Layer construction of three-dimensional topological states and String-String braiding statistics

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Outline

Part I

- 2D topological states and layer construction
- Generalization to 3D: a simplest example
- Layer construction of 3D topological states: general setting and examples
- Field theory description

Part II

- Some general results on string-string braiding.

- Ref: Chao-Ming Jian & XLQ, arXiv:1405.6688

(arriving here tonight…)

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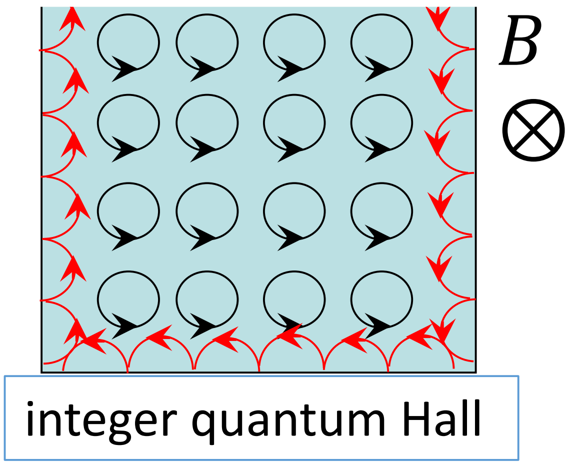
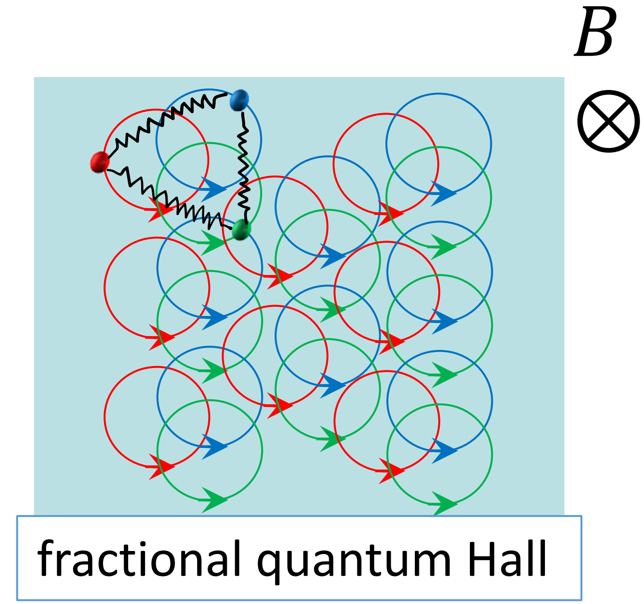
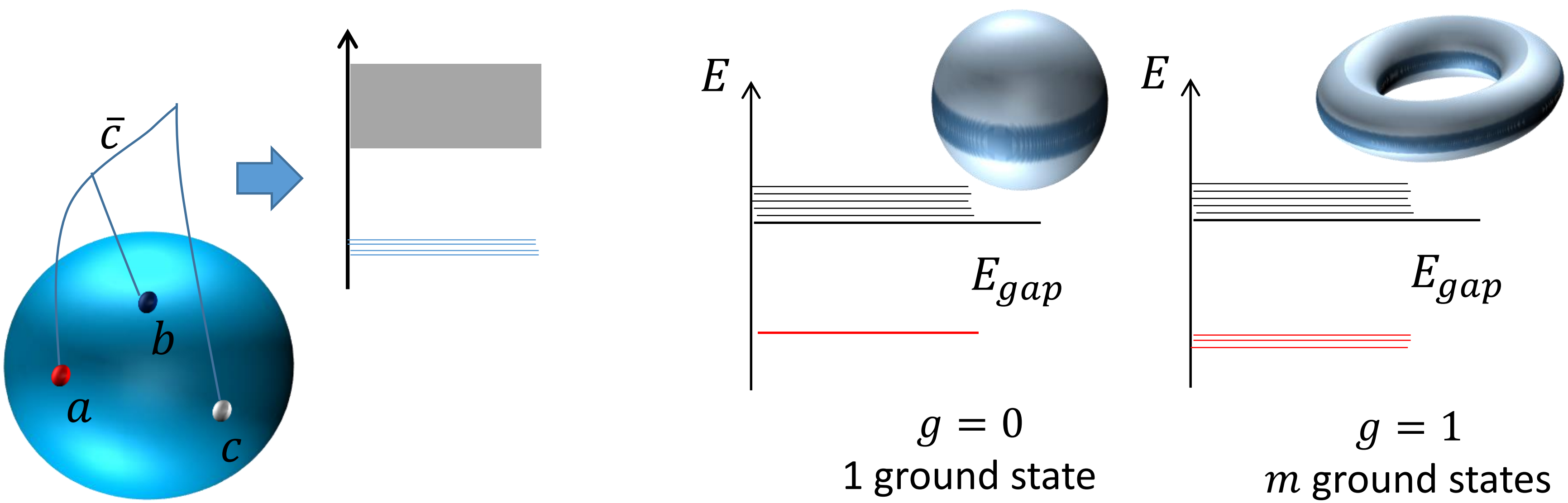
Topologically ordered states



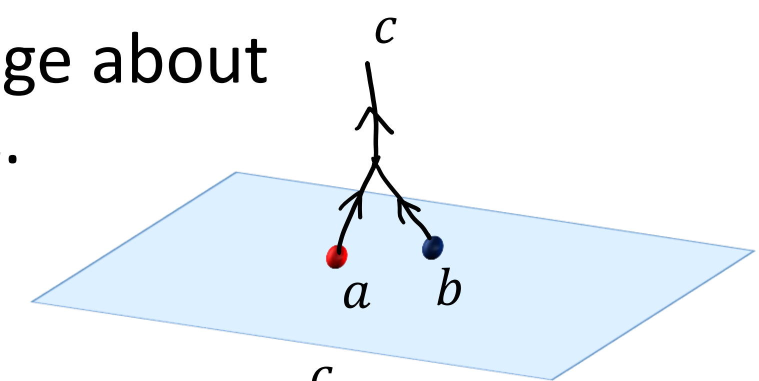


- Topological ground state degeneracy; quasiparticles with fractional quantum numbers and fractional statistics.

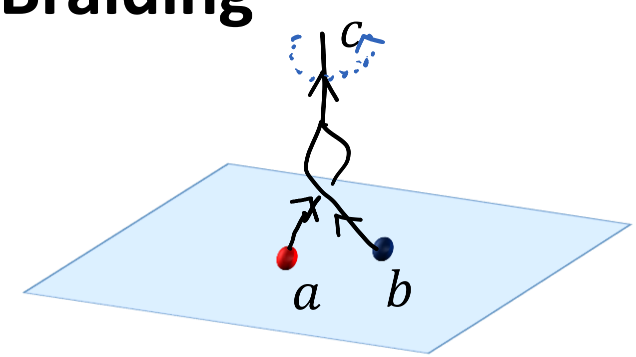


Key properties of topologically ordered states

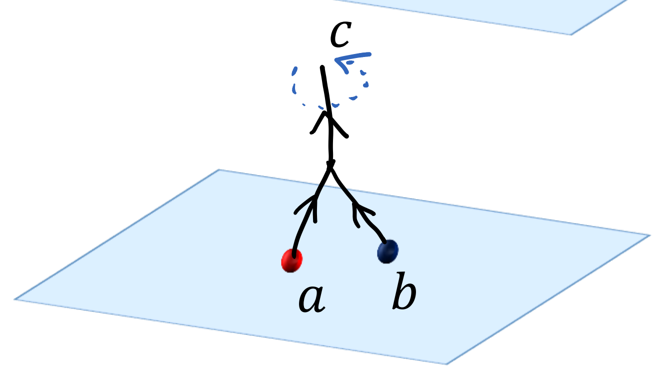
- Quasiparticles have no knowledge about distance. Only topology matters.



