Quantum entanglement, topological order, and tensor category theory

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Local unitary trans. defines gapped quantum phases

Two gapped states, $|\Psi(0)\rangle$ and $|\Psi(1)\rangle$ (or more precisely, two ground state subspaces), are in the same phase iff they are related through a local unitary (LU) evolution

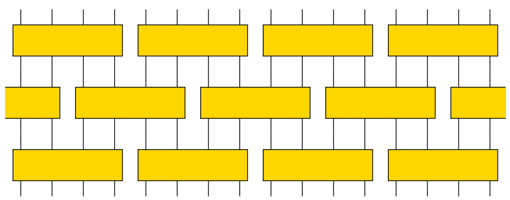
$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\int_0^1 dg'\, H(g')}\right)|\Psi(0)\rangle$$

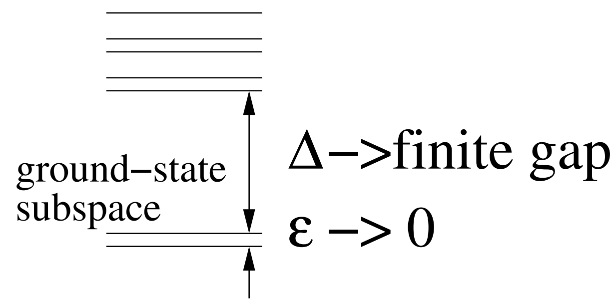
where $H(g) = \sum_{\mathbf{i}} O_{\mathbf{i}}(g)$ and $O_{\mathbf{i}}(g)$ are local hermitian operators.

Hastings, Wen 05; Bravyi, Hastings, Michalakis 10

- LU evolution = *local unitary transformation*:

$$|\Psi(1)\rangle = P\left(e^{-\mathrm{i}\,T\int_0^1 dg\, H(g)}\right)|\Psi(0)\rangle$$

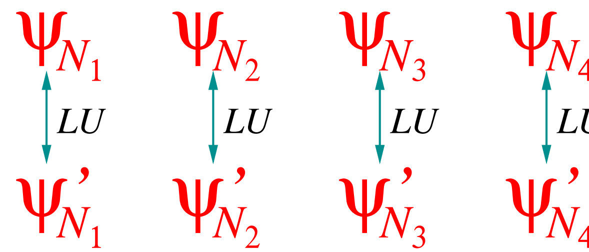
$=$  $|\Psi(0)\rangle$



- The local unitary transformations define an equivalence relation: *Two gapped states related by a local unitary transformation are in the same phase.*
  A gapped quantum phase is an equivalence class of local unitary transformations – a conjecture.

A gapped quantum liquid phase:

- **A gapped quantum phase:**
  $H_{N_1}, H_{N_2}, H_{N_3}, H_{N_4}, \cdots$
  $H'_{N_1}, H'_{N_2}, H'_{N_3}, H'_{N_4}, \cdots$
  $N_{i+1} = sN_i, \ s \sim 2$



