# Invariants of disordered topological insulators

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# What is a topological insulator?

- $d$-dimensional disordered system of independent Fermions with a combination of basic symmetries

TRS, PHS, SLS = time reversal, particle hole, sublattice symmetry

- Fermi level in a Gap or Anderson localization regime
- Topology of bulk (*e.g.* of Bloch bundles):

winding numbers, Chern numbers, $\mathbb{Z}_2$-invariants, higher invariants

- Delocalized edge modes with non-trivial topology
- Bulk-edge correspondence
- Toy models: tight-binding

**Aim:** index theory for invariants also for disordered systems

# Examples of topological insulators in $d = 2$:

- Integer quantum Hall systems (no symmetries at all)
- Quantum spin Hall systems (Kane-Mele 2005, odd TRS)

dissipationless spin polarized edge currents, charge-spin separation

- Dirty superconductors (Bogoliubov-de Gennes BdG models):

Thermal quantum Hall effect (even PHS)

Spin quantum Hall effect (SU(2)-invariant, odd PHS)

Majorana modes at Landau-Ginzburg vortices (even PHS)

- Examples in $d = 1$ and $d = 3$: chiral unitary systems

# Menu for the talk

- Some standard background on Fredholm operators
- Review of quantum Hall systems (focus on topology)
- Classification of $d = 2$ topological insulators by index theory
- Needed: Fredholm operators with symmetries
- More physics of $d = 2$ systems: QSH and BdG
- Index theory for topological invariants in any dimension $d$
- General bulk-edge correspondence principle

# Fredholm operators and Noether indices

**Definition** $T \in \mathcal{B}(\mathcal{H})$ bounded Fredholm operator on Hilbert space

$\iff T\mathcal{H}$ closed, $\dim(\mathrm{Ker}(T)) < \infty$, $\dim(\mathrm{Ker}(T^*)) < \infty$

Then: $\mathrm{Ind}(T) = \dim(\mathrm{Ker}(T)) - \dim(\mathrm{Ran}(T))$ Noether index

**Theorem** $\mathrm{Ind}(T)$ compactly stable homotopy invariant

**Noether Index Theorem** $f \in C(\mathbb{S}^1)$ invertible, $\Pi$ Hardy on $L^2(\mathbb{S}^1)$

$\Longrightarrow \mathrm{Wind}(f) = \int f^{-1}df = -\,\mathrm{Ind}(\Pi f \Pi)$

**Atiyah-Singer** index theorems in differential topology

**Alain Connes** non-commutative geometry and topology

**Applications in physics** Anomalies in QFT, Defects, etc.

**Solid state physics** robust labelling of different phases

**Problem** determine Fredholm operator in concrete situation

# Review of quantum Hall system (no symmetries)

Toy model: disordered Harper Hamiltonian on Hilbert space $\ell^2(\mathbb{Z}^2)$

$$H \; = \; U_1 + U_1^* + U_2 + U_2^* + \lambda_{\rm dis} V$$

$U_1 = e^{i\varphi X_2} S_1$ and $U_2 = S_2$ with magnetic flux $\varphi$ and $S_{1,2}$ shifts

random potential $V = \sum_{n\in\mathbb{Z}^2} V_n |n\rangle\langle n|$ with i.i.d. $V_n \in \mathbb{R}$

Fermi projection $P = \chi(H \leq \mu)$ with $\mu$ in And. localization regime

**Theorem** (Connes, Bellissard, Kunz, Avron, Seiler, Simon ...)

$$PFP \quad \text{Fredholm operator} \qquad , \qquad F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$$

Index equal to Chern number

$$\begin{aligned} \mathrm{Ind}(PFP) = \mathrm{Ch}(P) &= 2\pi i \, \mathbb{E} \, \langle 0| P \, [[X_1, P], [X_2, P]] |0\rangle \\ &= \int \frac{d^2k}{2\pi i} \, \mathrm{Tr}_q(P \, [\partial_1 P, \partial_2 P]) \end{aligned}$$

# Physical consequences

**Theorem**

(Thouless et.al. 1982, Avron, Seiler, Simon 1983-1994, Kunz 1987, Bellissard, van Elst, S-B 1994, ...)

