$ of a

$\mathcal{U}_n(\mathcal{A})$ = the set of invertibles in $\mathcal{A} \otimes \mathcal{M}_n(\mathbb{C})$

$$\mathcal{U}_\infty(\mathcal{A}) = \lim_{n \to \infty} \mathcal{U}_n(\mathcal{A}) \quad \text{(inductive limit)}$$

$$K_1^A(\mathcal{A}) = \mathcal{U}_\infty(\mathcal{A}) / [\mathcal{U}_\infty(\mathcal{A}), \mathcal{U}_\infty(\mathcal{A})]$$

$[\mathcal{U}_\infty(\mathcal{A}), \mathcal{U}_\infty(\mathcal{A})]$ = the normal subgroup of commutators (generated by $fgf^{-1}g^{-1}$)

Cyclic Cohomology

The objects are the cyclic $(n+1)$-linear functionals

$$\mathcal{A} \ni a_0, a_1, \ldots, a_n \longmapsto \phi(a_0, a_1, \ldots, a_n) \in \mathbb{C}$$

$$\phi(a_1, a_2, \ldots, a_n, a_0) = (-1)^n \phi(a_0, a_1, \ldots, a_n)$$

$C^n(\mathcal{A})$ = the space of cyclic $n+1$-linear functionals

The complex

$$\cdots \xrightarrow{b} C^{n-1}(A) \xrightarrow{b} C^{n}(A) \xrightarrow{b} C^{n+1}(A) \to \cdots \qquad (b^2 = 0)$$

$$C^n(A) \ni \phi \longrightarrow b\phi \in C^{n+1}(A) \quad (\text{Hochschild map})$$

$$(b\phi)(a_0, a_1, \dots, a_{n+1}) =$$

$$\sum_{j=0}^{n} (-1)^j \phi(a_0, \dots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \dots, a_{n+1})$$

$$+ (-1)^{n+1} \phi(a_{n+1} a_0, a_1, \dots, a_n)$$

Terminology:

$\phi \in C^n(A)$, $b\phi = 0$ <-> n-cyclic cocycle

$\phi' \sim \phi$ if $\phi - \phi' = b\varphi$ → cohomology classes $[\phi]$

cyclic cocycle is $\begin{cases} \text{even if } n \text{ is even} \\ \text{odd if } n \text{ is odd} \end{cases}$

Pairing of even cyclic cocycles with $K_0^A(\mathcal{A})$

Theorem: ϕ even cocycle. Then:

$$\mathcal{P}_\infty(\mathcal{A}) \ni p \longrightarrow (\phi \# Tr)(p, p, \ldots, p)$$

1) constant on the equivalence class $[p]$ of p in $K_0^A(\mathcal{A})$

2) insensitive if we change ϕ with other ϕ' from $[\phi]$.

In short, the even cyclic cohomology pairs well with K_0

$$\langle [p], [\phi] \rangle = (\phi \# Tr)(p, p, \ldots, p)$$

The pairing gives a numerical invariant which in general is not integer.

Pairing of odd cyclic cocycles with $K_1^A(A)$

Theorem: ϕ odd cyclic cocycle. Then:

$$\mathcal{U}_\infty(A) \ni v \to (\phi \# Tr)(v^{-1}-1, v-1, \dots, v^{-1}-1, v-1)$$

1) constant on the equivalence class $[v]$ of v in $K_1^A(A)$

2) insensitive if we change ϕ by any other ϕ' from $[\phi]$

In short, the odd cyclic cohomology pairs well with K_1

$$\langle [v], [\phi] \rangle = (\phi \# Tr)(v^{-1}-1, v-1, \dots, v^{-1}-1, v-1).$$

This pairing gives a numerical invariant which in general is not integer.

Quantized Calculus with Fredholm modules.

(to produce cyclic cocycles!)

Even Fredholm modules $(\mathcal{A}, \mathcal{H}, \pi, F, \gamma)$

- π is a representation of $\mathcal{A}$ on $\mathcal{H}$
- F is operator on $\mathcal{H}$, $F^+ = F$, $F^2 = 1$, $[F, \pi(a)]$ compact
- γ is a grading, $\gamma^+ = \gamma$, $\gamma^2 = 1$ and

$$\gamma \pi(a) = \pi(a)\gamma, \quad \gamma F = -F\gamma$$

Odd Fredholm modules $(\mathcal{A}, \mathcal{H}, \pi, F)$

- π is a representation of $\mathcal{A}$ on $\mathcal{H}$
- F is operator on $\mathcal{H}$, $F^+ = F$, $F^2 = 1$, $[F, \pi(a)]$ compact

A module is said to be (n+1)-summable if

$$\mathrm{Tr}\,|[F, \pi(a)]|^{n+1} < \infty$$

Differential forms

$$\Omega^k = \text{span}\{\Pi a_0 [F, \Pi a_1] \ldots [F, \Pi a_k],\ a_j \in A\}$$

The diferentiation:

$$\Omega^k \ni \eta \to d\eta = F\eta - (-1)^k \eta F$$

The integration

$$\Omega^n \ni \eta \to \int \eta = \begin{cases} \frac{1}{2}\,\mathrm{Tr}\{\gamma F d\eta\} & \text{even} \\ \frac{1}{2}\,\mathrm{Tr}\{F d\eta\} & \text{odd} \end{cases}$$

The Connes – Chern characters.

Theorem: Provided $(n+1)$-summability: (n = even or odd)

$$\tau_n(a_0, a_1, \dots, a_n) = \frac{(-1)^n}{2^{n+1}} \int \pi a_0 [F, \pi a_1] \dots [F, \pi a_n]$$

are cyclic cocycles.

Notation:

$$Ch_*(\mathcal{H}, F, \gamma) = [\tau_n] \quad n \text{ even}$$

$$\widetilde{Ch}_*(\mathcal{H}, F) = [\tau_n] \quad n \text{ odd}$$

The Connes–Chern characters

Theorem: The pairing of Ch_* with K_* is integral:

$$\langle [P], Ch_*(\mathcal{H}, F, \gamma) \rangle = \int \pi p [F, \pi(p)] \dots [F, \pi p] = \mathrm{Index}\{\pi^+ P \, F \, \pi^- P\}$$

$$\langle [v], \widetilde{Ch}_*(\mathcal{H}, F) \rangle = \int \pi(v^{-1}) [F, \pi v] \dots [F, \pi v^{-1}][F, \pi v] = \mathrm{Index}\{E \pi v E\}$$

THE LOCAL INDEX FORMULA IN NONCOMMUTATIVE GEOMETRY

Alain CONNES and Henri MOSCOVICI

Before running through the list of examples, let us state the mathematical problem:

compute by a local formula the cyclic cohomology Chern character of $(\mathcal{A}, \mathcal{H}, D)$.