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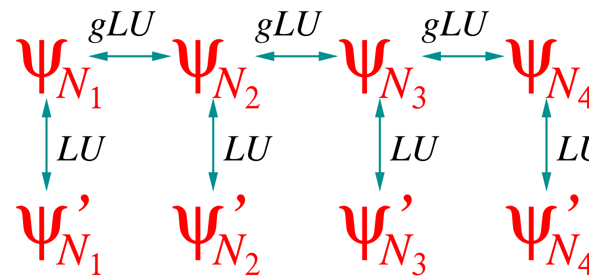
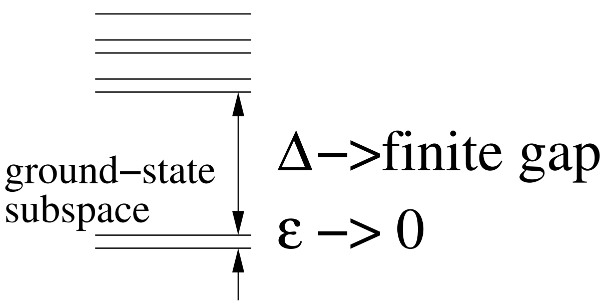
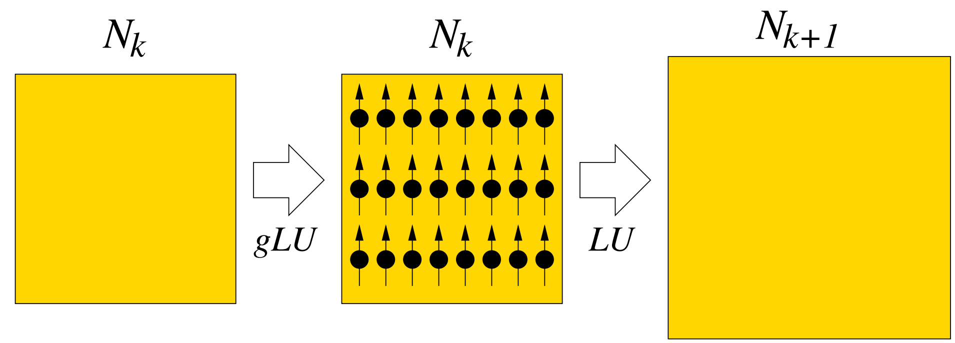


Generalized local unitary (gLU) trans.

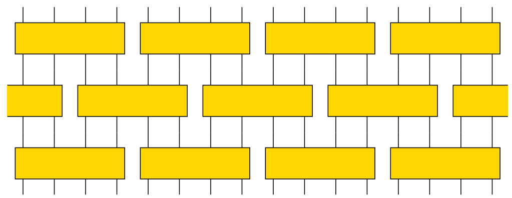




- 3+1D toric code model → a 3+1D gaped quantum liquid.
- Stacking 2+1D FQH states and Haah cubic model → gapped quantum state, but not liquids.

Bosonic/fermionic gapped quantum phases

Both local bosonic and fermionic systems have the following local property: $\mathcal{V}_{\text{tot}} = \otimes_i \mathcal{V}_i$

$|\Psi(1)\rangle =$  $|\Psi(0)\rangle$

- Bosonic gapped phases are the equivalent classes of LU transformation: $LU = \prod U_{ijk}$, which acts within $V_i \otimes V_j \otimes V_k$, and $[U_{ijk}, U_{i'j'k'}] = 0$, e.g. $U_{ijk} = e^{i(b_i b_j b_k^\dagger + h.c.)}$
- Fermionic gapped phases are the equivalent classes of fermionic LU transformation: $fLU = \prod U_{ijk}^f$, which does not act within $V_i \otimes V_j \otimes V_k$, but $[U_{ijk}^f, U_{i'j'k'}^f] = 0$, e.g. $U_{ijk}^f = e^{i(c_i c_j c_k^\dagger c_k + h.c.)}$

Gapped quantum liquids for bosons and fermions have very different mathematical structures

LU trans. defines long-range entanglement (ie topo. order)

For gapped systems with no symmetry:

- According to Landau theory, no symmetry to break
  $\rightarrow$ all systems belong to one trivial phase

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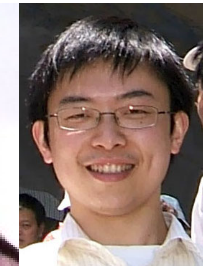

For gapped systems with no symmetry:

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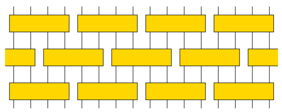
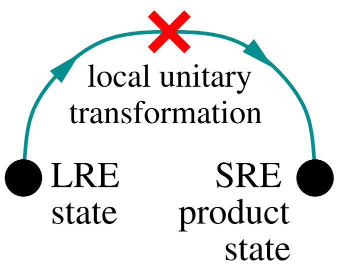
- Thinking about entanglement: Chen-Gu-Wen 2010
  - There are **long range entangled (LRE) states**
  - There are **short range entangled (SRE) states**