Kubo formula for zero temperature Hall conductivity $\sigma_H(\mu)$

$$\sigma_H(\mu) \;=\; \frac{e^2}{h}\,\mathrm{Ch}(P)$$

and $\mu \in \Delta \mapsto \sigma_H(\mu)$ constant if Anderson localization in $\Delta \subset \mathbb{R}$

**Theorem**

(Rammal, Bellissard 1985, Resta 2010, S-B, Teufel 2013)

$M(\mu) = \partial_B p(T = 0, \mu)$ orbital magnetization at zero temperature

$$\partial_\mu M(\mu) \;=\; \mathrm{Ch}(P) \qquad \mu \in \Delta$$

# Link to spectral flow (Laughlin argument 1981)

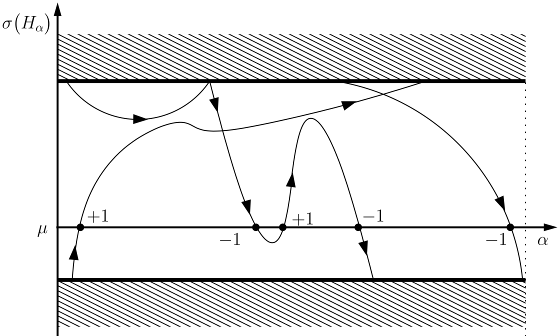
**Folk** involves adiabatics; for Landau see Avron, Pnuelli (1992)

**Theorem** (Macris 2002, Nittis, S-B 2014 )

Hamiltonian $H(\alpha)$ with extra flux $\alpha \in [0,1]$ through 1 cell of $\mathbb{Z}^2$

$H(\alpha) - H$ compact, so only discrete spectrum close to $\mu$ in gap

$$\mathrm{Ch}(P) \;=\; \mathrm{Spectral\ Flow}\Big(\alpha \in [0,1] \mapsto H(\alpha) \;\text{ through }\; \mu\Big)$$



# Bulk-edge correspondence

Edge currents in periodic systems: Halperin 1982, Hatsugai 1993

**Theorem** (S-B, Kellendonk, Richter 2000, 2002)

$\mu \in \Delta$ gap of $H$ and $\widehat{H}$ restriction to half-space $\ell^2(\mathbb{Z} \times \mathbb{N})$

With $g : \mathbb{R} \to [0, 1]$ increasing from 0 to 1 in $\Delta$

$$\widehat{\mathcal{T}}(g'(\widehat{H})\,\widehat{J}_1) \;=\; \mathrm{Ch}(P)$$

where $\widehat{J}_1 = i[X_1, \widehat{H}] = \nabla_1 \widehat{H}$ current operator and

$$\widehat{\mathcal{T}}(\widehat{A}) \;=\; \sum_{x_2 \geq 0} \mathbb{E}\,\langle 0, x_2|\,\widehat{A}\,|0, x_2\rangle \qquad \text{tracial state on edge ops}$$

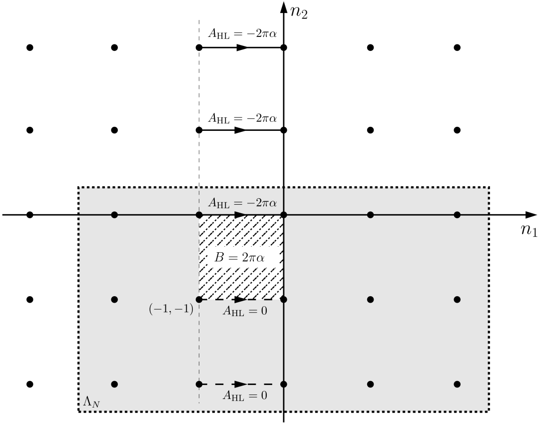
Moreover, link to winding number of $\widehat{V} = \exp(2\pi i\, g(\widehat{H}))$

$$\mathrm{Ch}(P) \;=\; i\,\widehat{\mathcal{T}}(\widehat{V}^* \nabla_1 \widehat{V})$$

without gap condition: Elgart, Graf, Schenker 2005

# Macris' argument for bulk-edge correspondence

$$\mathrm{Ch}(P) \;=\; \mathrm{Ind}(PFP) \;=\; -\int_0^1 d\alpha\;\mathrm{Tr}\big(g'(\widetilde{H}^N_\alpha)\,\partial_\alpha \widetilde{H}^N_\alpha\big)$$



