Theory of non-Abelian statistics: fusion space of topo. exc.

What are the most general properties of the topological excitations? can be boson, can be fermion, can be semion, ...

Consider a state with quasiparticles $|i_1, i_2, i_3, \cdots\rangle$ at $\vec{x}_1, \vec{x}_2, \vec{x}_3, \cdots$, which is a gapped ground state of

$$H + \delta H^{\text{trap}}_{i_1}(\vec{x}_1) + \delta H^{\text{trap}}_{i_2}(\vec{x}_2) + \delta H^{\text{trap}}_{i_3}(\vec{x}_3) + \cdots$$

- *The ground state subspace of the above Hamiltonian is the* **fusion space** $V^F(i_1, i_2, i_3, \cdots)$ *of the quasiparticles* $i_1, i_2, i_3, \cdots$.
- We assume the above ground state degeneracy is stable arbitary purterbations around $\vec{x}_1, \vec{x}_2, \vec{x}_3, \cdots$ and the traped quasiparticles are said to be **simple**.
- If the ground state subspace is not stable against any perturbations $\delta H(\vec{x}_1)$ near $\vec{x}_1$, then the quasiparticle i_1 at $\vec{x}_1$ is **composite**.
- If i_1 is composite, we can add $\delta H(\vec{x}_1)$ to split the ground state subspace:

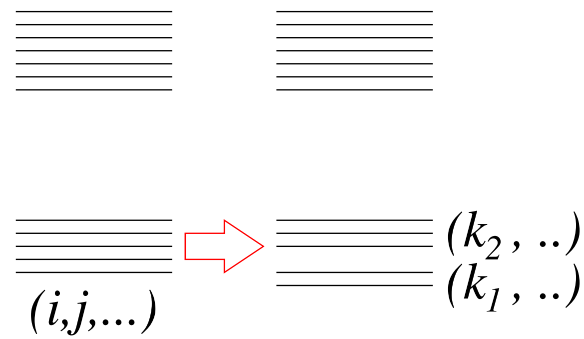
$$V^F(i_1, i_2, i_3, \cdots) \to V^F(j_1, i_2, i_3, \cdots) \oplus V^F(k_1, i_2, i_3, \cdots) \oplus \cdots$$

We denote $i_1 = j_1 \oplus k_1 \oplus \cdots$.

Fusion algebra of (non-Abelian) topological excitations

- For simple i, j, if we view (i, j) as one particle,
  it may correspond to a composite particle:

  $$V^F(i, j, l_1, l_2, \cdots) = \oplus_{\tilde{k}} V^F(\tilde{k}, l_1, l_2, \cdots)$$
  $$= \oplus_k \oplus_{\alpha_k^{ij}=1}^{N_k^{ij}} V^F_{\alpha_k^{ij}}(k, l_1, l_2, \cdots)$$

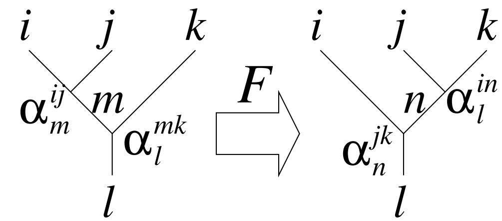
  $i \otimes j = \oplus_k N_k^{ij} k \to$ the *fusion algebra*.

  Associativity:
  $(i \otimes j) \otimes k = i \otimes (j \otimes k) = \oplus_l N_l^{ijk} l, \quad N_l^{ijk} = \sum_m N_m^{ij} N_l^{mk} = \sum_n N_n^{jk} N_l^{in}$

  Quantum dimension and vector space fractionalization:
- In general, we cannot view $V^F(i, j, k, \cdots)$ as
  $V(i) \otimes V(j) \otimes V(k) \otimes \cdots$, and $\dim[V^F(i, i, i, \cdots)] \neq d_i^n, \ d_i \in \mathbb{Z}$.
  Quasiparticle i may carry fractional degree freedom.
  $\dim[V^F(i, i, \cdots, i)] = \sum_{m_i} N_{m_1}^{ii} N_{m_2}^{m_1 i} \cdots N_1^{m_{n-2} i} = (\mathbf{N}^i)_{i1}^{n-1} \sim d_i^n$
  where the matrix $(\mathbf{N}^i)_{jk} = N_k^{ji}$, and d_i the largest eigenvalue of $\mathbf{N}^i$.
- d_i is called the *quantum dimension* of the quasiparticle i.
  Abelian particle $\to d_i = 1$. Non-Abelian particle $\to d_i \neq 1$.