$|LRE\rangle \neq$  $|\text{product state}\rangle = |SRE\rangle$



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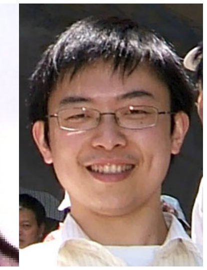

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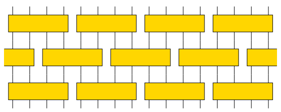
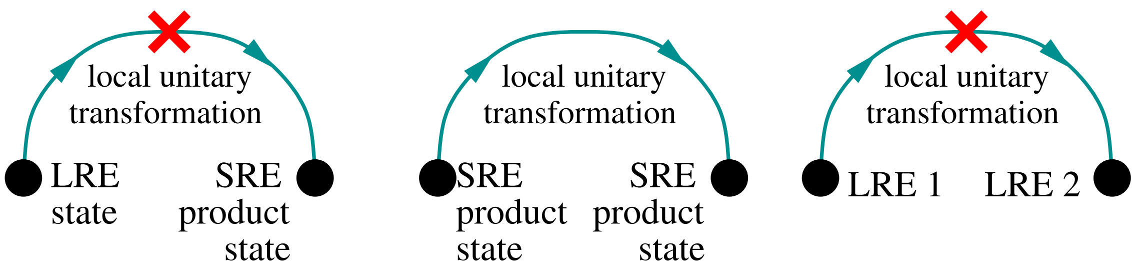
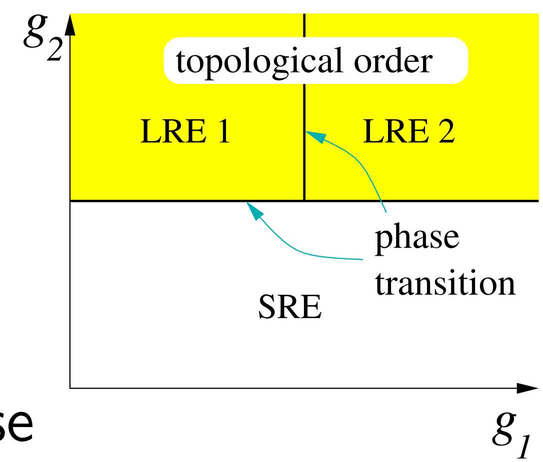
  - There are **short range entangled (SRE) states** $\to$ one phase

$|\text{LRE}\rangle \neq$  $|\text{product state}\rangle = |\text{SRE}\rangle$





- All SRE states belong to the same trivial phase
- LRE states can belong to many different phases
  = different **patterns of long-range entanglements** defined by the LU trans.
  = different **topological orders** Wen 1989
  $\to$ A classification by **tensor category theory** Levin-Wen 05, Chen-Gu-Wen 2010

Quantum entanglements through examples

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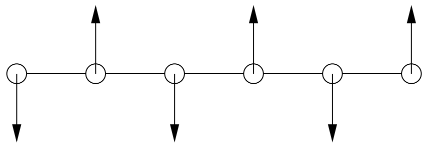
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ $\to$ unentangled