Corollary II.1. *Let D and U be as above. Then*

$$\text{Index } PUP =$$

$$\sum_{n \leq p} (-1)^{\frac{n-1}{2}} \left(\frac{n-1}{2}\right)! \sum_{k,q} \frac{(-1)^{|k|}}{k_1! \ldots k_n!} \ \alpha_k \ \frac{1}{q!} \ \sigma_{m-q}(m) \ \text{Res}_{s=0} \ s^q \ \zeta_{(k,n)}(s) \ ,$$

with $m = |k| + \frac{n-1}{2}$.

For our particular setting (torus and its NC version) the formula simplifies tremendously.

In fact, a more direct proof is possible, which remains valid in the regime of strong disorder

Recap :

Alain Connes Program

Algebra

$\Downarrow$

Fredholm module $(\mathcal{H}, \pi, F, \gamma)$

$\Downarrow$

Connes - Chern Characters

$\Downarrow$

Local formula

Algebra:

- Fermi projections P_{E_F} or the U's for AIII class are not in the C^*-algebra of covariant observables if E_F is in the energy spectrum.

1) Extend $\mathcal{A}$ by closing it in the norm

$$\|f\| = \underset{\mathbb{P}}{\text{ess}}\text{-}\sup_{\omega \in \Omega} \|\pi_\omega f\| \quad \text{(weak closure)}$$

2) In this extended algebra, consider only the covariant observables with:

$$\int_\Omega d\mathbb{P}(\omega)\, |f(\omega, \vec{x})| < A\, e^{-\gamma |\vec{x}|}$$

for some strictly positive A and γ.

These elements form an algebra called the algebra of localized observables $\mathcal{A}_{loc}$

Natural Fredholm modules.

d = even

$$\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathrm{Cliff}(d), \quad \pi_\omega f \to \pi_\omega f \otimes \mathrm{id}$$

$$F_{\vec{x}_0} = \frac{D_{\vec{x}_0}}{|D_{\vec{x}_0}|}, \quad D_{x_0} = \sum_{i=1}^{d} (X_i + x_0^i) \cdot \gamma^i \text{ (shifted Dirac Op)}$$

$$\gamma = i^d \gamma_1 \gamma_2 \dots \gamma_d$$

The result is a family of F-M $(\mathcal{A}_{loc}, \mathcal{H}, \pi_\omega, F_{\vec{x}_0}, \gamma)$

d = odd

$$\mathcal{H} = \ell^2(\mathbb{Z}^d) \otimes \mathrm{Cliff}(d), \quad \pi_\omega \to \pi_\omega \otimes \mathrm{id}$$

$$F_{\vec{x}_0} = \frac{D}{|D|}; \quad D_{\vec{x}_0} = \sum_{i=1}^{d} (X_i + x_0^i) \cdot \sigma_i \text{ (shifted Weyl op).}$$

The result is a family of F-M $(\mathcal{A}_{loc}, \mathcal{H}, \pi_\omega, F_{\vec{x}_0})$

Theorem: The Connes-Chern form

$$\tau_d(f_0, f_1, \dots, f_n) = \Lambda_d \int_{\text{cell}} d\vec{x}_0 \int_{\Omega} d\mathbb{P}(\omega) \int \pi_\omega f_0 [\pi_\omega f_1, F_{\vec{x}_0}] \dots [\pi_\omega f_n, F_{\vec{x}_0}]$$

associated to the family of Fredholm modules,

is a cyclic cocycle whic pairs integrally with the K_* groups:

$$\langle [p], [\tau_d] \rangle = \text{Index}\{\pi_\omega^+ p \; F_{\vec{x}_0} \; \pi_\omega^- p\} \quad (d = \text{even})$$

$$\langle [v], [\tau_d] \rangle = \text{Index}\{E_{x_0} \pi_\omega v \; E_{x_0}\} \quad (d = \text{odd})$$

Furthermore, τ_d accepts the local formula:

$$\tau_d(f_0, f_1, \dots, f_n) = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \; \tilde{\int} \left\{ f_0 \prod_{i=1}^{d} \partial_{\rho_i} f_i \right\}$$

Cartan	d = 0	1	2	3	4	5	6	7	8	9	10	11	...
Complex case:													
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
Real case:													
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	...
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	...
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	...
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	...
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	...
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	...
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	...

For class A (d = even)

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$$Ch_d(P) = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \, \tilde{\mathcal{T}} \left\{ p \prod_{i=1}^{d} \partial_{\rho_i} p \right\} \qquad (\text{NC form})$$

For class AIII (d = odd)

$$\widetilde{Ch}_d(U) = \tilde{\Lambda}_d \int_{\mathbb{T}^d} \mathrm{Tr}\left\{ (U^{-1} dU)^d \right\} \qquad (k\text{-space})$$

$$\widetilde{Ch}_d(u) = \tilde{\Lambda}_d \sum_{\rho \in S_d} (-1)^\rho \, \tilde{\mathcal{T}} \left\{ \prod_{i=1}^{d} u^{-1} \partial_{\rho_i} u \right\} \qquad (\text{NC-form})$$

Computational Non-Commutative Geometry

$$H_\omega : \ell^2(\mathbb{Z}^D, \mathbb{C}^N) \to \ell^2(\mathbb{Z}^D, \mathbb{C}^N)$$

$$(H_\omega \psi)(\boldsymbol{x}) = \sum_{\boldsymbol{y}\in\mathbb{Z}^D} e^{\imath \boldsymbol{x}\wedge\boldsymbol{y}}\, \hat{t}_{\boldsymbol{x},\boldsymbol{y}}(\omega)\, \psi(\boldsymbol{y})$$