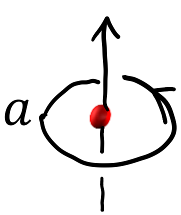
- **Fusion** $a \times b = N_{ab}^{c} c$
- **Braiding**



$= R_{ab}^{c} \cdot$



- Braiding a, b and spinning a, b is equivalent to spinning c. **Topological spin** of particles h_a

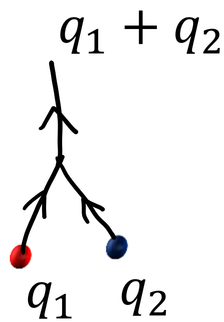


$= e^{i2\pi h_a}$

$$R_{ab}^{c} R_{ba}^{c} = e^{i2\pi(h_a + h_b - h_c)}$$

Examples of topologically ordered states

- 1. Laughlin state $\Psi(\{z_i\}) = \prod_{i<j}(z_i - z_j)^m e^{-\sum_i |z_i|^2}$
- Quasiparticles labeled by $q = 0, \frac{1}{m}, \frac{2}{m}, \ldots, 1 - \frac{1}{m}$
- Fusion rule

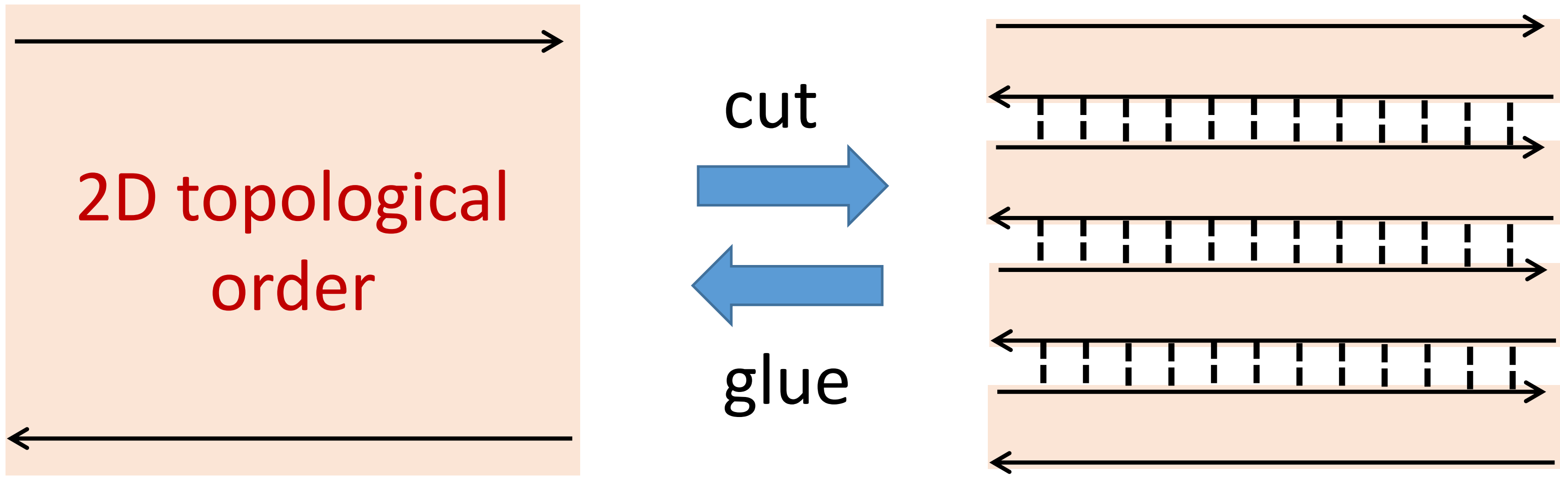


braiding $R^{q_1+q_2}_{q_1 q_2} = \exp\left[i\pi \frac{q_1 q_2}{m}\right]$

- Spin $h_q = \frac{q^2}{2m}$
- 2. Z_2 gauge theory (toric code)
- Quasiparticles include charge e, flux m and their boundstate $\psi = e \times m$.
- Nontrivial braiding $R^{\psi}_{em} = i$

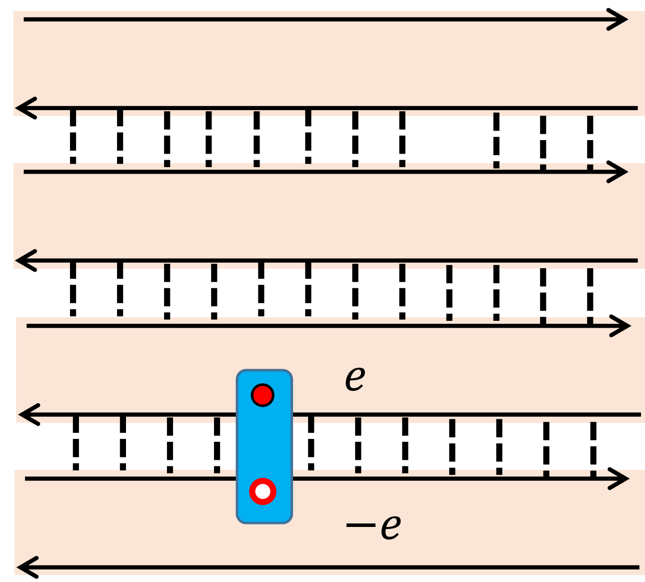
- Goal of this work: understanding 3D topological states from 2D ones

Part I: Layer construction



- 2D topological states can be constructed from coupled 1D chains (Sondhi&Yang '01, Kane et al '02, Teo&Kane, '10)
- Weakly coupled chains as a controlled limit that can realize these topological states.
- Both integer and fractional quantum Hall states can be realized.

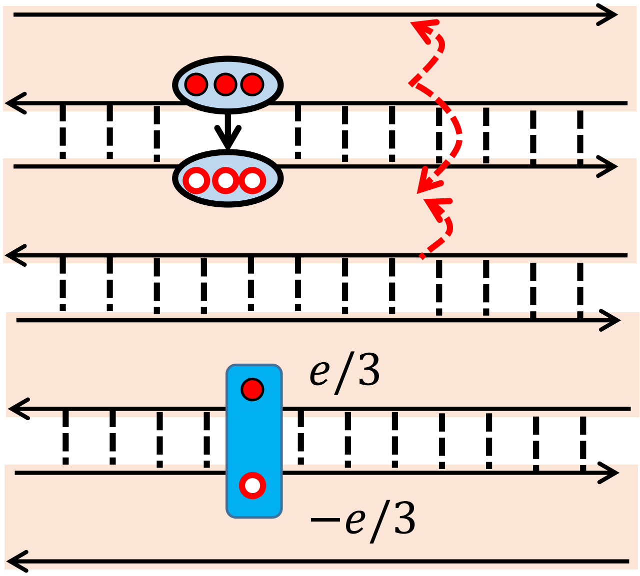
Layer construction of 2D topological states



- Example 1: integer quantum Hall (Sondhi&Yang '01)
- Electron tunneling between edge states of each strip: $\langle c_{nL}^{+} c_{n+1,R} \rangle \neq 0$,
- Electron tunneling can be equivalently viewed as exciton condensation
- Condensation of the exciton (particle-hole pair) leads to coherent tunneling between quasi-1D strips
- The strips are glued to a quantum Hall state

Layer construction of 2D topological states

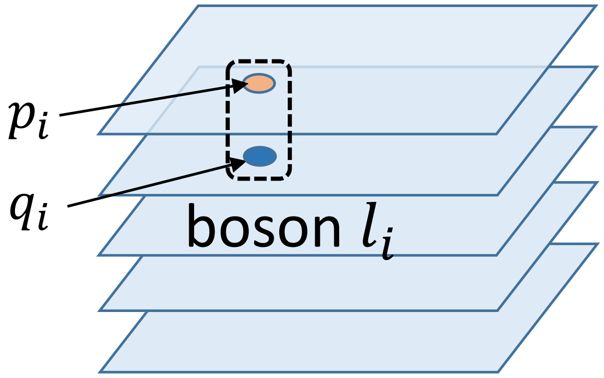
- Example 2: Laughlin 1/3 state
  (Kane et al '02)
- Electron tunneling between ν =1/3 edges of chiral Luttinger liquids $e = 3 \times \frac{e}{3}$,
  $c_{nL}^{+} c_{n+1,R} = \left(e^{i(\phi_{nL} - \phi_{n+1,R})} \right)^3$,