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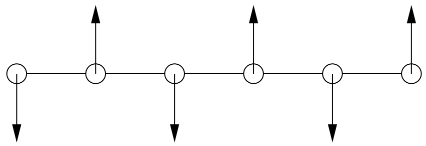
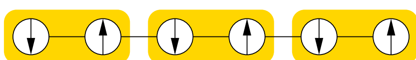
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ →
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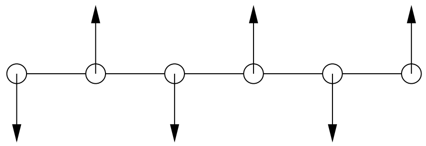
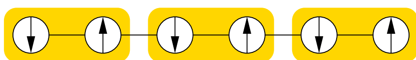
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ →
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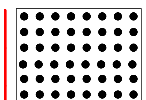
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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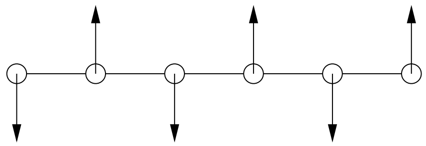
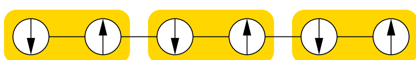
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \ldots$ $\to$ unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes \ldots \to$
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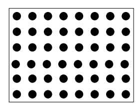
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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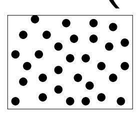
- Particle condensation (superfluid)
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle$

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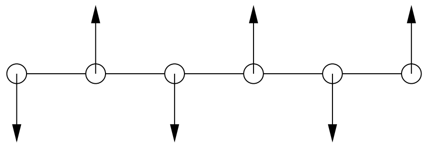
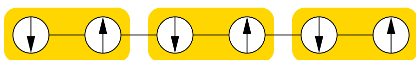
-  $= |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle \otimes |\uparrow\rangle \otimes |\downarrow\rangle ...$ → unentangled
-  $= (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes (|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \otimes ...$ →
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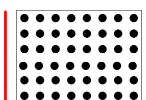
- Crystal order: $|\Phi_{\text{crystal}}\rangle = \Big|$  $\Big\rangle = |0\rangle_{x_1} \otimes |1\rangle_{x_2} \otimes |0\rangle_{x_3} \cdots$
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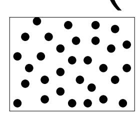
  $|\Phi_{\text{SF}}\rangle = \sum_{\text{all conf.}} \Big|$  $\Big\rangle = (|0\rangle_{x_1} + |1\rangle_{x_1} + ..) \otimes (|0\rangle_{x_2} + |1\rangle_{x_2} + ..) ...$
  = direct-product state → unentangled state (classical)
  - Superfluid, as an exemplary quantum state of matter, is actually very classical and unquantum from entanglement point of view.

Scramble the phase: local rule $\to$ global dancing pattern

$\Phi_{\text{SF}}(\{z_1, ..., z_N\}) = 1$ $\to$ unentangled product state

- Local dancing rules of a FQH liquid:

  (1) every electron dances around clock-wise

  (Φ_{FQH} only depends on $z = x + \mathrm{i}y$)

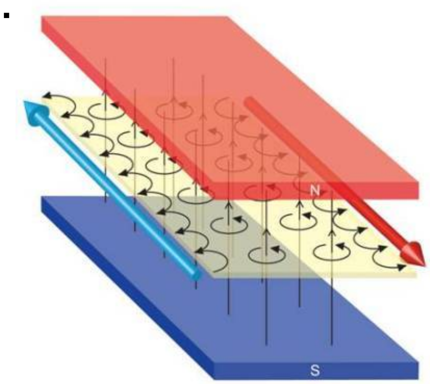
  (2) takes exactly three steps to go around any others

  (Φ_{FQH}'s phase change 6π)

  $\to$ Global dancing pattern $\Phi_{\text{FQH}}(\{z_1, ..., z_N\}) = \prod(z_i - z_j)^3$

- A general theory of multi-layer Abelian FQH state:

$$\prod_{I;i<j}(z_i^I - z_j^I)^{K_{II}} \prod_{I<J;i,j}(z_i^I - z_j^J)^{K_{IJ}} e^{-\frac{1}{4}\sum_{i,I}|z_i^I|^2}$$