# Tight-binding toy models in dimension $d = 2$

Hilbert space $\ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^L$

Fiber $\mathbb{C}^L = \mathbb{C}^{2s+1} \otimes \mathbb{C}^r$ with spin $s$ and $r$ internal degrees

e.g. $\mathbb{C}^r = \mathbb{C}^2_{\rm ph} \otimes \mathbb{C}^2_{\rm sl}$ particle-hole space and sublattice space

Typical Hamiltonian

$$H \;=\; \sum_{i=1}^{4} \left( W_i^* U_i + W_i U_i^* \right) \,+\, \lambda_{\rm dis}\, V$$

$U_1 = e^{i\varphi X_2} S_1$ and $U_2 = S_2$ with magnetic flux $\varphi$ and $S_{1,2}$ shifts

next nearest neighbor $U_3 = U_1^* U_2$ and $U_4 = U_1 U_2$

$W_i$ matrices $L \times L$ (e.g. for spin orbit coupling, pair creation)

Matrix potential $V = V^* = \sum_{n \in \mathbb{Z}^2} V_n |n\rangle\langle n|$ random (i.i.d.)

$P = \chi(H \leq \mu)$ Fermi projection, $PFP$ still Fredholm operator

# Implementing symmetries

$K_{\rm sl}$ unitary on fiber $\mathbb{C}^2_{\rm sl}$ with $K_{\rm sl}^2 = \mathbf{1}$

$$\begin{aligned} \text{SLS (Chiral)}: \qquad & K_{\rm sl}^* \, H \, K_{\rm sl} \; = \; -H \\ \text{TRS}: \qquad & I_{\rm s}^* \, \overline{H} \, I_{\rm s} \; = \; H \\ \text{PHS}: \qquad & K_{\rm ph}^* \, \overline{H} \, K_{\rm ph} \; = \; -H \end{aligned}$$

$I_{\rm s}$, $K_{\rm ph}$ real unitaries on fibers $\mathbb{C}^{2s+1}$, $\mathbb{C}^2_{\rm ph}$ which are even/odd:

$$I_{\rm s}^2 \; = \; \pm\mathbf{1} \qquad K_{\rm ph}^2 \; = \; \pm\mathbf{1}$$

**Example:** $I_{\rm s} = e^{i\pi s^y}$ even/odd = integer/half-integer spin

Note: TRS + PHS $\Longrightarrow$ SLS with $K_{\rm sl} = I_{\rm s} K_{\rm ph}$ or $K_{\rm sl} = i \, I_{\rm s} K_{\rm ph}$

10 combinations of symmetries: none (1), one (5), three (4)

10 Cartan-Altland-Zirnbauer classes, 2 complex and 8 real

# Classification of $d = 2$ topological insulators

Schnyder, Ryu, Furusaki, Ludwig 2008, reordering Kitaev 2008

Nittis, S-B 2014: classification with $T = PFP$ (strong invariants)

| CAZ | TRS | PHS | SLS | Phase/Ind | System | symmetry of $T$ |
|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | $\mathbb{Z}$ | QHE | none |
| AIII | 0 | 0 | 1 | 0 | | $K_{\rm sl}^* T K_{\rm sl} = T^c$ |
| D | 0 | $+1$ | 0 | $\mathbb{Z}$ | TQH | none |
| DIII | $-1$ | $+1$ | 1 | $\mathbb{Z}_2$ | SCS | two |
| AII | $-1$ | 0 | 0 | $\mathbb{Z}_2$ | QSH | $I_{\rm s}^* T^t I_{\rm s} = T$ |
| CII | $-1$ | $-1$ | 1 | 0 | | two |
| C | 0 | $-1$ | 0 | $2\,\mathbb{Z}$ | SQH | Ker($T$) quat. |
| CI | $+1$ | $-1$ | 1 | 0 | | two |
| AI | $+1$ | 0 | 0 | 0 | | $I_{\rm s}^* T^t I_{\rm s} = T$ |
| BDI | $+1$ | $+1$ | 1 | 0 | | two |

# $\mathbb{Z}_2$ indices of odd symmetric Fredholm operators

$I = I_s$ real unitary on Hilbert space $\mathcal{H}$ with real structure, $I^2 = -\mathbf{1}$

**Definition** $T$ odd symmetric $\iff I^* T^t I = T$ with $T^t = (\overline{T})^*$

**Theorem** (S-B 2013) Ind of odd symm. Fredholm vanishes, but:
$\mathbb{F}_2(\mathcal{H}) = \{$odd symmetric Fredholm operators$\}$ has 2 connected components labeled by the compactly stable homotopy invariant:

$$\mathrm{Ind}_2(T) = \dim(\mathrm{Ker}(T)) \bmod 2 \;\in\; \mathbb{Z}_2$$

**Class AII (QSH):** $H$ odd TRS $\iff I^*\overline{H}I = H \iff I^* H^t I = H$

So: $H$ odd symmetric $\implies H^n$ odd sym. $\implies f(H)$ odd sym.