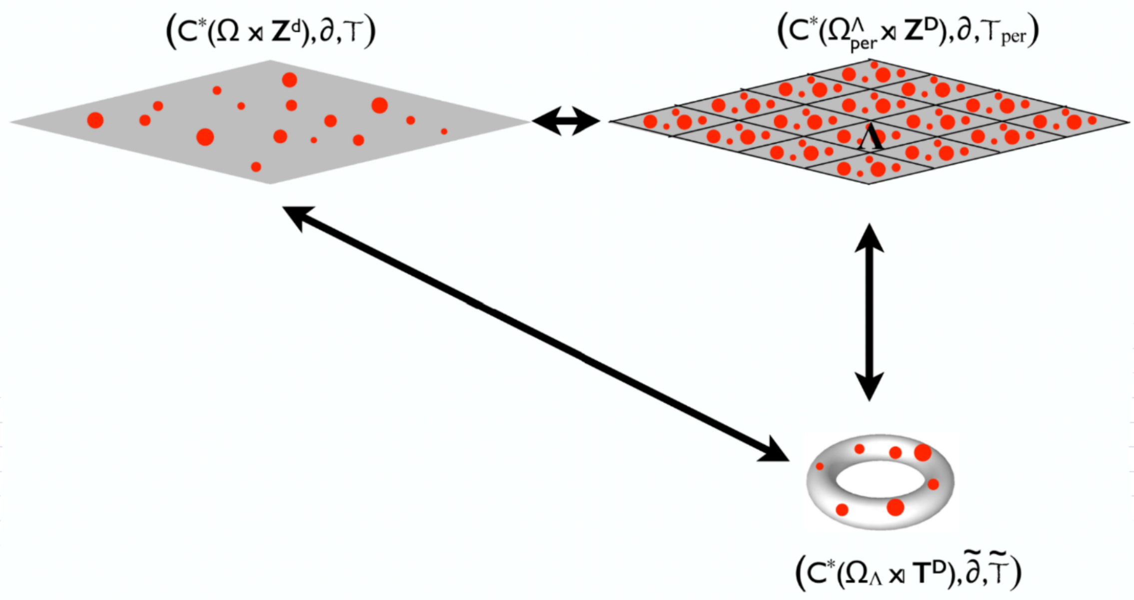
Where:

- $e^{\imath \boldsymbol{x}\wedge\boldsymbol{y}}$ is the Peierls factor due to a uniform magnetic field.
- Random hopping matrices: $\hat{t}_{\boldsymbol{x},\boldsymbol{y}}(\omega) = (1 + \lambda\, \omega_{\boldsymbol{x},\boldsymbol{y}})\hat{t}_{\boldsymbol{x}-\boldsymbol{y}}$
- Disorder configuration space: $\omega \equiv \{\omega_{\boldsymbol{x},\boldsymbol{y}}\}$, $\omega \in \Omega = [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{Z}^D} \times [-\frac{1}{2}, \frac{1}{2}]^{\mathbb{Z}^D} \times \ldots$
- Natural probability measure: $d\boldsymbol{P}(\omega) = \prod_{\boldsymbol{x},\boldsymbol{y}} d\omega_{\boldsymbol{x},\boldsymbol{y}}$
- Ergodic translations: $(\mathrm{t}_{\boldsymbol{a}}\omega)_{\boldsymbol{x},\boldsymbol{y}} = \omega_{x-\boldsymbol{a},\boldsymbol{y}-\boldsymbol{a}}$

These models:

- Can be generated from first-principle calculations or from purely empirical data.
- They can be tuned to accurately describe the physics of a material.

The concept of Approximating Noncommutative Spaces



Real-space picture:

Infinite Volume $\rightarrow$ Periodic $\longrightarrow$ Finite torus

The non-commutative calculus over the torus.

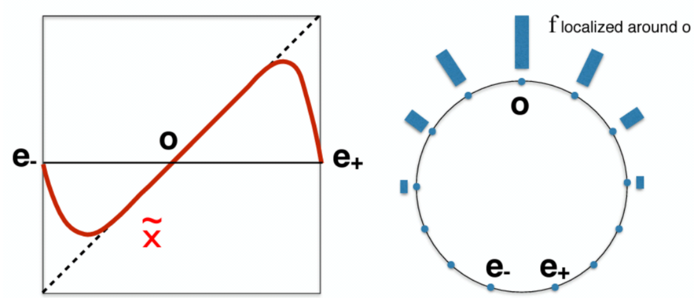
Integration:

$$\mathcal{T}_{\mathbb{T}}(\tilde{f}) = \int_{\Omega_N} d\boldsymbol{P}_{\mathbb{T}}(\omega)\tilde{f}(\omega,\boldsymbol{o}) \ \text{(a proper trace)}.$$

Derivations (approximate):

$$(\tilde{\partial}_j \tilde{f})(\omega,\boldsymbol{p}) = \imath \tilde{x}_j(\boldsymbol{p})\tilde{f}(\omega,\boldsymbol{p}),$$

where $\tilde{\boldsymbol{x}}(\boldsymbol{p})$ is an approximation of $\boldsymbol{x}_{\boldsymbol{p}}$ which is exact near $\boldsymbol{o}$ and closes continuously on $\mathbb{T}^D_N$.



Proposition Consider the discrete Fourier transform:

$$\tilde{\boldsymbol{x}}(\boldsymbol{p}) = \sum_{\lambda^{2N+1}=1} \boldsymbol{b}_\lambda \lambda^{\boldsymbol{x}_p}$$

Then:

$$\tilde{\pi}_\omega(\tilde{\boldsymbol{\nabla}}\tilde{f}) = i\sum_{\lambda^{2N+1}=1} \boldsymbol{b}_\lambda \lambda^{-\boldsymbol{X}}(\tilde{\pi}_\omega \tilde{f})\lambda^{\boldsymbol{X}}. \ \text{(this replaces } i[X,F_\omega]\text{!!)}$$

Infinite volume $\longrightarrow$ Periodic Case

Theorem. (E.P., Appl. Math. Res. Express 2013) Let h be compact supported from $C^*(\Omega \times \mathbb{Z}^D)$ and let $h_{\rm per}$ from $C^*(\Omega^{\mathcal{N}}_{\rm per} \times \mathbb{Z}^D)$ obtained by restricting h to $\Omega^{\mathcal{N}}_{\rm per}$. Then, for any analytic functions Φ_j around $\sigma(h)$:

$$\left| \mathcal{T}\left(\prod_{j=1}^{M} \partial^{\alpha_j}\Phi_j(h)\right) - \mathcal{T}_{\rm per}\left(\prod_{j=1}^{M} \partial^{\alpha_j}\Phi_j(h_{\rm per})\right)\right| \leqslant \mathfrak{B}_1(\xi,\{\boldsymbol{\alpha}\})\left(\prod_{j=1}^{M}\bar{\Phi}_j\right)\mathcal{N}^{-1}e^{-\xi\mathcal{N}}.$$