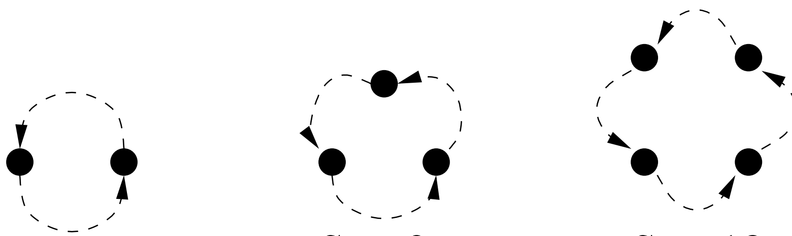


  Low energy effective theory is the K-matrix Chern-Simons theory $L = \frac{K_{IJ}}{4\pi} a_I \mathrm{d} a_J$

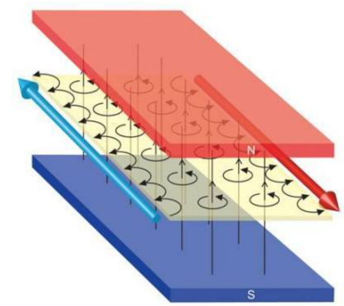
- An integer number of edge modes $c = \dim(K)$ = number of layers.

Even K-matrix classifies all 2+1D Abelian topological order (but not one-to-one)

- A systematic theory of single-layer non-Ableian FQH state:
  Pattern of zeros S_a:
  a-electron cluster has a relative angular momentum S_a Wen-Wang 08



$S_2 = 3$	$S_3 = 9$	$S_4 = 18$	ν=1/3 Laughlin
$S_2 = 1$	$S_3 = 5$	$S_4 = 10$	ν=1/2 Pfaffian



- Local dancing rules are enforced by Hamiltonian to lower energy.
- Only certain sequences S_a correspond to valid FQH states. Which?
- Different POZ S_a give rise to different topological properties

- A fractional number of edge modes $c \neq$ integer.

Sum over a subset of product states

To make topological order, we need to sum over many different product states, but we should not sum over everything.

$\sum_{\text{all spin config.}} |\uparrow\downarrow ..\rangle = |\rightarrow\rightarrow ..\rangle$

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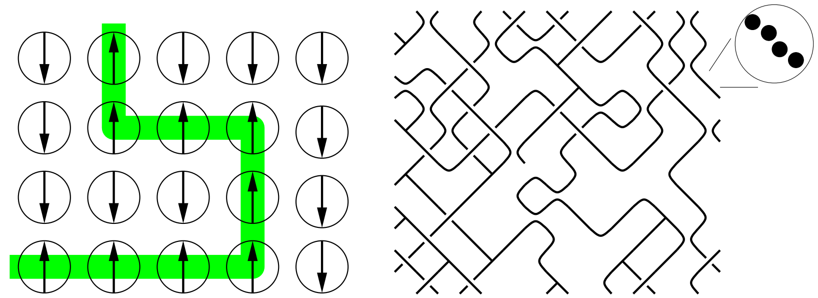
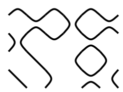
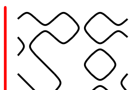
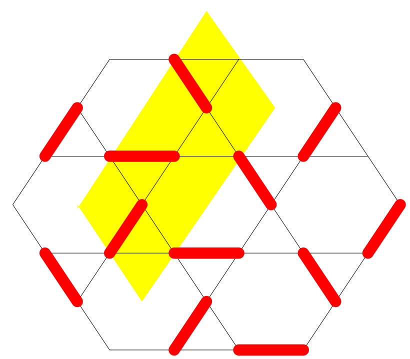
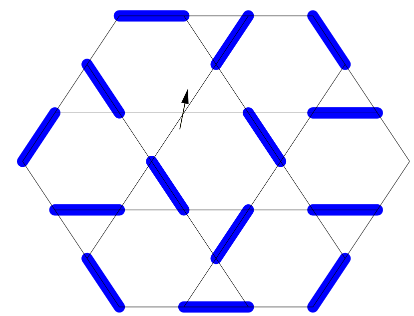