Fermi projection $P$ odd sym. and $PFP$ odd sym. Fredholm

$$\mathrm{Ind}_2(PFP) \in \mathbb{Z}_2 \text{ well-defined} \qquad , \qquad F = \frac{X_1 + iX_2}{|X_1 + iX_2|}$$

Also for Fermi level in region of dynamically localized states!

# Proofs for $\mathbb{Z}_2$ indices (S-B 2013)

**Proposition** Even degeneracies for odd symmetric matrices.

**Proof:** odd symmetry $I^* T^t I = T \implies (IT)^t = -IT$
$\implies \det(T - z\,\mathbf{1}) = \det(IT - z\,I) = \mathrm{Pf}(IT - z\,I)^2$ □

Similar to Kramers' degeneracy, but no invariance under $\psi \mapsto I\overline{\psi}$

**Proposition** $K$ compact odd symmetric

$\implies \mathbf{1} + K$ even degeneracies and $\mathrm{Ind}_2(\mathbf{1} + K) = 0$

This is a weak form of compact stability, namely at $T = \mathbf{1}$

**Theorem** (Siegel) $T$ odd symmetric $\iff T = I^* A^t I A$

**Proof** of connectedness:

$\mathrm{Ind}_2(T) = 0 \implies T$ invertible (mod $\mathcal{K}$) $\implies A$ invertible

$s \in [0,1] \mapsto A_s$ homotopy to $\mathbf{1}$

$\implies s \in [0,1] \mapsto T_s = I^*(A_s)^t I A_s$ path to $\mathbf{1}$ in odd symmetrics

# Link to Atiyah-Singer classifying spaces (1969)

$\mathbb{F}_k^{\mathbb{R}} =$ skew-adjoint Freds on $\mathcal{H}_{\mathbb{R}}$ with $\pm i \in \sigma_{\rm ess}$ commuting $C_{k-1}$

**Fact:** $\mathbb{F}_1^{\mathbb{R}}$ and $O$ have same homotopy type and $\pi_k(O) = \pi_0(\mathbb{F}_k^{\mathbb{R}})$

**Example:** $T \in \mathbb{F}_1^{\mathbb{R}} \implies \sigma(T) = \overline{\sigma(T)} \subset i\,\mathbb{R}\,,\ \ 0 \notin \sigma_{\rm ess}(T)$

$\implies \ \mathrm{Ind}_1(T) = \dim(\mathrm{Ker}(T)) \bmod 2$ invariant

Only few index theorems in $\mathbb{F}_1^{\mathbb{R}}$ (Kervaire invariant), none in $\mathbb{F}_2^{\mathbb{R}}$

**Theorem** Identifications with Freds on complex Hilbert space:

$$\mathbb{F}_0^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, \overline{T} = T\} \qquad \mathbb{F}_1^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F} \,|\, \overline{T} = -T\}$$

$$\mathbb{F}_2^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, I^* T^t I = T\} \qquad \mathbb{F}_3^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F}_* \,|\, I^* \overline{T} I = T\}$$

$$\mathbb{F}_4^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, I^* \overline{T} I = T\} \qquad \mathbb{F}_5^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F} \,|\, I^* \overline{T} I = -T\}$$

$$\mathbb{F}_6^{\mathbb{R}} \cong \{T \in \mathbb{F} \,|\, T^t = T\} \qquad \mathbb{F}_7^{\mathbb{R}} \cong \{T = T^* \in \mathbb{F}_* \,|\, \overline{T} = T\}$$