Relation between fusion spaces and the F-matrix

- Two different ways to fuse $i, j, k \to l$:

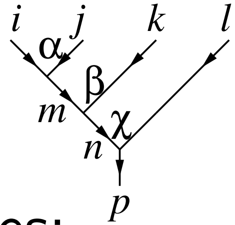


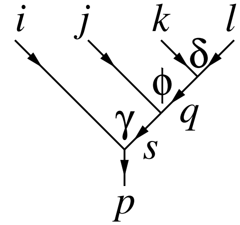
$$V^F(i,j,k,\cdots) = \oplus_m \oplus_{\alpha^{ij}_m=1}^{N^{ij}_m} V^F_{\alpha^{ij}_m}(m,k,\cdots)$$
$$= \oplus_m \oplus_{\alpha^{ij}_m=1}^{N^{ij}_m} \oplus_l \oplus_{\alpha^{mk}_l=1}^{N^{mk}_l} V^F_{\alpha^{ij}_m;\alpha^{mk}_l,m}(l,\cdots)$$
$$= \oplus_l \{|l;\alpha^{ij}_m,\alpha^{mk}_l,m\rangle\} \otimes V^F(l,\cdots)$$
$$V^F(i,j,k,\cdots) = \oplus_n \oplus_{\alpha^{jk}_n=1}^{N^{jk}_n} V^F_{\alpha^{jk}_n}(i,n,\cdots)$$
$$= \oplus_n \oplus_{\alpha^{jk}_n=1}^{N^{jk}_n} \oplus_l \oplus_{\alpha^{in}_l=1}^{N^{in}_l} V^F_{\alpha^{jk}_n;\alpha^{in}_l,n}(l,\cdots)$$
$$= \oplus_l \{|l;\alpha^{jk}_n,\alpha^{in}_l,n\rangle\} \otimes V^F(l,\cdots)$$

- $|l;\alpha^{jk}_n,\alpha^{in}_l,m\rangle = \sum_{n,\alpha^{jk}_n,\alpha^{in}_l} F^{ijk;m,\alpha^{ij}_m,\alpha^{mk}_l}_{l;n,\alpha^{jk}_n,\alpha^{in}_l} |l;\alpha^{jk}_n,\alpha^{in}_l,n\rangle$

  where $\mathbf{F}^{ijk}_l$ is an unitary matrix.

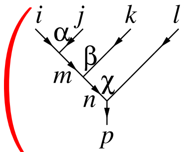
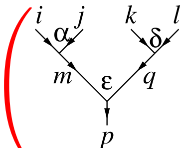
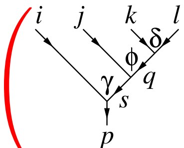
Consistent conditions for $F^{ijk;m\alpha\beta}_{l;n\chi\delta}$ and UFC

Two different ways of fusion 

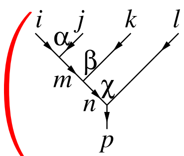
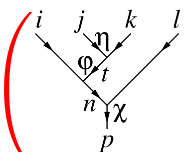
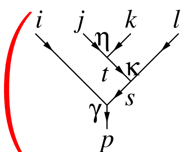
and 

are related via two different paths of F-moves:

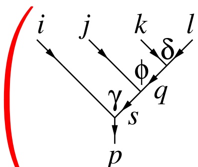
$$\Phi\left(\right) = \sum_{q,\delta,\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon}\Phi\left(\right) = \sum_{q,\delta,\epsilon;s,\phi,\gamma} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}\Phi\left(\right),$$

$$\Phi\left(\right) = \sum_{t,\eta,\varphi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi}\Phi\left(\right) = \sum_{t,\eta,\varphi;s,\kappa,\gamma} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma}\Phi\left(\right)$$

$$= \sum_{t,\eta,\kappa;\varphi;s,\kappa,\gamma;q,\delta,\phi} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi}\Phi\left(\right).$$



The two paths should lead to the same unitary trans.:

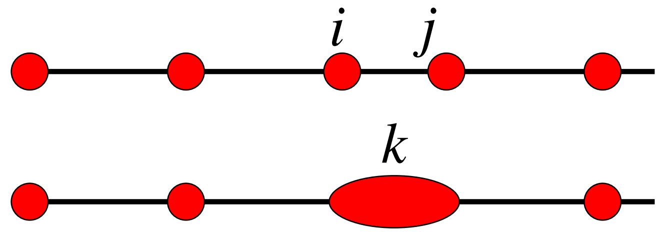
$$\sum_{t,\eta,\varphi,\kappa} F^{ijk;m\alpha\beta}_{n;t\eta\varphi} F^{itl;n\varphi\chi}_{p;s\kappa\gamma} F^{jkl;t\eta\kappa}_{s;q\delta\phi} = \sum_{\epsilon} F^{mkl;n\beta\chi}_{p;q\delta\epsilon} F^{ijq;m\alpha\epsilon}_{p;s\phi\gamma}$$

Such a set of non-linear algebraic equations is the famous pentagon identity. Moore-Seiberg 89

$N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\chi\delta} \to$ **Unitary fusion category (UFC)**

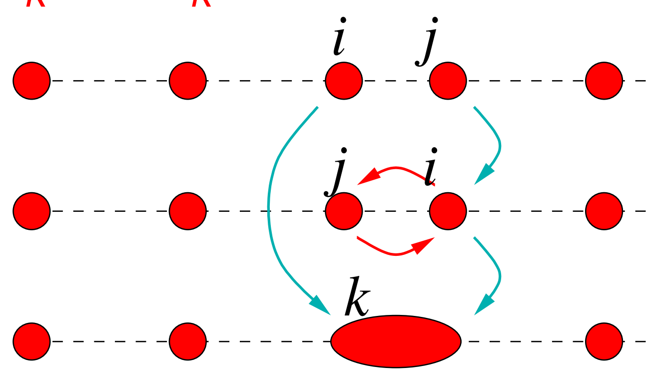
UFC and topological quasiparticles in different dimensions

- Topological excitations in 1+1D are described/classified by (non-Abelian) UFC.



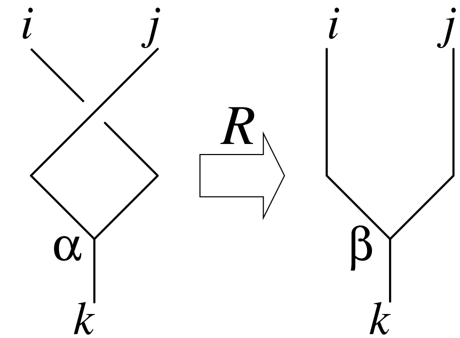
Consider topological excitations described by an arbitary UFC, can we realize them via a 1+1D lattice model?

- Topological excitations in 2+1D (and beyond) are described by Abelian (symmetric) UFC: $N_k^{ij} = N_k^{ji}$.



In higher dimension, topological excitations also have non-trivial braiding properties.

Braiding and R-matrix



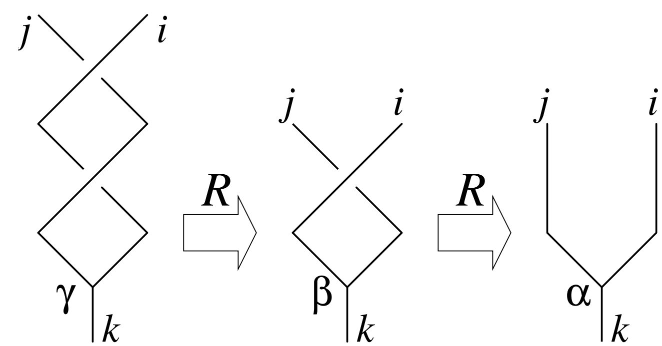
- Two ways to fuse:

$$V^F(i,j,\cdots) = \oplus_{k,\alpha} \tilde{V}^F_\alpha(k,\cdots)$$
$$= \oplus_k \{|k;\alpha\rangle'\} \otimes V^F(k,\cdots)$$
$$V^F(i,j,\cdots) = \oplus_{k,\beta} V^F_\beta(k,\cdots)$$
$$= \oplus_k \{|k;\beta\rangle\} \otimes V^F(k,\cdots)$$