Periodic case $\longrightarrow$ Torus

Theorem. (E.P., Appl. Math. Res. Express 2013) Let h be a compactly supported Hamiltonian and $\tilde{h} = P(h_{\rm per})$. Let Φ_j $(j = 1,\ldots,M)$ be analytic functions in the neighborhood of $\sigma(h)$. Then

$$\left| \mathcal{T}_{\rm per}\left(\prod_{j=1}^{M} \partial_{\boldsymbol{\alpha}_j}\Phi_j(h_{\rm per})\right) - \mathcal{T}_{\mathbb{T}}\left(\prod_{j=1}^{M} \tilde{\partial}_{\boldsymbol{\alpha}_j}\Phi_j(\tilde{h})\right)\right| < \mathfrak{B}(\xi,\{\boldsymbol{\alpha}\})\left(\prod_{j=1}^{M}\bar{\Phi}_j\right)e^{-\xi\mathcal{N}}.$$

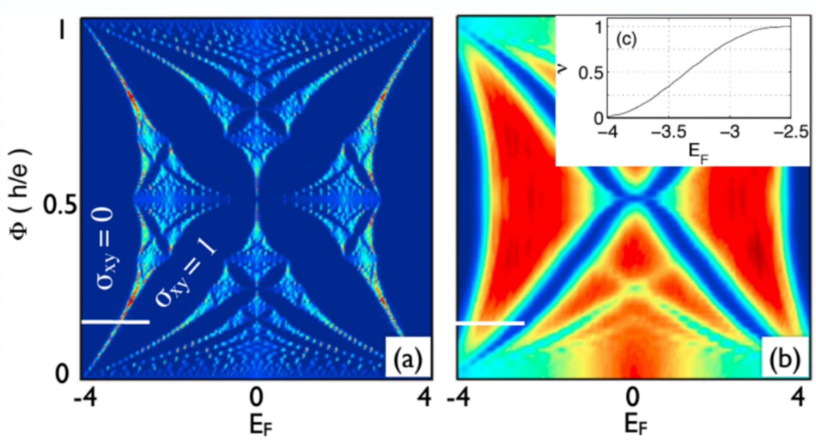
Non-commutative Kubo-Formula

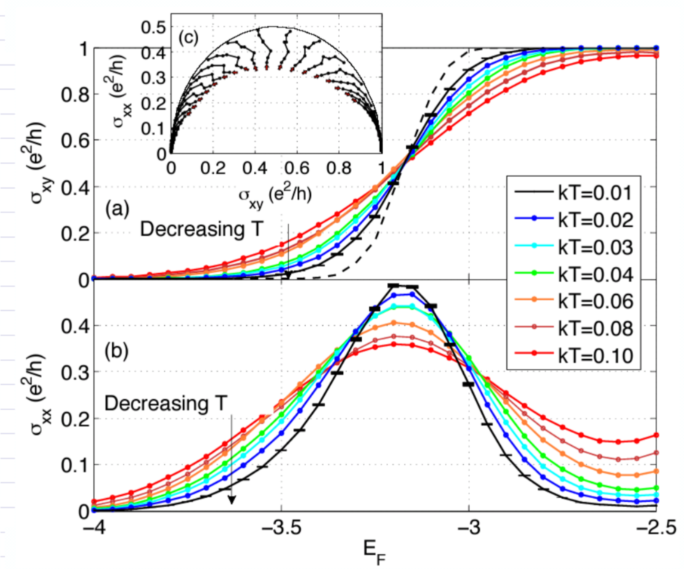
$$\sigma_{ij} = -\mathcal{T}\left((\partial_i h) * (1/\tau + \mathcal{L}_h)^{-1} * \partial_j \Phi_{FD}(h) \right)$$

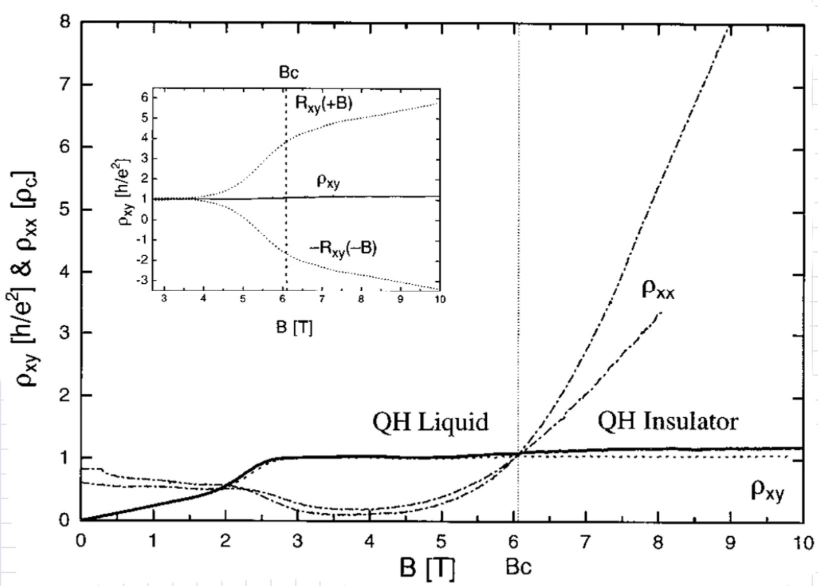
E_F	80×80	100×100	120×120	140×140	Exact
0.0	4.0339628247	4.0339630615	4.0339630708	4.0339630712	4.0339630712
-0.4	3.9394154619	3.9394154735	3.9394154621	3.9394154624	3.9394154624
-0.8	3.7040304262	3.7040301193	3.7040301310	3.7040301307	3.7040301307
-1.3	3.3684805414	3.3684801617	3.3684801517	3.3684801516	3.3684801516
-1.7	2.9522720814	2.9522713926	2.9522714007	2.9522714009	2.9522714009
-2.2	2.4678006935	2.4678005269	2.4678005093	2.4678005104	2.4678005104
-2.6	1.9239335953	1.9239338070	1.9239338090	1.9239338089	1.9239338089
-3.1	1.3274333126	1.3274333067	1.3274333084	1.3274333085	1.3274333085
-3.5	0.6854442914	0.6854442923	0.6854442923	0.6854442923	0.6854442923
-4.0	0.1086465150	0.1086465150	0.1086465150	0.1086465150	0.1086465150