**Example** QSH provides an index theorem in $\pi_0(\mathbb{F}_2^{\mathbb{R}}) = \mathbb{Z}_2$

# Quantum spin Hall system (odd TRS, Class AII)

Disordered Kane-Mele model on hexagon lattice and with $s = \frac{1}{2}$

$$H \;=\; \Delta_{\mathrm{hexagon}} + H_{\mathrm{SO}} + H_{\mathrm{Ra}} + \lambda_{\mathrm{dis}} V$$

Pseudo-gap at Dirac point opens non-trivially due to

$$H_{\mathrm{SO}} \;=\; i\,\lambda_{\mathrm{SO}} \sum_{i=1,2,3} \left(S_i^{\mathrm{nn}} - (S_i^{\mathrm{nn}})^*\right) s^z$$

No $s^z$-conservation due to Rashba term $H_{\mathrm{Ra}}$, but odd TRS

**Non-trivial topology:**

Kane-Mele (2005): $\mathbb{Z}_2$ invariant for periodic system from Pfaffians

Haldane et al. (2005): spin Chern numbers for $s^z$ invariant systems

Prodan (2009): spin Chern number from $P_{\mathrm{s}} = \chi(|Ps^zP - \frac{1}{2}| < \frac{1}{2})$

$$\mathrm{SCh}(P) = \mathrm{Ch}(P_{\mathrm{s}}) \;\in\; \mathbb{Z}$$

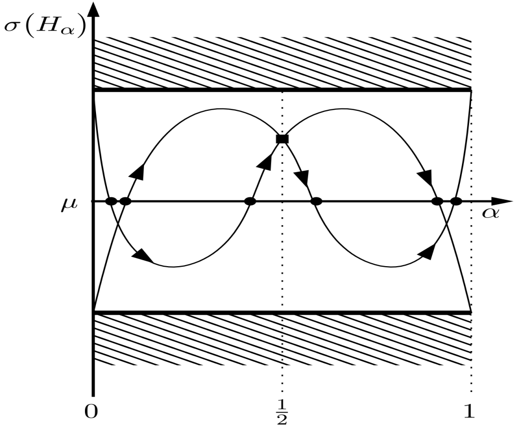
Systems periodic in one direction: Graf, Porta 2013

# $\mathbb{Z}_2$ invariant and spin-charge separation

**Theorem** $\mathrm{Ind}_2(PFP)$ phase label for odd TRS

**Theorem** (Nittis, S-B, 2014) $\alpha \in [0,1] \mapsto H(\alpha)$ inserted flux

$\mathrm{Ind}_2(PFP) = 1 \implies H(\alpha = \frac{1}{2})$ has TRS + Kramers pair in gap



# Spin filtered helical edge channels for QSH

**Theorem** (S-B 2013) Small Rashba term

$\mathrm{Ind}_2(PFP) = 1 \implies$ spin Chern numbers $\mathrm{SCh}(P) \neq 0$

**Remark** Non-trivial topology $\mathrm{SCh}(P)$ persists TRS breaking!

**Theorem** (S-B 2012)

Spin filtered edge currents in $\Delta \subset$ gap stable w.r.t. perturbations by magnetic field and disorder: $g : \Delta \to [0, 1]$ with $\int g = 1$

$$\widehat{\mathcal{T}}\left(g(\widehat{H})\,\tfrac{1}{2}\{\widehat{J}_1, s^z\}\right) \;=\; \mathrm{SCh}(P) \;+\; \mathcal{O}(\|g\|_{C^4}\|[H, s^z]\|)$$

**Resumé:** $\mathrm{Ind}_2(PFP) = 1 \implies$ no Anderson loc. for edge states

Rice group of Du (since 2011): QSH stable w.r.t. magnetic field

Here spin Chern number is relevant and not $\mathbb{Z}_2$ invariant!

# BdG Hamiltonian for dirty superconductor

Disordered one-electron Hamiltonian $h$ on $\mathcal{H} = \ell^2(\mathbb{Z}^2) \otimes \mathbb{C}^{2s+1}$

$\mathfrak{c} = (\mathfrak{c}_{n,l})$ anhilation operators on fermionic Fock space $\mathcal{F}_-(\mathcal{H})$