- *sum* over a subset of spin config.:

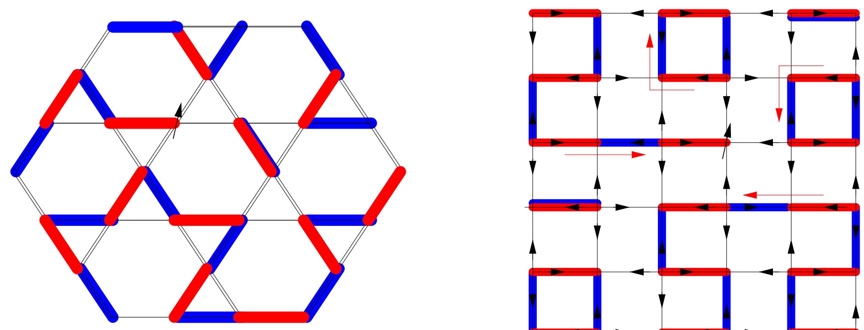
$|\Phi^{Z_2}_{\text{loops}}\rangle = \sum \Big|$  $\Big\rangle$

$|\Phi^{DS}_{\text{loops}}\rangle = \sum (-)^{\#\text{ of loops}} \Big|$  $\Big\rangle$

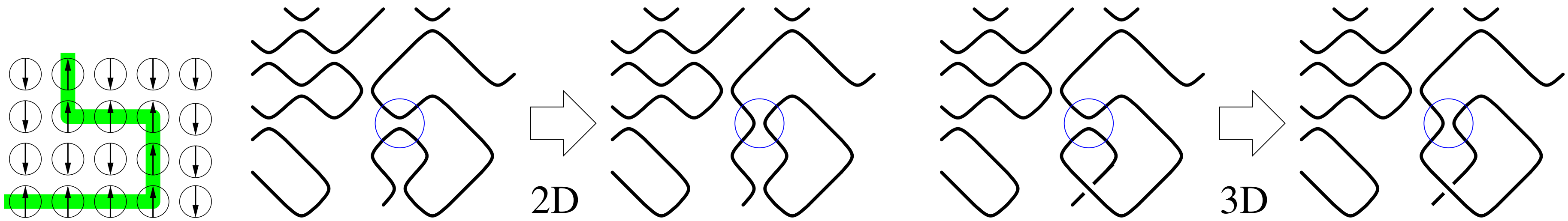


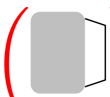
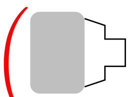
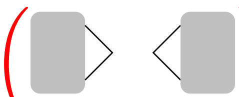
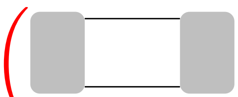
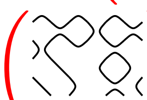


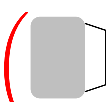
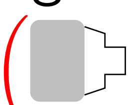
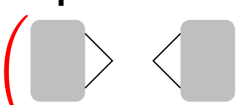
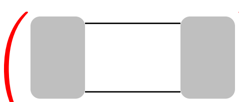
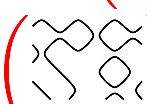
- Can the above wavefunction be the ground states of local Hamiltonians?



Sum over a subset: local rule → global dancing pattern



- Local dancing rules of a string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str}() $= \Phi_{str}$() , Φ_{str}() $= \Phi_{str}$()
  → Global dancing pattern Φ_{str}() $= 1$

- Local dancing rules of another string liquid:
  (1) Dance while holding hands (no open ends)
  (2) Φ_{str}() $= \Phi_{str}$() , Φ_{str}() $= -\Phi_{str}$()
  → Global dancing pattern Φ_{str}() $= (-)^{\#\text{ of loops}}$

- Two string-net condensations → two topological orders: Levin-Wen 05
  Z_2 topo. order Sachdev Read 91, Wen 91 and double-semion topo. order.