σ_{11} at $1/\tau = kT = 0.1$ for a clean 2D lattice model

Song, Prodan, EuroPhys Lett 2014

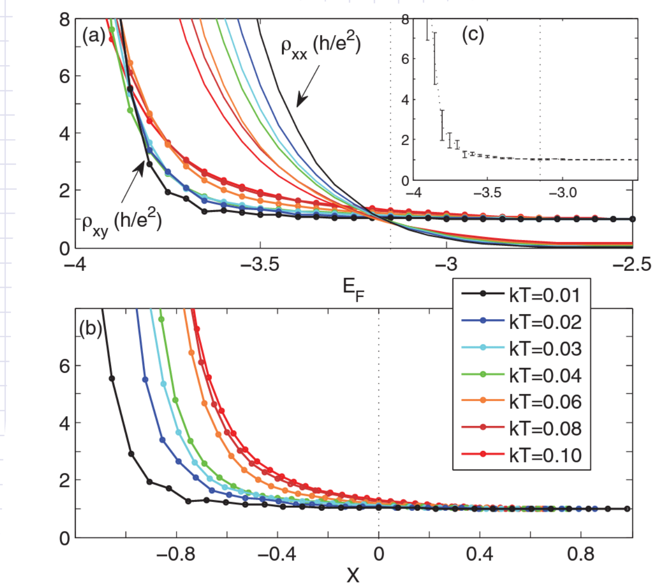




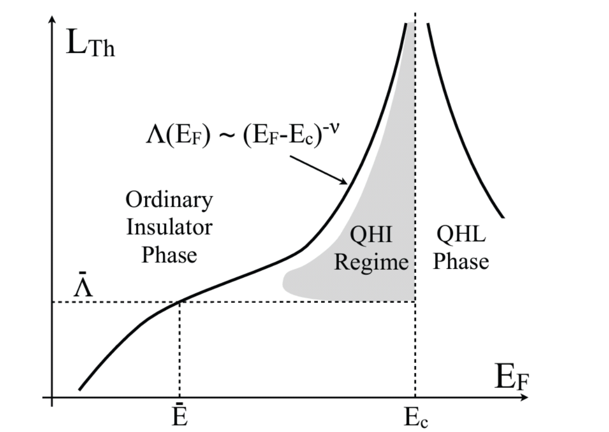


Quantized Hall Ins

Tsui & co, Nature 1998



$$\rho_{xx}(X) = e^{-X-\gamma X^3}, \quad \rho_{xy} = 1 + (T/T_1)^y \rho_{xx}(X).$$



$$X = (E_F - E_c)\left(\frac{T}{T_0}\right)^{-\kappa}$$

Science express

Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator (thin films of Cr-doped (Bi,Sb)2Te3)

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Minghua Guo,[1] Kang Li,[2] Yunbo Ou,[2] Pang Wei,[2] Li-Li Wang,[2] Zhong-Qing Ji,[2]
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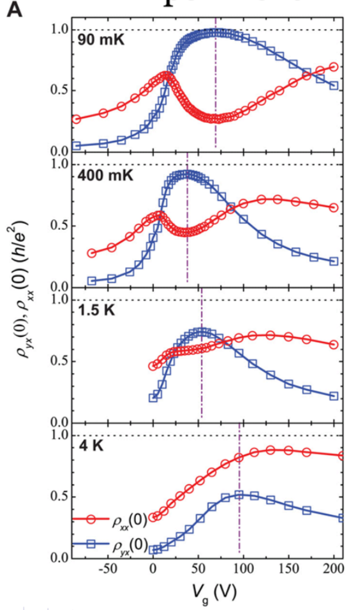
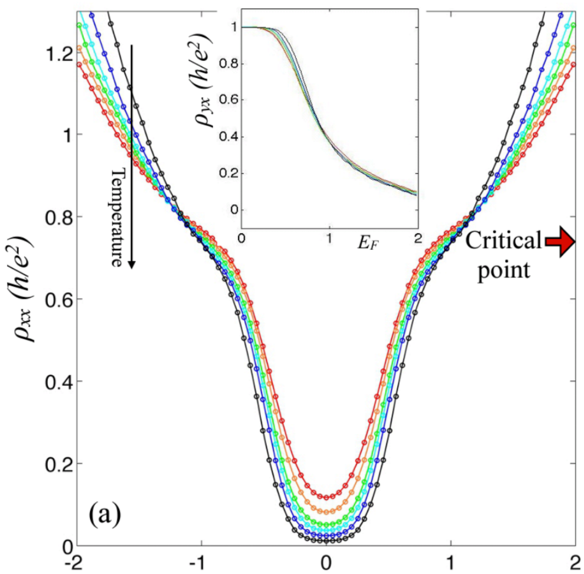
Experiment