Hamilt. on $\mathcal{F}_-(\mathcal{H})$ with mean field pair creation $\Delta^* = -\overline{\Delta} \in \mathcal{B}(\mathcal{H})$

$$\begin{aligned} \mathbf{H} - \mu\,\mathbf{N} &= \mathfrak{c}^*\,(h - \mu\,\mathbf{1})\,\mathfrak{c} \;+\; \frac{1}{2}\,\mathfrak{c}^*\,\Delta\,\mathfrak{c}^* \;-\; \frac{1}{2}\,\mathfrak{c}\,\overline{\Delta}\,\mathfrak{c} \\ &= \frac{1}{2}\begin{pmatrix}\mathfrak{c}\\ \mathfrak{c}^*\end{pmatrix}^* \begin{pmatrix} h-\mu & \Delta \\ -\overline{\Delta} & -\overline{h}+\mu \end{pmatrix} \begin{pmatrix}\mathfrak{c}\\ \mathfrak{c}^*\end{pmatrix} \end{aligned}$$

Hence BdG Hamiltonian on $\mathcal{H}_{\rm ph} = \mathcal{H} \otimes \mathbb{C}^2_{\rm ph}$

$$H_\mu \;=\; \begin{pmatrix} h-\mu & \Delta \\ -\overline{\Delta} & -\overline{h}+\mu \end{pmatrix}$$

Even PHS (Class D)

$$K_{\rm ph}^*\,\overline{H_\mu}\,K_{\rm ph} \;=\; -H_\mu \qquad , \qquad K_{\rm ph} = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

# Class D systems (even PHS)

**Proposition** $\sigma(H_\mu) = -\sigma(H_\mu)$

**Proposition** Gibbs (KMS) state for observable $\mathbf{Q} = d\Gamma(Q)$

$$\frac{1}{Z_{\beta,\mu}} \mathrm{Tr}_{\mathcal{F}_-(\mathcal{H})} \left( \mathbf{Q}\, e^{-\beta(\mathbf{H}-\mu\,\mathbf{N})} \right) \;=\; \mathrm{Tr}_{\mathcal{H}_{\mathrm{ph}}}(f_\beta(H_\mu)\,Q)$$

**Thus:** $P = \chi(H_\mu \leq 0)$ can have $\mathrm{Ch}(P) = \mathrm{Ind}(PFP) \neq 0$

**Example** $p + ip$ wave superconductor with $\mathcal{H} = \ell^2(\mathbb{Z}^2)$

$$h = S_1 + S_1^* + S_2 + S_2^* \qquad \Delta_{p+ip} \;=\; \delta\,(S_1 - S_1^* + i(S_2 - S_2^*))$$

**Quantized Wiedemann-Franz** (Sumiyoshi-Fujimoto 2013)

$$\kappa_H \;=\; \frac{\pi}{8}\,\mathrm{Ch}(P)\;T \;+\; \mathcal{O}(T^2)$$

**Theorem** $\mathrm{Ind}(PFP)$ odd $\Longrightarrow$ $0 \in \sigma(H(\alpha = \frac{1}{2}))$ Majorana state

# Spin quantum Hall effect in Class C (odd PHS)

**Theorem** (Altland-Zirnbauer 1997)

SU(2) spin rotation invariance $[\mathbf{H}, \mathbf{s}] = 0$

$\Longrightarrow H = H_{\mathrm{red}} \otimes \mathbf{1}$ with odd PHS (Class C)

$$K_{\mathrm{ph}}^* \,\overline{H_{\mathrm{red}}}\, K_{\mathrm{ph}} = -H_{\mathrm{red}} \qquad , \qquad K_{\mathrm{ph}} = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

**Theorem** (Nittis, S-B 2014) $H$ odd PHS $\Longrightarrow \mathrm{Ind}(PFP) \in 2\,\mathbb{Z}$

**Example** $d + id$ wave superconductor

$$\Delta_{d+id} = \delta \left( i(S_1 + S_1^* - S_2 - S_2^*) + (S_1 - S_1^*)(S_2 - S_2^*) \right) s^2$$