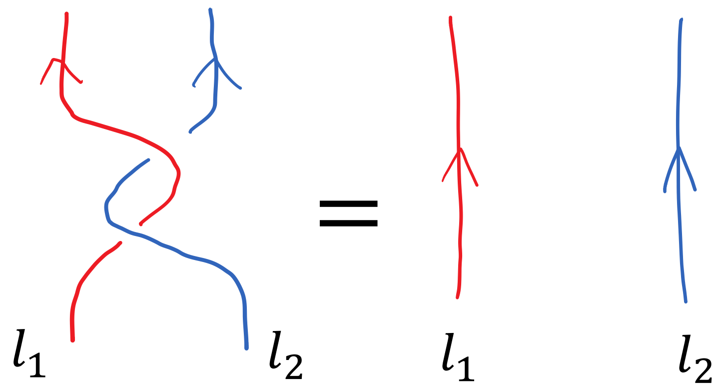
- Electron tunneling effectively generates coherent quasiparticle tunneling→ 2D topological order.
- The coherent tunneling can be understood as a "boson condensation" of the quasiparticle exciton with charge $\left(\frac{e}{3}, -\frac{e}{3} \right)$

Generalization of the layer construction to 3D

- General principle: Inter-layer coupling by boson condensation



Wang&Senthil '2013



- Abelian states: Chern-Simons theory and K matrix (Wen)

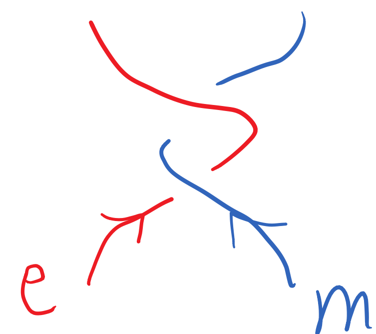
$$\mathcal{L} = \frac{1}{4\pi} K^{IJ} a_{I\mu} \epsilon^{\mu\nu\tau} \partial_\nu a_{J\tau} - l^I a_{I\mu} j^\mu$$

- Quasiparticles labeled by integer vectors l
- Equation of motion $j^\mu l^I = \frac{1}{2\pi} K^{IJ} \epsilon^{\mu\nu\tau} \partial_\nu a_{J\tau}$
- A quasiparticle carries flux $\nabla \times a^{\mathrm{I}} = 2\pi (K^{-1} l)^{\mathrm{I}}$

Examples of K-matrix theory

- Mutual statistics of l_1, l_2 given by $\theta_{12} = 2\pi l_1^T K^{-1} l_2$
- Local particles given by $\lambda = Kl$ (bosons or fermions)

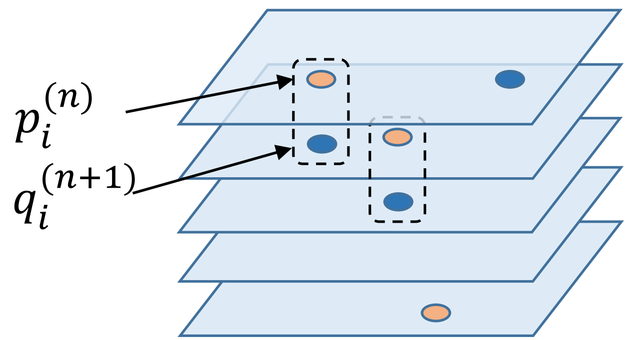
- Examples:
- Laughlin $1/m$ state $K = m$. Quasiparticle braiding $\theta_{12} = \frac{2\pi q_1 q_2}{m}$. Local particle (electron) $q = m$



- Z_N gauge theory $K = \begin{pmatrix} 0 & N \\ N & 0 \end{pmatrix}$
- Charge $e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Quasiparticle braiding $\theta_{em} = 2\pi [K^{-1}]_{12} = \frac{2\pi}{N}$

General setting of the layer construction

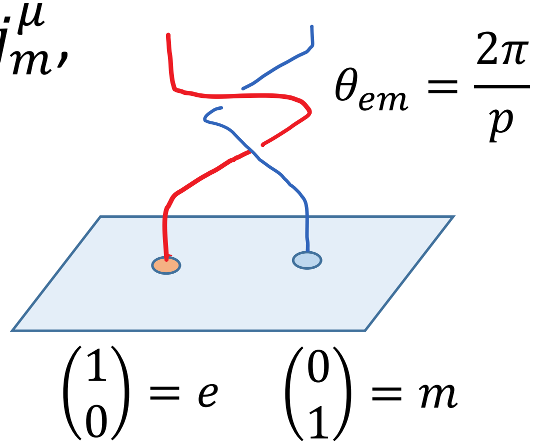
- L layers of 2D Abelian states, each with a K matrix
- Find quasiparticles p_i, q_i in each layer, so that the bound state are bosonic and mutually bosonic.
- In 2D language,
- Requirements $p_i^T K^{-1} p_j + q_i^T K^{-1} q_j = 0$, $p_i^T K^{-1} q_j = 0$.

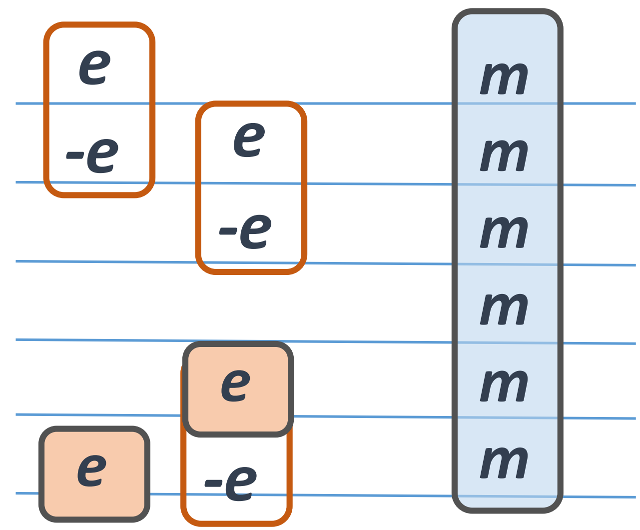


- Number of condensed particles: $i = 1,2,\ldots,N$ when $\dim K = 2N$.
- This is an "almost complete" set of *null vectors*. (Haldane '95, Levin '13, Barkeshli et al '13) There may be remaining particles, responsible for the topological order.
- With open boundary, q_i at top surface is always deconfined.