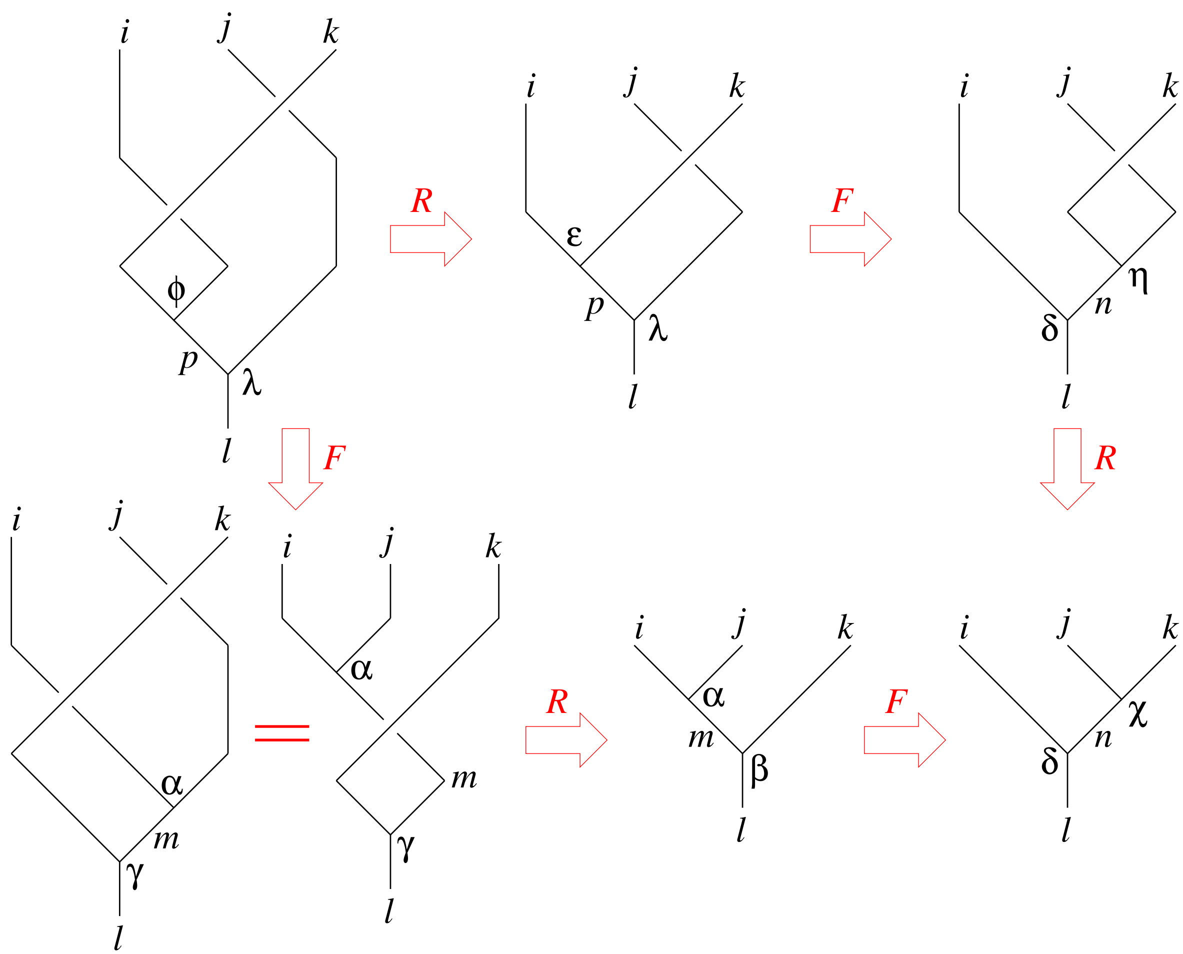
- $|k,\alpha\rangle' = \sum_\beta R^{ij;\alpha}_{k;\beta}|k,\beta\rangle$
  where $R^{ij;\alpha}_{k;\beta}$ is an unitray matrix.
- Relation to the spin $\theta_i = e^{i2\pi s_i}$ of the particle:

  2π rotation of (i,j) = 2π rotation of k
  2π rotation of (i,j) = 2π rotation
  of i and j and exchange i,j twice

$$\theta_i\theta_j R^{ij;\gamma}_{k;\beta} R^{ji;\beta}_{k;\alpha} = \theta_k \delta_{\gamma\alpha}$$