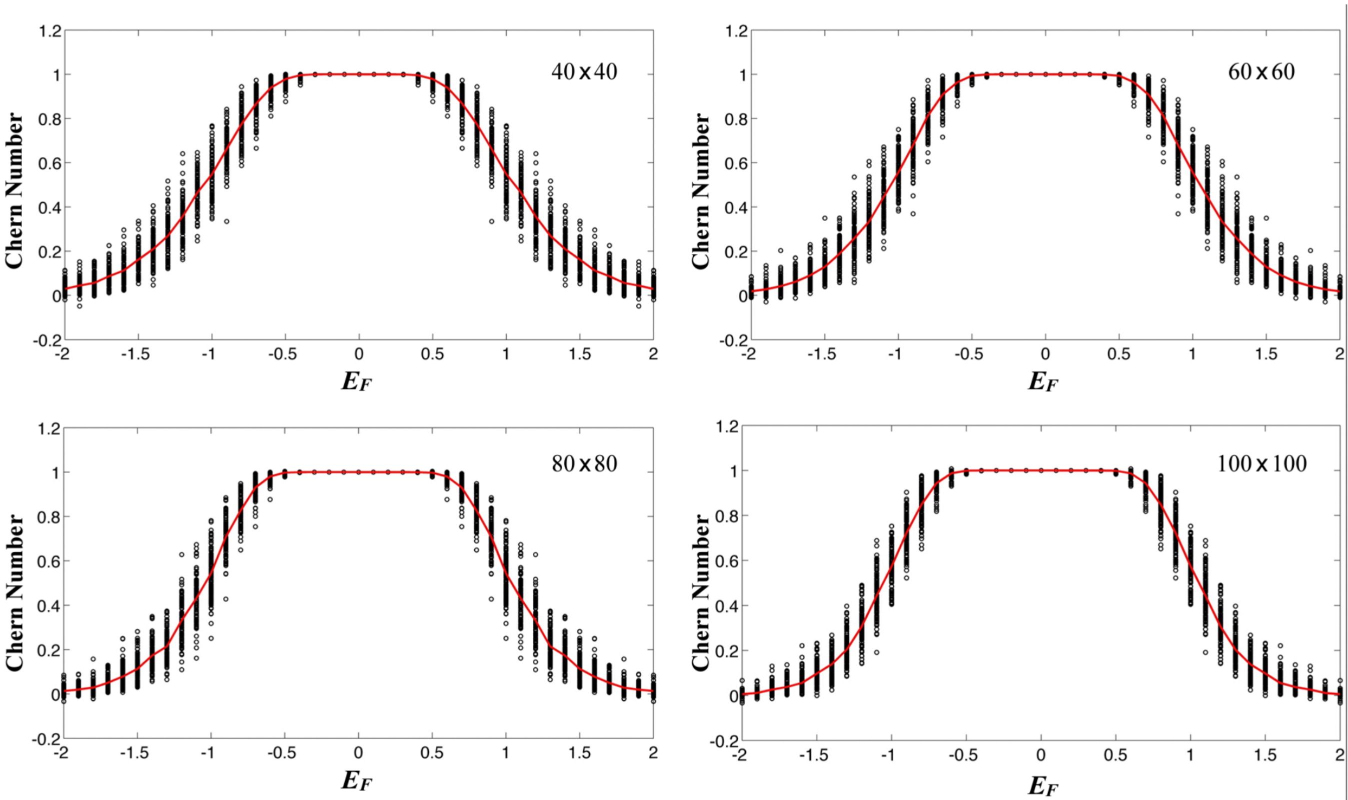


Simulation via NCG



Computation of the 1st Chern number

spin-up sector of the Kane Mele model (zero Rashba), strong disorder regime (spectral gap closed!!)

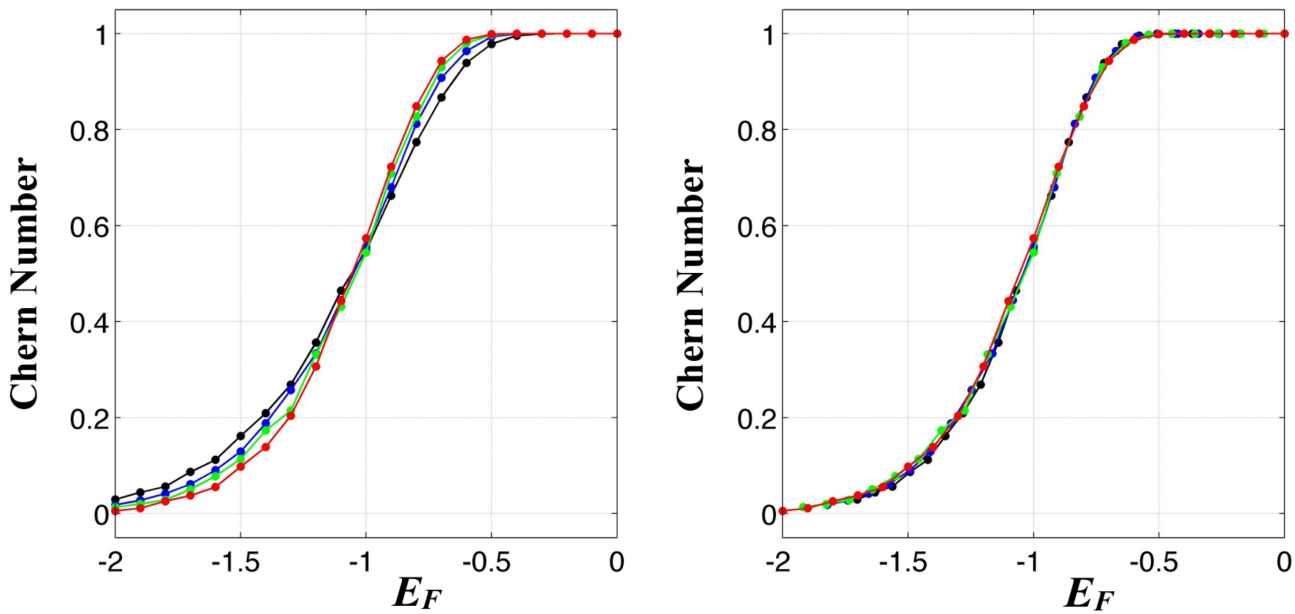


Quantization with MACHINE PRECISION

Table 1: Numerical values for average Chern numbers

Energy	40 × 40	60 × 60	80 × 80	100 × 100
-2.0000000000000000	0.0293885304649968	0.0183147848896676	0.0134785966919230	0.0055726403061233
-1.8999999999999999	0.0442301583027775	0.0274502505545331	0.0200229343621875	0.0112501012411246
-1.8000000000000000	0.0563736772645283	0.0416811880195335	0.0285382576963500	0.0259995275657507
-1.7000000000000000	0.0868202901241971	0.0612803850743208	0.0506852078002088	0.0377798251819264
-1.6000000000000001	0.1121154018269069	0.0905166860071905	0.0781754600177580	0.0554182299457663
-1.5000000000000000	0.1617580454580226	0.1291516191502659	0.1133966598848624	0.0977984662347778
-1.3999999999999999	0.2093536896403097	0.1883311262238442	0.1733092018533850	0.1386844139850113
-1.3000000000000000	0.2687556358733589	0.2575144956897765	0.2146703753513447	0.2040079233029510
-1.2000000000000000	0.3565352143319771	0.3333569482253110	0.3319133571108642	0.3066419928551302
-1.1000000000000001	0.4646789224167249	0.4444784219466996	0.4310440221933989	0.4427699238861748
-1.0000000000000000	0.5479958396159215	0.5561471440680733	0.5442615536532044	0.5738596277941682
-0.9000000000000000	0.6624275864985472	0.6798953821199148	0.7086514094234754	0.7228749266484203
-0.8000000000000000	0.7742005453064691	0.8124137607528051	0.8270271100278364	0.8487923693232788
-0.7000000000000000	0.8672349391630054	0.9079791895178040	0.9301639459675241	0.9432234611493278
-0.6000000000000000	0.9392873717233425	0.9636994770114942	0.9802652381114992	0.9872940741308633
-0.5000000000000000	0.9784417158133359	0.9935074963179980	0.9974987656403326	0.9988846769813913
-0.4000000000000000	0.9958865415757685	0.9992024708366942	0.9998527876642247	0.9999656328302596
-0.3000000000000000	0.9998184404341747	0.9999824660477071	0.9999988087144891	0.9999996457562911
-0.2000000000000000	0.9999952917010211	0.9999977443008894	0.9999999997655000	0.9999999999862120
-0.1000000000000000	0.9999999046002306	0.9999999998972079	0.9999999999998473	0.9999999999999849
0.0000000000000000	0.9999999963422543	0.9999999999988873	0.9999999999999996	0.9999999999999999

Finite-Size scaling of the Chern number near transition



The Chern lines overlap almost perfectly after a rescaling of the energy axis

$$E \to E_c + (E - E_c) * (L/L_0)^{\nu}$$

($\nu = 2.6$, as it should for the unitary class).