Example 1: 3D Z_p gauge theories

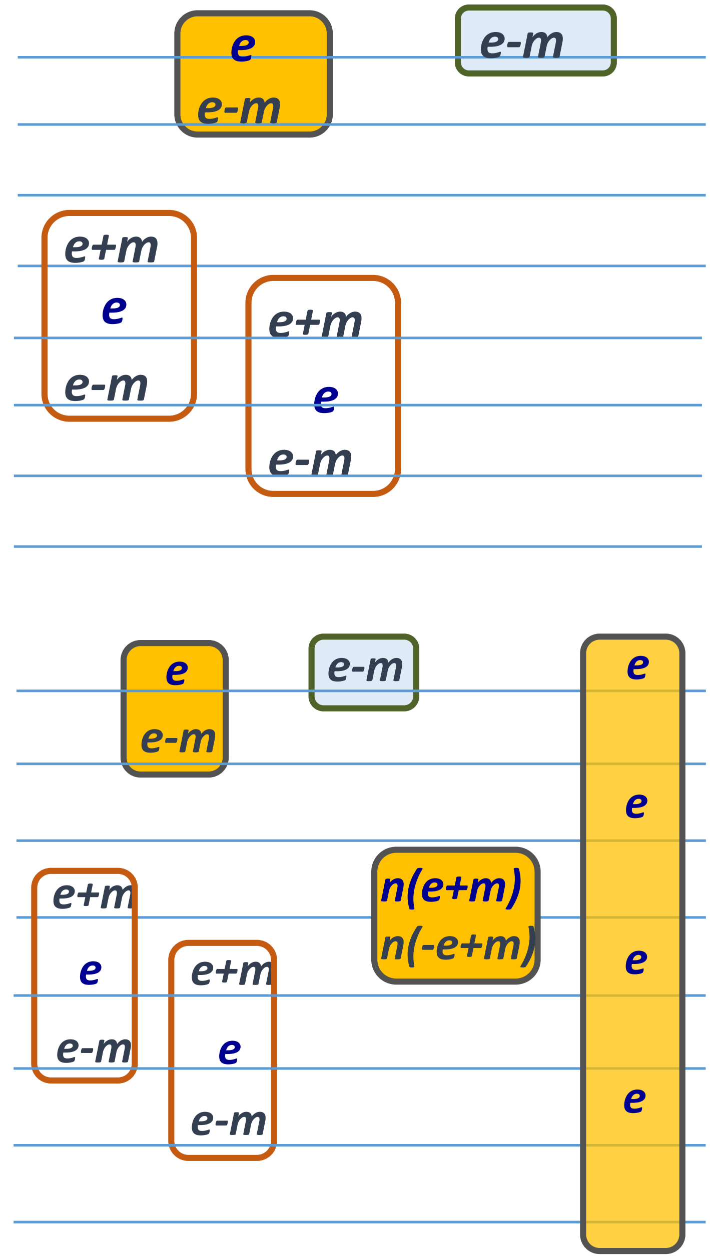
- Starting from layers of 2D Z_p gauge theories
  $K=\begin{bmatrix}0 & p\\ p & 0\end{bmatrix}, \mathcal{L}=\frac{p}{2\pi}a_\mu\epsilon^{\mu\nu\tau}\partial_\nu b_\tau + a_\mu j_e^\mu + b_\mu j_m^\mu,$
- Coupling the neighbor layers by condensation of $\binom{e}{-e}$ pair
- $p=\binom{1}{0}, q=\binom{-1}{0}$
- Particles with nontrivial braiding with the condensed particle are confined.
- Particles different by a condensed particle are identified
- Deconfined particles: e in 3D, and m string (flux tube)➔ 3D Z_p gauge theory



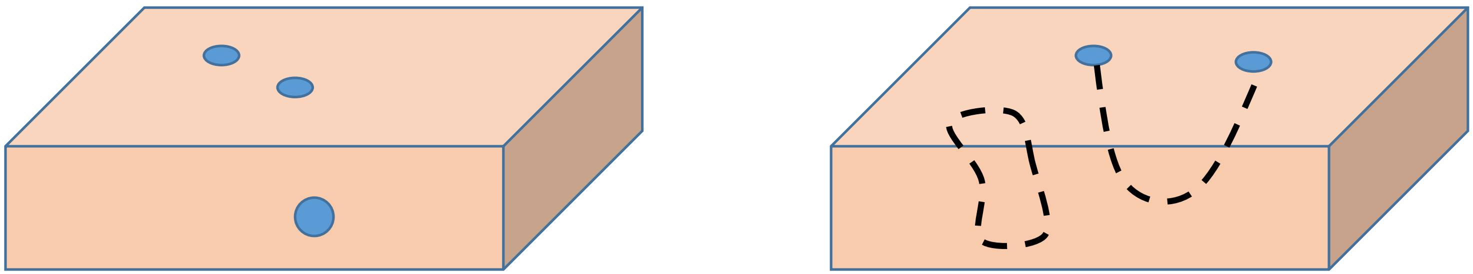


Example 2: Surface and bulk topological order

- Z_p toric code with tri-layer coupling
- A variation of the construction in Wang&Senthil '13
- $p \neq 3n$
  All bulk particles are confined.
  purely 2D topological order
- Surface central charge $c = 4$ for $p = 3n - 1$. ($p = 2$: surface theory of a 3D bosonic TI Vishwanath&Senthil '13)
- $p = 3n$
  Bulk deconfined particles coexisting with surface particles. Z_3 bulk topological order
- Surface central charge $c = 2$



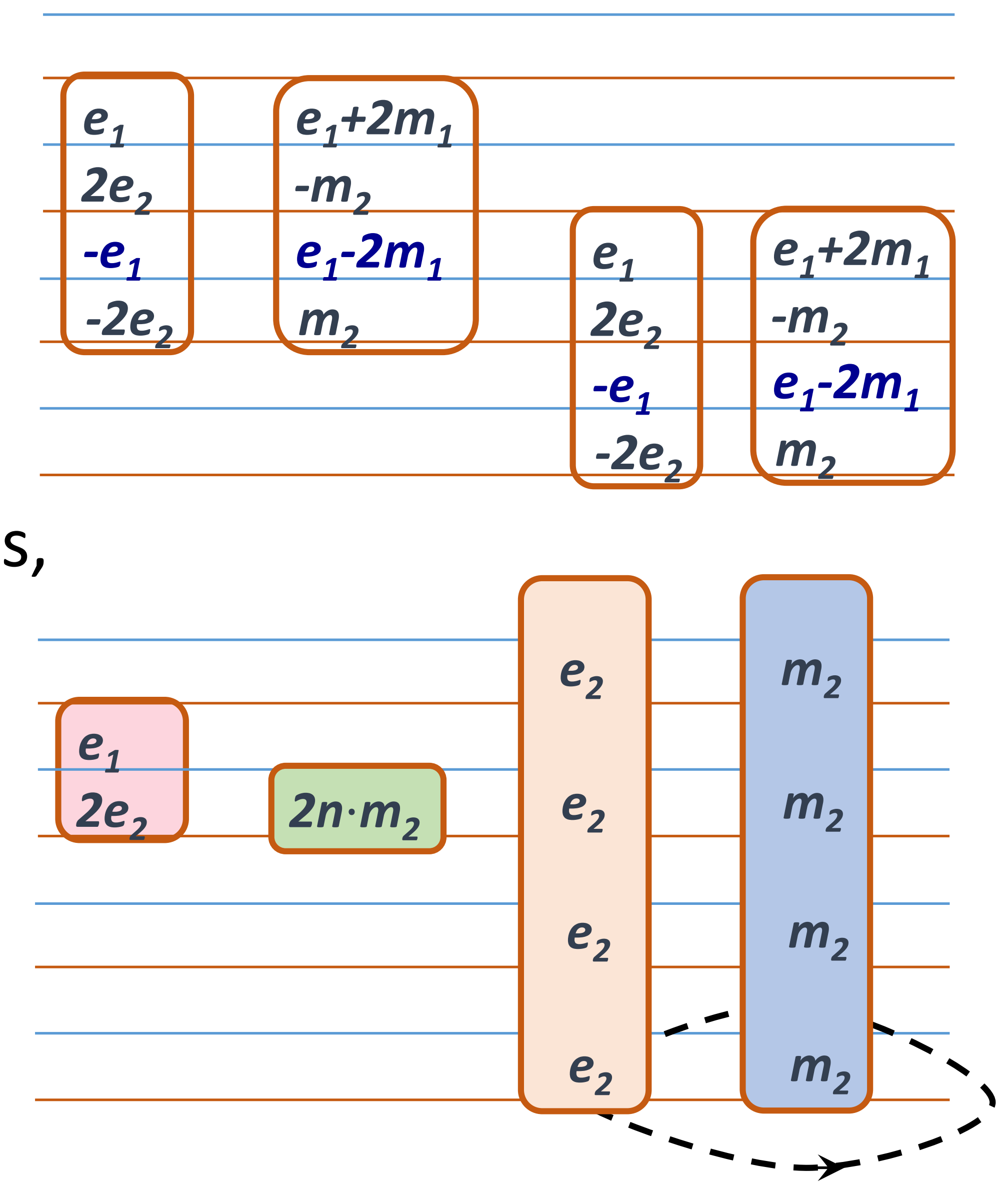
General criteria of surface-only topological order