Consistent conditions for $R^{ij;\alpha}_{k;\beta}$ and UMTC



Hexagon identity:

$$R^{ik;\phi}_{p;\epsilon} F^{ikj;p\epsilon\lambda}_{l;n\eta\delta} R^{jk;\eta}_{n;\chi} = \sum_{m\alpha\beta} F^{kij;p\phi\lambda}_{l;m\alpha\gamma} R^{mk;\gamma}_{l;\beta} F^{ijk;m\alpha\beta}_{l;n\chi\delta}$$

$N^{ij}_k, F^{ijk;m\alpha\beta}_{l;n\chi\delta}, R^{ij;\alpha}_{k;\beta} \rightarrow$ Unitary modular tensor category (UMTC) which describes non-Abelian statistics of 2+1D topo. excitations.

Boundary of topological order $\rightarrow$ gravitational anomaly

- Boundary of (some) topologically ordered states is gapless

Boundary of topological order → gravitational anomaly

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- Boundary of topologically ordered states has gravitational anomaly

Boundary of topological order $\rightarrow$ gravitational anomaly

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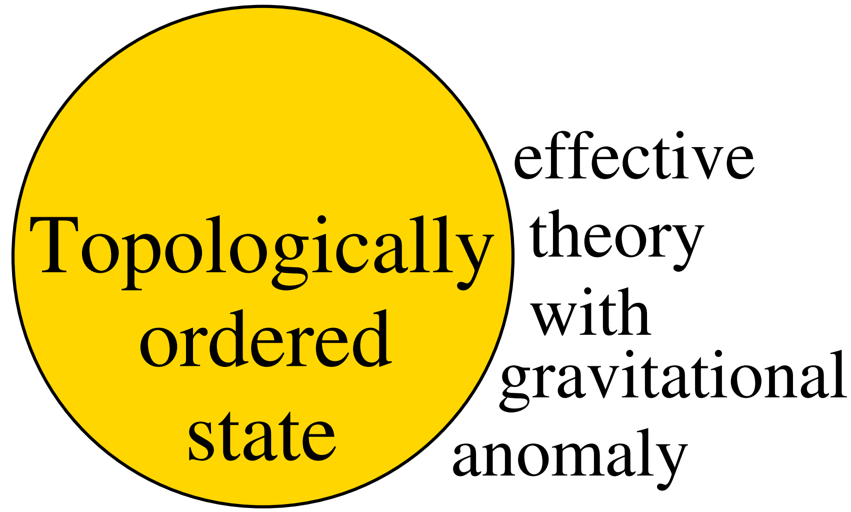
There is an one-to-one correspondence between d-dimensional topological orders and $d-1$-dimensional gravitational anomalies



Example 1 (gapless):

- 1+1D chiral fermion $L = i(\psi^\dagger \partial_t \psi - \psi^\dagger \partial_x \psi) \rightarrow \epsilon(k) = vk$. Gravitational anomalous, cannot appear as low energy effective theory of any well-definded local 1+1D lattice model.
- But the above chiral fermion theory cannot appear as low energy effective theory for the boundary of a 2+1D topologically ordered state – the $\nu = 1$ IQH state (which has no *topological excitations*).
- The same bulk $\rightarrow$ many different boundary of the same gravitational anomaly, e.g. 3 edge modes $(v_1 k, -v_2 k, v_3 k)$

Example 2 (gapless):

- 1+1D chiral boson (8 modes $c = 8$)

$L = \frac{K^{E_8}_{IJ}}{2\pi}\partial_x\phi_I\partial_t\phi_J - V_{IJ}\partial_x\phi_I\partial_x\phi_J.$

$$K^{E_8} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

- Gravitational anomalous.
  Realized as edge of
  8-layer bosonic QH state:
  $\Psi_{E_8} = \prod (z_i^I - z_j^J)^{K_{IJ}}$
  Filling fraction $\nu = 4$
  $det(K^{E_8}) = 1$ → no topo. exc.

Example 3 (gapped):

- 2+1D theory with excitations $(1, e, m, \epsilon)$. Fusion: $e \times e = m \times m = \epsilon \times \epsilon = 1$, $e \times m = \epsilon$. Braiding: e, m, ϵ have mutual π statistics, e, m are boson ϵ is fermion.
- No gravitational anomaly. Can be realized by the toric code model.