Then $\mathrm{Ch}(P) = \mathrm{Ind}(PFP) = 2$ for $\delta > 0$ and $\mu > 0$

**Theorem** (Nittis, S-B 2014) Spin Hall conductance

(given by Kubo formula) and spin edge currents quantized

# Periodic table (Schnyder et. al., Kitaev 2008)

Complex $K$-theory (2 periodic), Real $K$-theory (8-periodic)

| CAZ | TRS | PHS | SLS | $d=1$ | $d=2$ | $d=3$ | $d=4$ |
|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | | $\mathbb{Z}$ | | $\mathbb{Z}$ |
| AIII | 0 | 0 | 1 | $\mathbb{Z}$ | | $\mathbb{Z}$ | |
| D | 0 | $+1$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | | |
| DIII | $-1$ | $+1$ | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | |
| AII | $-1$ | 0 | 0 | | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |
| CII | $-1$ | $-1$ | 1 | $2\,\mathbb{Z}$ | | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| C | 0 | $-1$ | 0 | | $2\,\mathbb{Z}$ | | $\mathbb{Z}_2$ |
| CI | $+1$ | $-1$ | 1 | | | $2\,\mathbb{Z}$ | |
| AI | $+1$ | 0 | 0 | | | | $2\,\mathbb{Z}$ |
| BDI | $+1$ | $+1$ | 1 | $\mathbb{Z}$ | | | |

Focus on complex cases: chirality and bulk-edge correspondence

# Class A systems (dimension $d$ even)

**Given:** covariant Hamiltonian $(H_\omega)_{\omega\in\Omega}$ on $\ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^L$

$P=(P_\omega)_{\omega\in\Omega}$ Fermi projection with localization condition

$$\mathbf{E}_\omega \,\|\,\langle n|P_\omega|0\rangle\,\| \;\leq\; A_\gamma\, e^{-\gamma|n|}$$

**Aim:** Index theorem for strong invariant (generalizing QHE)

**Construction:** following Prodan, Leung, Bellissard (2013)

$\sigma_1,\ldots,\sigma_d$ irrep of Clifford $C_d$ on $\mathbb{C}^{2^{d/2}}$, Dirac phase:

$$D \;=\; \sum_{j=1}^{d} X_j\otimes\sigma_j \qquad F \;=\; \frac{D}{|D|} \qquad \text{on } \ell^2(\mathbb{Z}^d)\otimes\mathbb{C}^L\otimes\mathbb{C}^{2^{d/2}}$$

Grading $\gamma=-i^{-d/2}\sigma_1\cdots\sigma_d$ so that $F\gamma=-\gamma F$

# Index theorem for even dimension $d$

Extend $P$ on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L$ to $P \otimes \mathbf{1}$ on $\ell^2(\mathbb{Z}^d) \otimes \mathbb{C}^L \otimes \mathbb{C}^{2^{d/2}}$

**Theorem** (Prodan, Leung, Bellissard 2013) In grading of $\gamma$, upper right comp. $(P_\omega F P_\omega)_{+,-}$ Fredholm with index a.s. equal

$$\mathrm{Ch}_d(P) = \frac{(2i\pi)^{\frac{d}{2}}}{\frac{d}{2}!} \sum_{\rho \in S_d} (-1)^\rho \, \mathbf{E} \, \mathrm{Tr} \, \langle 0| \left( P \prod_{j=1}^{d} [X_{\rho_j}, P] \right) |0\rangle$$

**Remark** Real space formula of $k$-space version for periodic system

$$\mathrm{Ch}_d(P) = \frac{1}{(-2i\pi)^{\frac{d}{2}} \, \frac{d}{2}!} \int_{\mathbb{T}^d} \mathrm{Tr} \left( \left[ P(k) dP(k) \wedge dP(k) \right]^{\frac{d}{2}} \right)$$

**Proof:** higher dimensional version of Connes' triangle identity

# Chiral unitary systems (dimension $d$ odd)

$K_{\mathrm{sl}}^* H K_{\mathrm{sl}} = -H$ with $K_{\mathrm{sl}} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, thus $H = \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}$

$K_{\mathrm{sl}}^* f(H) K_{\mathrm{sl}} = -f(H)$ for any odd function $H$, so $f(H)$ off-diagonal

In particular, flat band Hamiltonian $Q = 2P - \mathbf{1} = \mathrm{sgn}(H)$ is odd

As $Q^2 = \mathbf{1}$ there is unitary $U$ with $Q = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}$

**Resumé:** Fermi projection $P = \chi(H \leq 0)$ encoded in unitary $U$

Dirac phase $F = \frac{D}{|D|}$ from $D = \sum_{j=1}^{d} X_j \otimes \sigma_j$, and $E = \frac{1}{2}(F + \mathbf{1})$