- p_i, q_i expand all quasiparticles in a layer➔ $\binom{p_i}{q_i}$ condensation leads to surface-only topological order. Surface particles are q_i at top surface, p_i at bottom surface
- Surface K matrix $K_S = \left[q_i^T K^{-1} q_j\right]^{-1}$
- The same topological order at the side surfaces
- Bulk has nontrivial particle when $\{p_i\} \cap \{q_i\} \neq \phi$
- Relation to Walker-Wang model (K Walker & Z Wang, '12):
  modular tensor category ➔ Surface-only topological order
  Pre-modular tensor category➔ Bulk nontrivial topological order

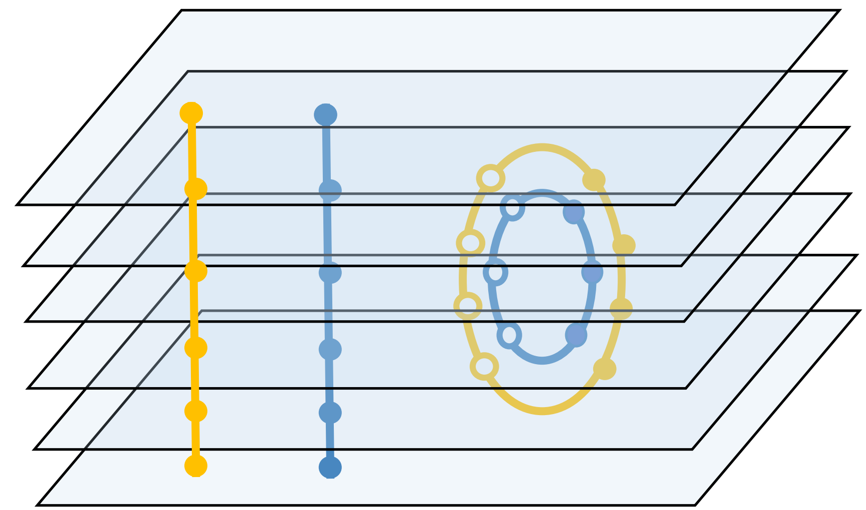
Example 3: String-String braiding

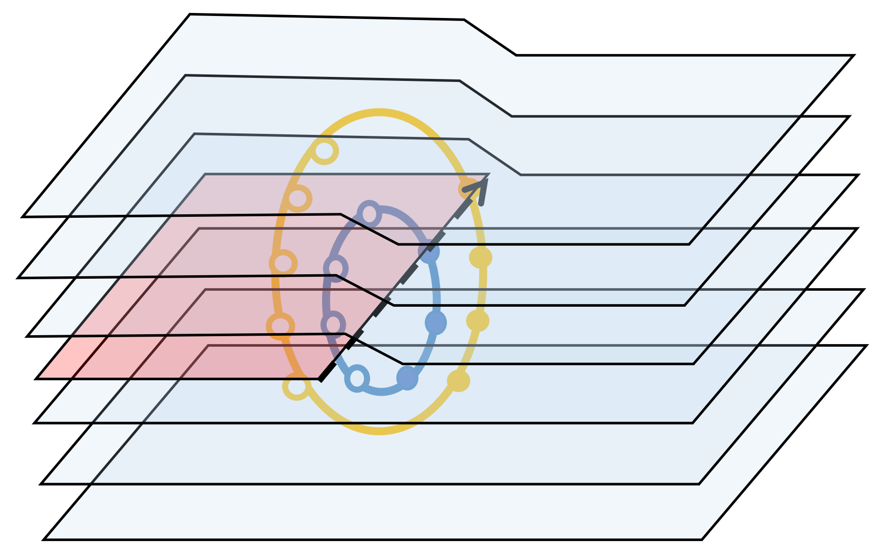
- Z_{4n} toric code theories with 4-layer coupling
- Condensed particles: hybridization of the red and blue layers
- Bulk deconfined particles: 2 point particles, 2 strings
- String-particle braiding
- String-string braiding phase $\omega_{em} = \frac{2\pi L}{4n}$ proportional to the number of layers



String-String braiding and dislocations

- Strings wraping around z direction have braiding proportional to system size
- Contractible strings have trivial braiding
- The more fundamental process of string braiding can be defined at presence of an edge dislocation
- Braiding at presence of the dislocation $\omega_{em}^{d} = \frac{2\pi}{4n} b_z$, proportional to the Burgers vector b_z

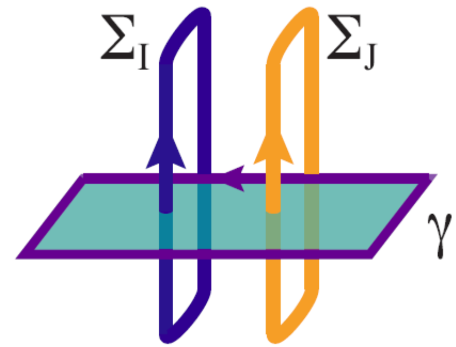




Topological field theory description

- A generalized BF theory can be written down to characterize the string-particle braiding and string-string braiding

$$\mathcal{L} = \frac{Q_{IJ}}{2\pi}\epsilon^{\mu\nu\sigma\tau} b^I_{\mu\nu}\partial_\sigma a^J_\tau + \frac{\Theta}{8\pi^2} R_{IJ}\epsilon^{\mu\nu\sigma\tau}\partial_\mu a^I_\nu \partial_\lambda a^J_\sigma + j^I_\mu a^\mu_I + J^I_{\mu\nu} b^{\mu\nu}_I$$



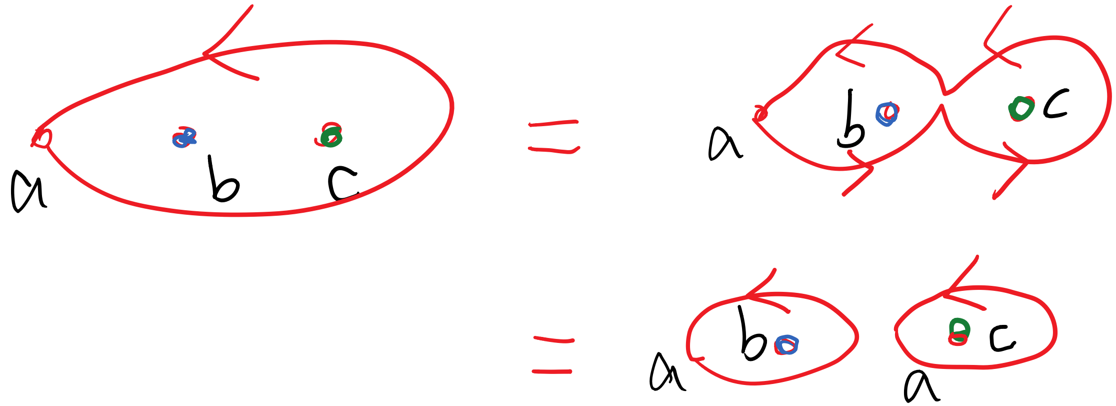
- j^I_μ: particle current; $J^I_{\mu\nu}$: string current
- Q_{IJ}: string-particle braiding
- R_{IJ}: string-string braiding when strings are linked with Θ vortex loop.
- Difference from BF theory for TI (Cho&Moore '11, Vishwanath&Senthil '12, Keyserlingk et al '13): Θ is a dynamical field
- Winding number $2\pi n$ of Θ ➔ Chern-Simons term of a with $K = nR$. ➔ String braiding $\omega^n_{IJ} = 2\pi n\left(Q^{-1\,T} R Q^{-1}\right)_{IJ}$

Topological field theory description

- Ordinary Z_p gauge theory: $Q = p, R = 0$
- Example 3: $Q = \begin{pmatrix} 2n & 0 \\ 0 & 2 \end{pmatrix}, R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- General structure of string braiding: two strings braid nontrivially only if they are not contractible.
- Consistent with other recent works on 3-string braiding (Wang&Levin 1403.7435, Jiang et al 1404.1062, Wang&Wen 1404.7854, Moradi&Wen 1404.4618)
- The dislocation is described by a Θ vortex string, which is an extrinsic defect.
- Intrinsic 3-string braiding can possibly be realized by deconfinement of the dislocations.