Example 4 (gapped):

- 2+1D theory with excitations $(1, e)$. $e \times e = 1$. e is a boson.
- Grav. anomalous. Cannot be realized by any 2D lattice model.
  But can be realized as the 2D boundary of 3+1D toric code model.

Example 5 (gapped):

- 2+1D theory with excitations $(1, e)$. $e \times e = 1$. e is a semion.
  No grav. anomaly. Can be realized by $\nu = 1/2$ bosonic Laughlin state.

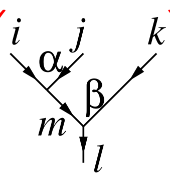
Entanglement = Geometry

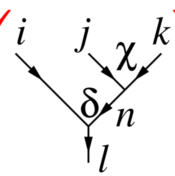
- The boundary of topologically ordered states has *gravitational anomaly*. Topological orders (patterns of long-range entanglement) classify gravitational anomalies in one lower dimension.
  long-range entanglement $\leftrightarrow$ **geometry**

Classify long-range entanglement and topological order

- 1+1D: there is no topological order Verstraete-Cirac-Latorre 05

- 2+1D: Abelian topological order are classified by K-matrices
  2+1D: topological orders are classified by $(UMTC, c) = (T, S, c)$?
  2+1D: topo. order with gappable edge are classified by unitary fusion categories (UFC): $\mathcal{Z}(UFC) = UMTC$ Levin-Wen 05

$\Phi\Bigg($ 

$\Bigg) = \sum F^{ijk;m\alpha\beta}_{l;n\chi\delta} \Phi\Bigg($ 

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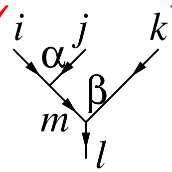
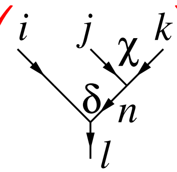
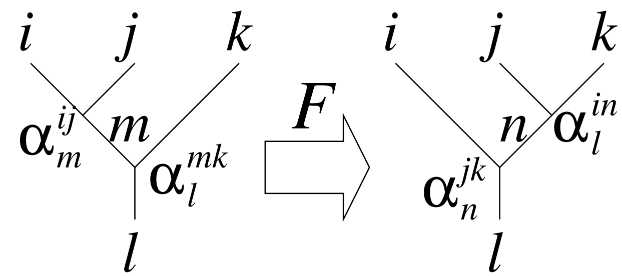
Φ () $= \sum F^{ijk;m\alpha\beta}_{l;n\chi\delta}\,\Phi$ ()



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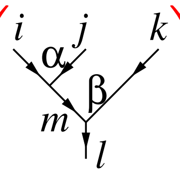
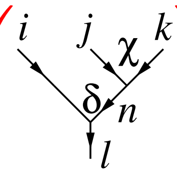
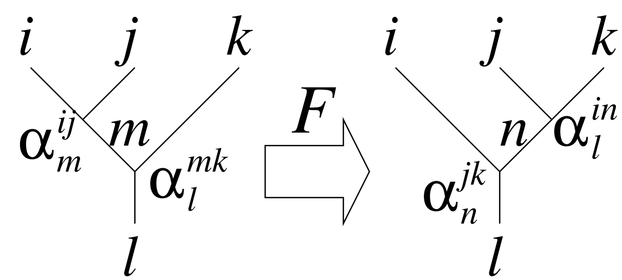
$\Phi\Big($  $\Big) = \sum F^{ijk;m\alpha\beta}_{l;n\chi\delta}\,\Phi\Big($  $\Big)$

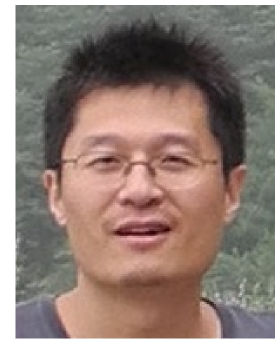


- **Topo. order with no non-trivial topo. excitations**: Kong-Wen 14

	$1+1D$	$2+1D$	$3+1D$	$4+1D$	$5+1D$	$6+1D$
Boson:	0	$\mathbb{Z}$ $_{E_8}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}\oplus\mathbb{Z}$
Fermion:	$\mathbb{Z}_2$	$\mathbb{Z}$ $_{p+ip}$	?	?	?	?