**Theorem** (Prodan, S-B 2014)

$EUE$ Fredholm operator with almost sure index equal to

$$\mathrm{Ch}_d(U) = \frac{(i\pi)^{\frac{d-1}{2}}}{d!!} \sum_{\rho \in S_d} (-1)^{\rho} \, \mathbf{E} \, \mathrm{Tr} \, \langle 0 | \left( \prod_{j=1}^{d} U^{-1} [X_{\rho_j}, U] \right) | 0 \rangle$$

# Chiral systems: comments and example

**Remark** $k$-space version (Schnyder, Ryu, Furusaki, Ludwig 2008)

$$\mathrm{Ch}_d(U) = \frac{(\frac{1}{2}(d-1))!}{d!} \left(\frac{i}{2\pi}\right)^{\frac{d+1}{2}} \int_{\mathbb{T}^d} \mathrm{Tr}\left(\left[U^{-1}dU\right]^d\right)$$

New phase label generalizing higher winding numbers

**Remark** Phase stable under small breaking of chiral symmetry (as long as off-diagonal entry of $Q$ invertible)

**Example** $d = 1$ (Mondragon-Shem, Song, Hughes, Prodan 2013):

$$H \;=\; \frac{1}{2}(\sigma_1 + \imath\sigma_2)\,S^* + \frac{1}{2}(\sigma_1 - \imath\sigma_2)\,S + m\,\sigma_2$$

$\mathrm{Ch}_1(U) \neq 0$ for $|m| < 1$, only localized states for random coeffs

Divergence of localization length at $E = 0$ at transition point

# General bulk-edge correspondence (Prodan S-B)

**Hypothesis:** gap in bulk system in dimension $d$ (even or odd)

**Exact sequence:** edge — half space — bulk

$$0 \longrightarrow \mathcal{A}_{d-1} \otimes \mathcal{K} \longrightarrow \mathcal{T}(\mathcal{A}_d) \longrightarrow \mathcal{A}_d \longrightarrow 0$$

**Crucial fact:** $\mathrm{Ch}_{d-1}$ extends to edge operators in $\mathcal{A}_{d-1} \otimes \mathcal{K}$

$$\begin{array}{ccccc} K_0(\mathcal{A}_{d-1}) & \longrightarrow & K_0(\mathcal{T}(\mathcal{A}_{d-1})) & \longrightarrow & K_0(\mathcal{A}_d) \\ \mathrm{Ind} \ \uparrow & & & & \downarrow \ \exp \\ K_1(\mathcal{A}_d) & \longleftarrow & K_1(\mathcal{T}(\mathcal{A}_{d-1})) & \longleftarrow & K_1(\mathcal{A}_{d-1}) \end{array}$$

**Class A system in even $d$:** $\mathrm{Ch}_d(P) = \mathrm{Ch}_{d-1}(\exp(P))$

**Chiral system in odd $d$:** $\mathrm{Ch}_d(U) = \mathrm{Ch}_{d-1}(\mathrm{Ind}(U))$

# Example in $d = 3$ (Schnyder *et. al.*, Prodan S-B)

$(\sigma_j)_{j=1,\ldots,5}$ irrep of Clifford algebra $C_5$ on $\mathbb{C}^4$, *e.g.* with Pauli mats

Hamiltonian on $\ell^2(\mathbb{Z}^3) \otimes \mathbb{C}^4$

$$H \;=\; \sum_{j=1}^{3} \frac{1}{2\imath}(S_j - S_j^*) \otimes \sigma_j \;+\; \left( m + \sum_{j=1}^{3} \frac{1}{2}(S_j + S_j^*) \right) \otimes \sigma_4$$

Chiral symmetry $\sigma_5 \, H \, \sigma_5 \;=\; -H$

Closed gap at $m = -3, -1, 1, 3$, between $\mathrm{Ch}_3(U) = 0, -1, 2, -1, 0$

$d = 2$ surface state have Dirac points adding up to $\mathrm{Ch}_3(U)$

Split in magnetic field (as for Dirac or on honeycomb)

$\widehat{P}$ spectral projection on central band of surface states has QHE

**Theorem** $\mathrm{Ind}([U]_1) = [\widehat{P} J]_0$ and $\mathrm{Ch}_2(\widehat{P} J) = \mathrm{Ch}_3(U)$

## Resumé

- $\mathbb{Z}_2$ indices of Fredholm operators
- Invariants and indices in higher dimesion
- General bulk-edge correspondence
- Non-trivial topology persists if symmetries slightly broken