Part II: General results on string-string braiding

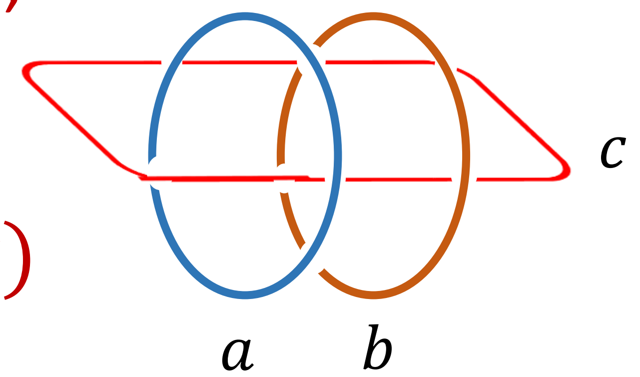
- General structure of 3D topologically ordered states are not understood yet.
- In 2D, we know the braiding phase R^{d}_{ab} is not arbitrary. There are some identities satisfied by braiding and fusion, such as the hexagon identity.



- In 3D, some similar identities may exist as a property of the general structure of topologically ordered states

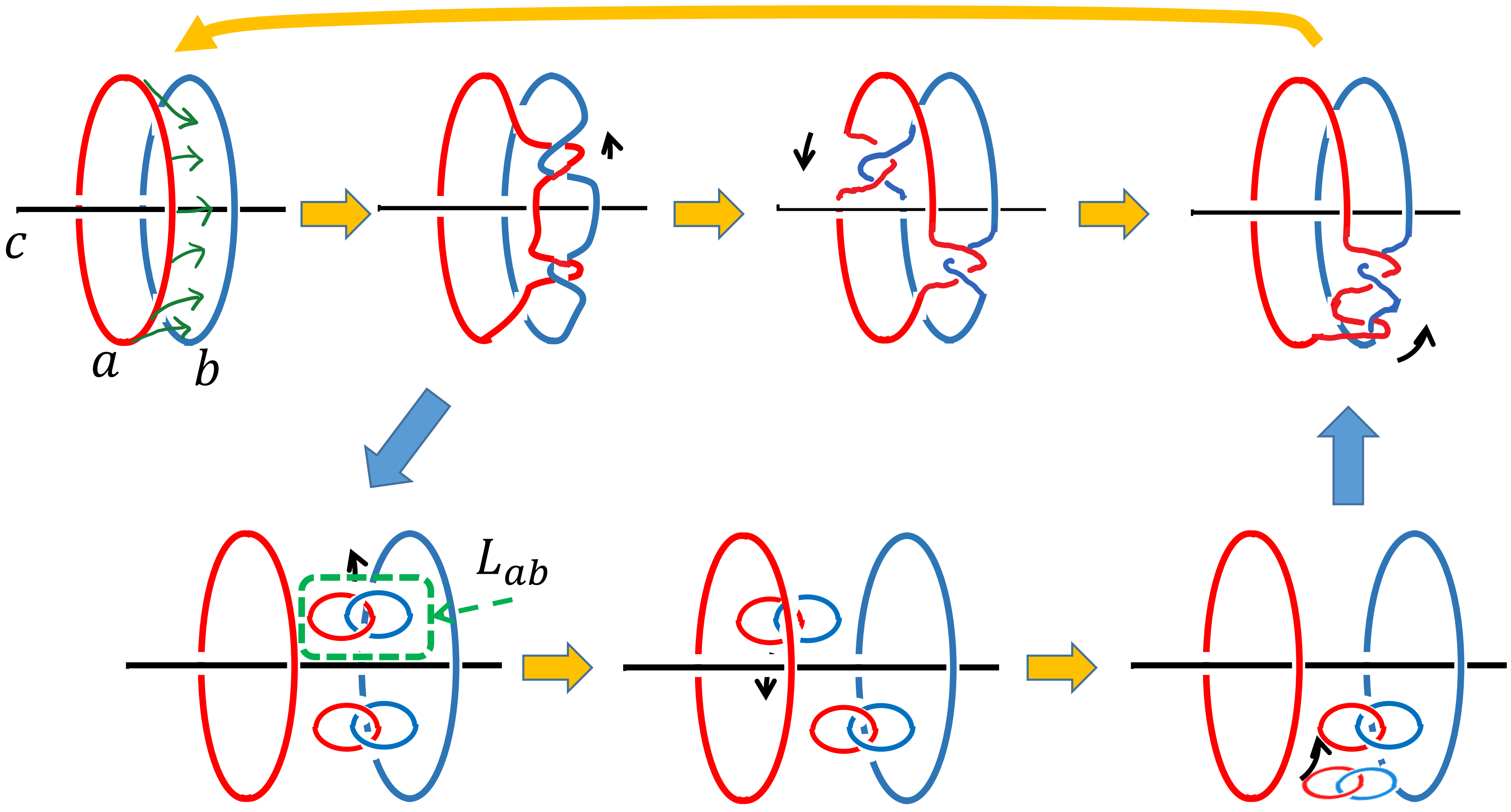
General results on string-string braiding

- Wang&Levin 1403.7435 proposed an identity of the 3-string braiding in twisted Z_p gauge theories
  $p(\omega_{ab}^{c} + \omega_{bc}^{a} + \omega_{ca}^{b}) = 0 \ (\mathrm{mod}\ 2\pi)$,
- Here we give a more general proof to a stronger identity
  $\omega_{ab}^{c} + \omega_{bc}^{a} + \omega_{ca}^{b} = 0 \ (\mathrm{mod}\ 2\pi)$
  with the general conditions
  1) Strings can fuse and split without additional phase;
  2) Strings are Abelian;
  3) Strings are not marked.