AIII unitary chiral class (1D)

$$H_\omega = \sum_{n\in\mathbb{Z}} \left\{ (1+W_1\,\omega_n)\left[\tfrac{1}{2} c_n^\dagger(\tau_1 + i\tau_2)\, c_{n+1} + h.c.\right] + (m + W_2\,\omega_n')\, c_n^\dagger\, \tau_2\, c_n \right\}$$

$$[H_\omega, \tau_3] = -H_\omega, \qquad E_F = 0.$$

Calculation of the localization length

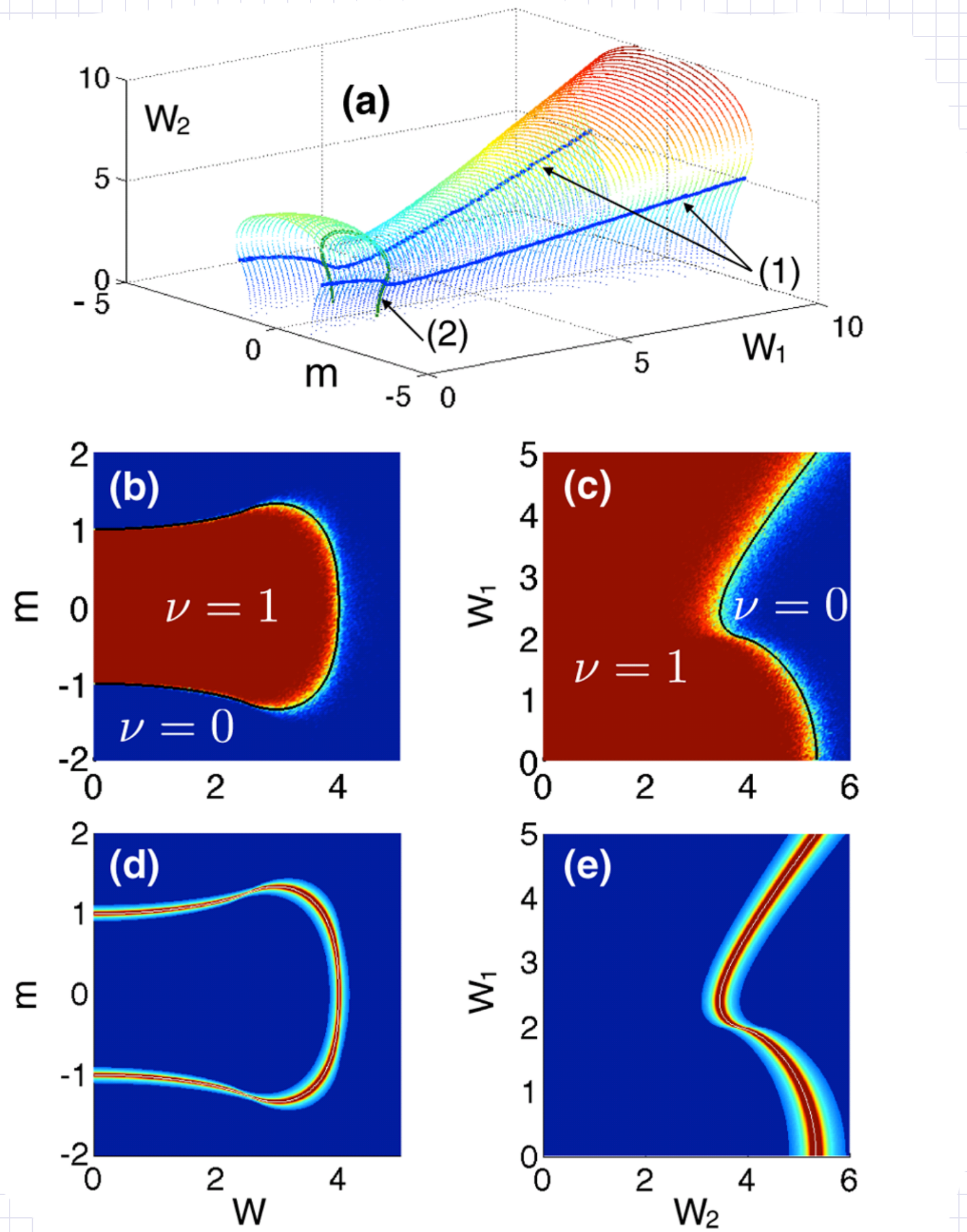
$$H_\omega \Psi = 0 \Rightarrow \quad t_n \Psi_{n-\alpha,\alpha} + i\alpha\, m_n \Psi_{n,\alpha} = 0 \qquad (\alpha = \pm 1)$$

$$\Psi_{n+\xi_\alpha,\alpha} = i^n \prod_{j=1}^{n} \left(\frac{t_j}{m_j}\right)^\alpha \Psi_{\xi_\alpha,\alpha} \qquad (\xi_\alpha = 0,1).$$

$$\Lambda^{-1} = \max_{\alpha=\pm1}\left[-\lim_{n\to\infty}\frac{1}{n}\log|\Psi_{n+\xi_\alpha,\alpha}|\right] = \left|\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} \ln|t_j| - \ln|m_j|\right|$$

$$\text{Birkhoff} \rightarrow \quad = \left|\int_{-1/2}^{1/2} d\omega \int_{-1/2}^{1/2} d\omega' \left(\ln|1+W_1\omega| - \ln|m + W_2\,\omega'|\right)\right|$$