Step 1 of the proof: $\omega_{ab}^{c} = \Omega_{ab}^{c}$

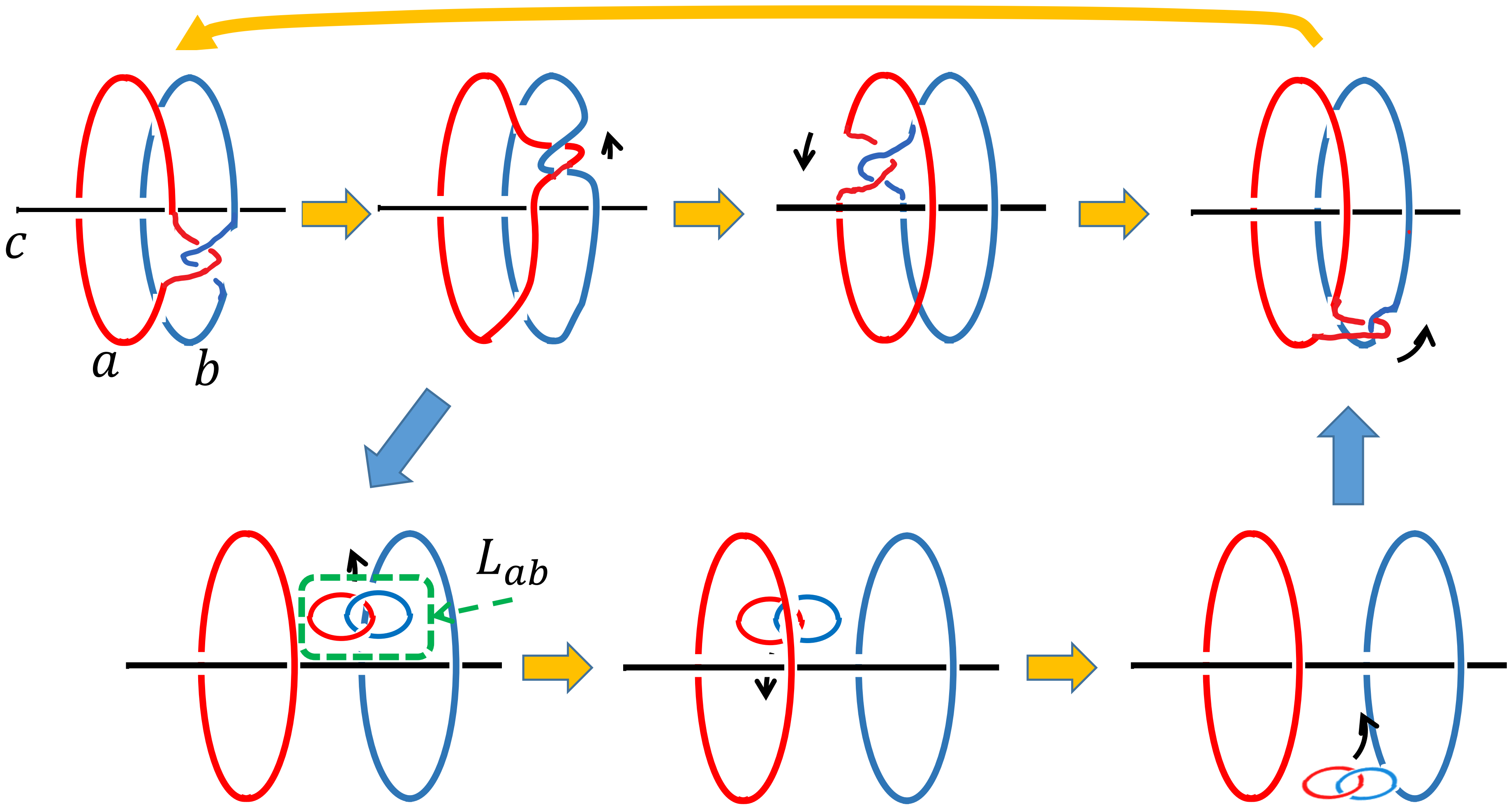
String braiding ω_{ab}^{c}



String-particle braiding Ω_{ab}^{c} between link of a, b and string c

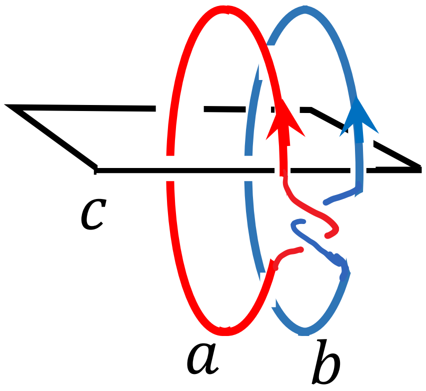
Step 2 of the proof: $\widetilde{\omega}_{ab}^{c} = \Omega_{ab}^{c}$

"linked" string braiding $\widetilde{\omega}_{ab}^{c}$, for 3 mutually-linked strings



String-particle braiding Ω_{ab}^{c} between link of a, b and string c

Step 3 of the proof: $\widetilde{\omega}^{c}_{ab} + \widetilde{\omega}^{a}_{bc} + \widetilde{\omega}^{b}_{ca} = 0$



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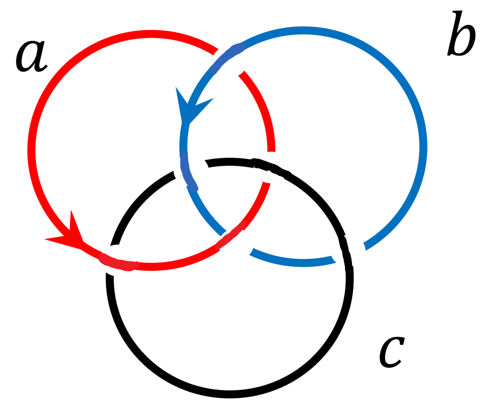
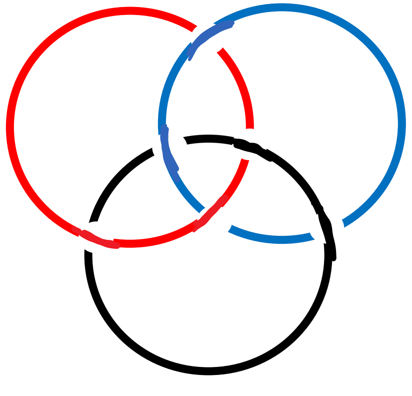


- $\widetilde{\omega}^{c}_{ab}$: 2π rotation of a and b around c
- ➔ $\widetilde{\omega}^{c}_{ab} + \widetilde{\omega}^{a}_{bc} + \widetilde{\omega}^{b}_{ca} \simeq$ global 4π rotation $\simeq$ trivial



String braiding identities

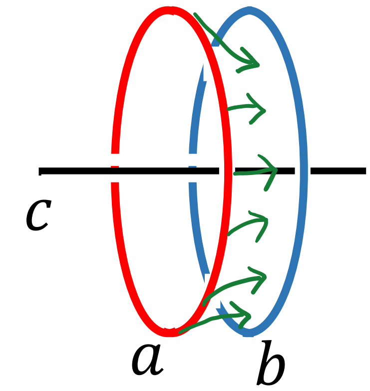
- Using this proof we obtain three identities

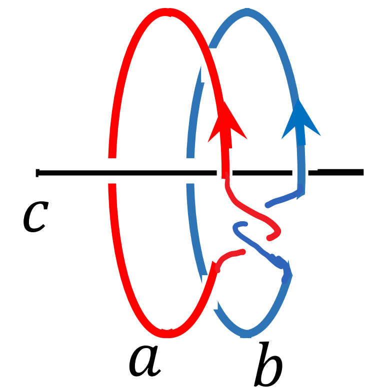
$$\tilde{\omega}_{ab}^{c} + \tilde{\omega}_{bc}^{a} + \tilde{\omega}_{ca}^{b} = 0$$

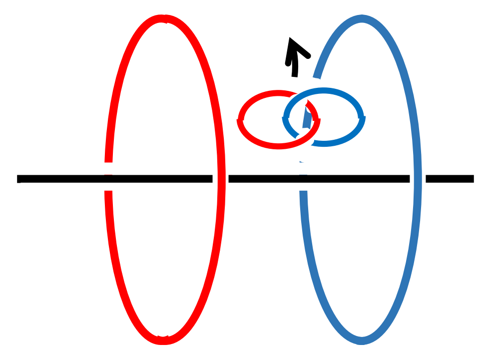
$$\omega_{ab}^{c} + \omega_{bc}^{a} + \omega_{ca}^{b} = 0$$

$$\Omega_{ab}^{c} + \Omega_{bc}^{a} + \Omega_{ca}^{b} = 0$$

- A new feature of 3D topological order that is qualitatively distinct from 2D case
- Open question: In general, is it always possible to require the strings to be *unmarked*, i.e., translation invariant along the string direction?

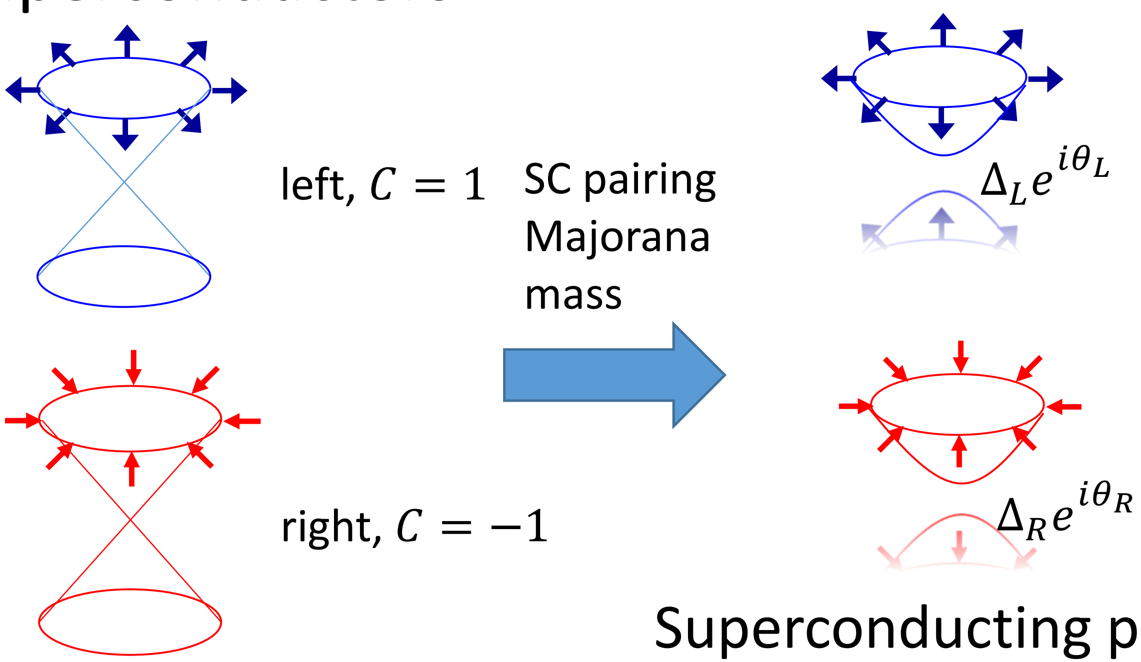






A non-Abelian example of string-string braiding

- Little is known about non-Abelian strings.
- However, an example can be found in 3D topological superconductors



Superconducting pairing

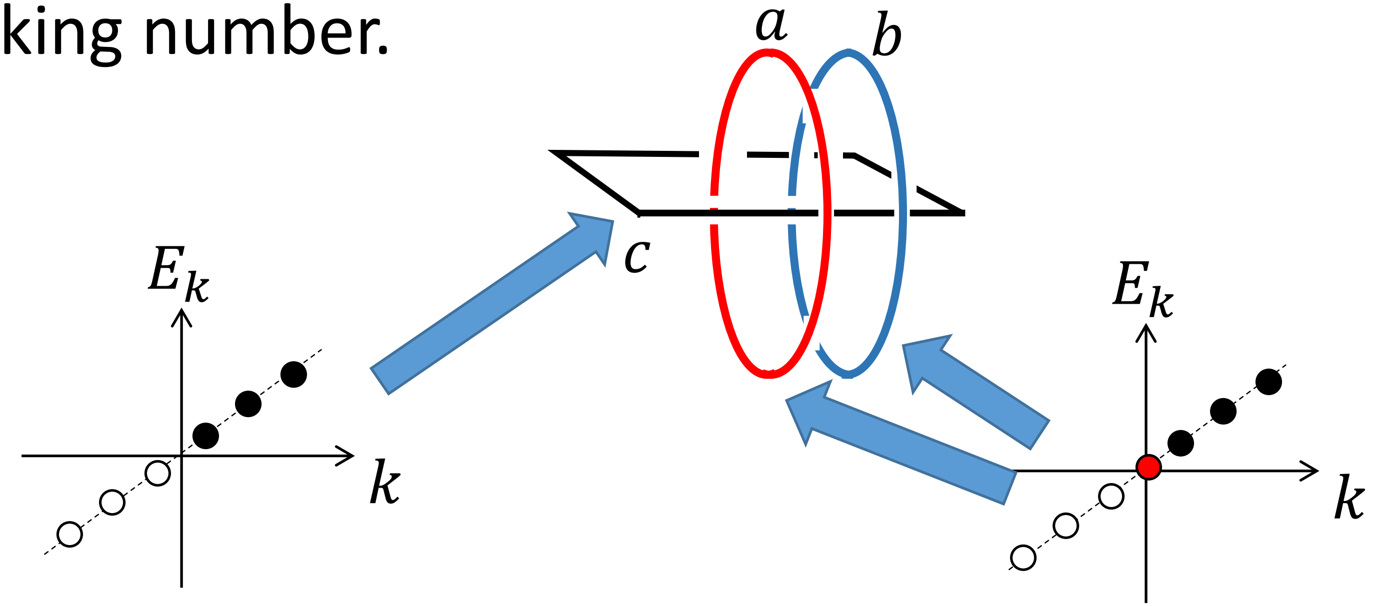
$$\int d^3x \left(\Delta e^{i\theta_L} \psi_L^+ \sigma_y \psi_L^+ + \Delta e^{i\theta_R} \psi_R^+ \sigma_y \psi_R^+\right)$$

Weyl fermions

$$\mathrm{H} = d^3x\, v(\psi_L^+ \sigma \cdot p \psi_L - \psi_R^+ \sigma \cdot p \psi_R)$$

A non-Abelian example of string-string braiding

- Chiral vortex strings: vortex loops of θ_L or θ_R
- Each vortex string is an *axion string*, carrying a 1+1 Majorana-Weyl fermion (Callan&Harvey '85, XLQ&Witten&Zhang '12)
- Majorana zero modes carried by vortices with odd linking number.

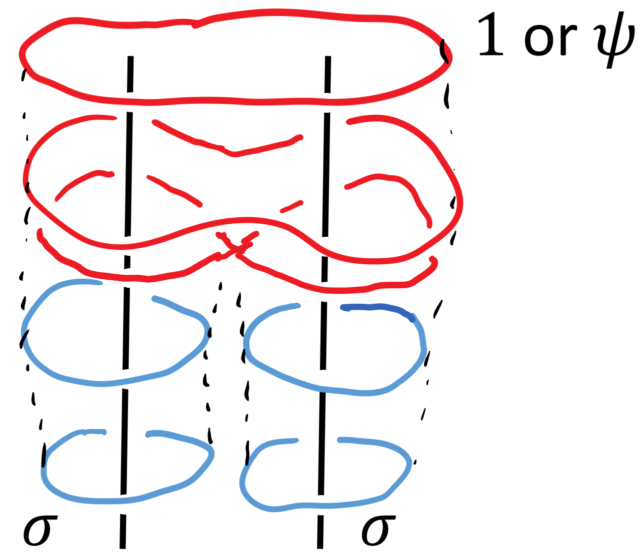


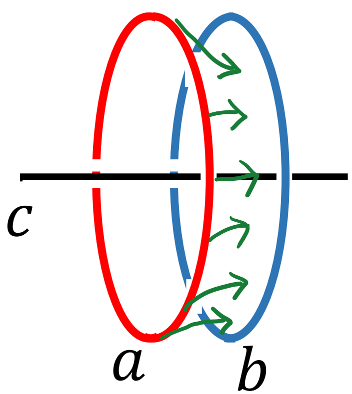
- Non-Abelian braiding of a, b similar to $(p + ip)$ vortices (Read&Green '2000)

(see also M Sato, Physics Letters B 575 (2003) 126–130)

A non-Abelian example of string-string braiding

- Key difference from Abelian string: splitting/fusion of string is not adiabatic.
- Non-Abelian strings can fuse to Abelian strings.
- Braiding depends on the fusion channel.





2 zero modes on a, b

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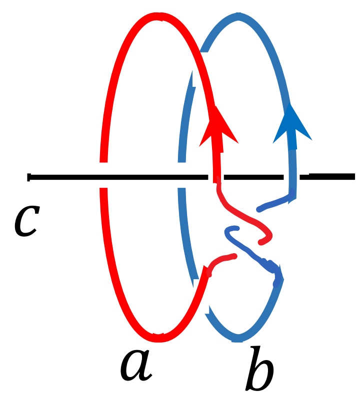

no zero mode

Summary

- Layer construction provides an explicit approach to 3D topological states.
- Different types of 3D topological states can be generated, with surface-only topological order and/or bulk topological order
- String-string braiding can be induced in system with periodic boundary condition or dislocations
- General identity proved for Abelian string-string braiding
- Non-Abelian 3D topological order: An example can be found in topological superconductors. There are a lot of open questions for more general cases.