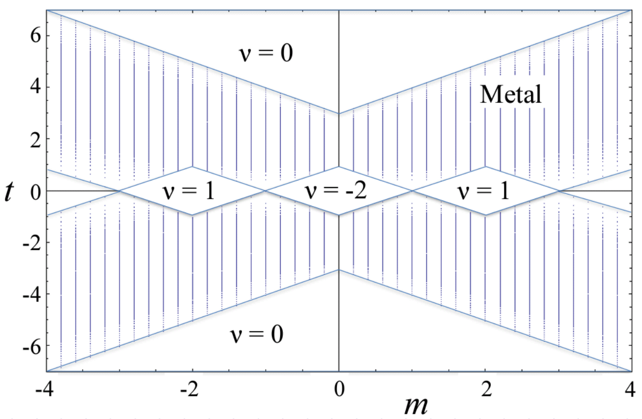
$$\Lambda^{-1} = \left|\ln\left[\frac{|2+W_1|^{1/W_1 + 1/2}\,|2m - W_2|^{m/W_2 - 1/2}}{|2-W_1|^{1/W_1 - 1/2}\,|2m + W_2|^{m/W_2 + 1/2}}\right]\right|$$

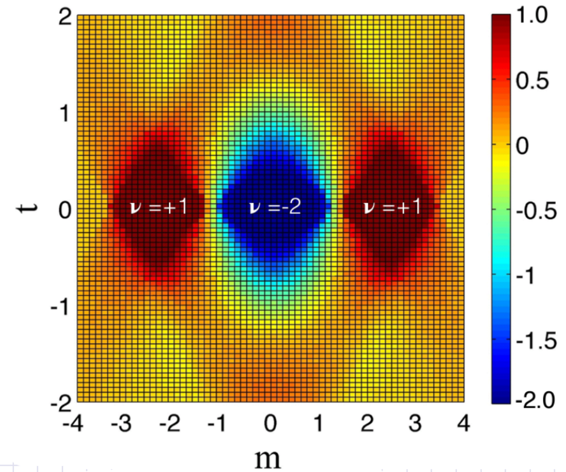

(a)
W2
W1
m
(1)
(2)
(b)
(c)
(d)
(e)
ν = 1
ν = 0
W
W2
W1
m

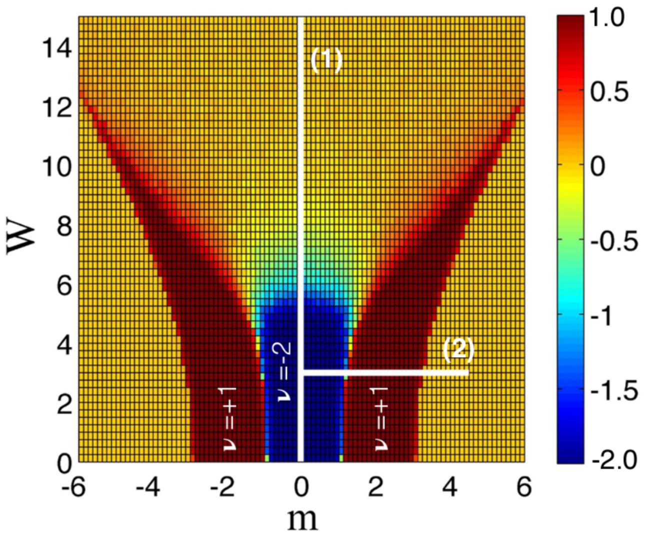
AIII unitary chiral class (3D)

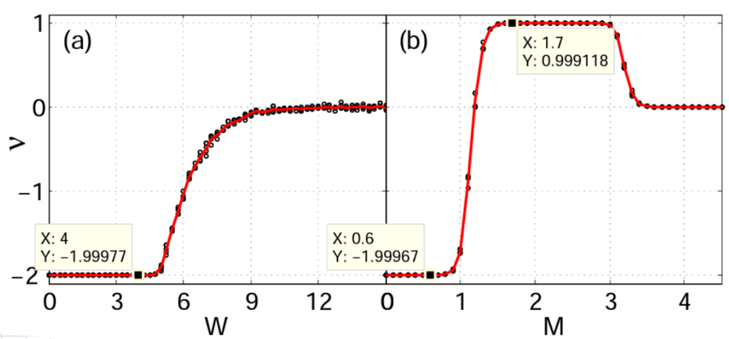
$$H_\omega \Psi_x = \frac{1}{2}\sum_{j=1}^{3}\left\{ i\Gamma_j\left(\Psi_{x-e_j} - \Psi_{x+e_j}\right) + \Gamma_4\left(\Psi_{x+e_j}+\Psi_{x-e_j}\right)\right\}$$

$$+\left[(m + W\omega_x)\Gamma_4 + it\,\Gamma_1\Gamma_3\Gamma_4\right]\Psi_x$$









list of useful non-commutative formulas

► **Chern Numbers** (Bellissard et al, JMP 1994; EP, Leung and Bellissard 2013, EP and Schulz-Baldes (2014):

$$C_D^{even}(p) = \Lambda_D \sum_{\sigma\in S_D} (-1)^\sigma \mathcal{T}\big(p \prod_{i=1}^{D} \partial_{\sigma_i} p\big), \quad C_D^{odd}(u) = \Lambda_D \sum_{\sigma\in S_D} (-1)^\sigma \mathcal{T}\big(\prod_{i=1}^{D} u^* \partial_{\sigma_i} u\big) \tag{1}$$

► **Spin-Chern number** (EP, PRB 2009): $(p_\pm = \chi_\pm(p s_z p))$

$$C_s = 1/2(C_+ - C_-),\ C_\pm = 2\pi i\, \mathcal{T}\left(p_\pm\left[\partial_1 p_\pm, \partial_2 p_\pm\right]\right). \tag{2}$$

► **Kubo-formula** (Schulz-Baldes & Bellissard in 1990's):

$$\sigma_{ij} = -\mathcal{T}\left((\partial_i h) * (\Gamma + \mathcal{L}_h)^{-1} \partial_j \Phi_{\mathrm{FD}}(h)\right). \tag{3}$$

► **Electric polarization** (Schulz-Baldes and Teufel in Comm. Math. Phys. 2012):

$$\Delta \boldsymbol{P} = \int_0^T dt\, \mathcal{T}\left(p(t)[\partial_t p(t), \nabla p(t)]\right) \tag{4}$$

► **Orbital magnetization** (Schulz-Baldes and Teufel in Comm. Math. Phys. 2012):

$$M_j = \frac{i}{2}\mathcal{T}\left(|h - \epsilon_F|[\partial_{j+1}p, \partial_{j+2}p]\right) \tag{5}$$

► **Magneto-Electric Response in 3D** (Leung and EP in JPA 2013):

$$\Delta\alpha = 1/2 \int dt \sum_{\sigma\in S_4} (-1)^\sigma \mathcal{T}\big(p \prod_{i=1}^{4} \partial_{\sigma_i} p\big) \tag{6}$$