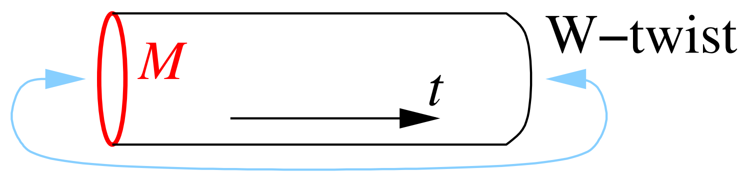
Volume-ind. partition function – Universal topo. inv.

- Assume the space-time $= M \rtimes S^1_t$ (a fiber bundle over S^1_t).
  Such a fiber bundle is described an element in $\widehat{W} \in \text{MCG}(M)$.
  So we denote space-time $= M \rtimes_{\widehat{W}} S^1_t$



- Volume-ind. (fixed-point) partition function Kong-Wen 14
  $$Z(M \rtimes_{\widehat{W}} S^1_t) = Z_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) e^{-\epsilon_{\text{grnd}} V_{\text{space-time}}}$$
  $$Z_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) = \mathrm{Tr}(W)$$

- $Z_{\text{vol-ind}}(M \times S^1_t) =$ the ground state degeneracy on space M.

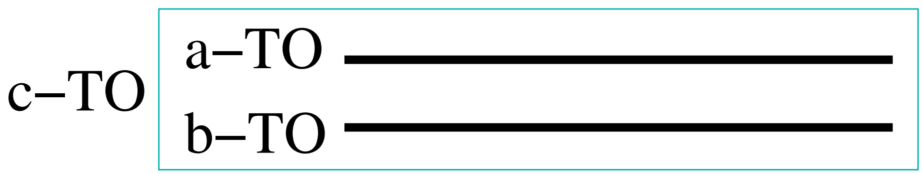


$Z_{\text{vol-ind}}(S^d \times S^1_t) = 1$
$Z_{\text{vol-ind}}(S^{d-1} \times S^1 \times S^1_t) =$ number of topological particle types.

Volume-ind. partition function, universal wave function overlap, and non-Abelian geometric phases are the same type of topological invariants for topologically ordered states

Monoid and group structures of topological orders

- Let $\mathcal{C}_d = \{a, b, c, \cdots\}$ be a set of topologically ordered phases in d dimensions.
  Stacking a-TO state and b-TO state $\to$ a c-TO state:
  $a \boxtimes b = c, \quad a, b, c \in \mathcal{C}_d$



- $\boxtimes$ make $\mathcal{C}_d$ a monoid (a group without inverse).
  Consider topological order a and topological order a^*
  $Z^{a^*}_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) = [Z^{a}_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t)]^*$, then
  $Z^{a\boxtimes a^*}_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) = Z^{a}_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) Z^{a^*}_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t)$
  In general, $Z^{a}_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) Z^{a^*}_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) \neq 1 \to a \boxtimes a^*$ is a non trivial topological order, and a-TO has no inverse.
- A topological order is invertible iff its $Z_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) = e^{i\theta}$
  A topological order is invertible iff it has no topological excitations.

Kong-Wen 14

Classify invertible bosonic topo. order (with no topo. exc.)

In 2+1D:

- $Z_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) = e^{i\frac{2\pi c}{24}\int_{M\rtimes_{\widehat{W}} S^1_t}\omega_3(g_{\mu\nu})}$ where ω_3 is the gravitational Chern-Simons term: $d\omega_3 = p_1$ and p_1 is the first Pontryagin class.
- The quantization of the topological term: $c = 8 \times \text{int.} \to \mathbb{Z}$-class: $\int_M \omega_3(g_{\mu\nu}) = \int_{N,\partial N=M} p_1 = \int_{N',\partial N'=M} p_1 \text{ mod } 3$, since $\int_{N_{\text{closed}}} p_1 = 0 \text{ mod } 3$.
- Relation to gravitational anomaly on the boundary B^2:
  (1) $Z = e^{i\int_{B^2} L^{\text{bndry}}_{\text{eff}}(g_{\mu\nu})} e^{i\frac{2\pi c}{24}\int_{M^3,\partial M^3=B^2}\omega_3(g_{\mu\nu})}$
  $e^{i\frac{2\pi c}{24}\int_{M^3,\partial M^3=B^2}\omega_3(g_{\mu\nu})}$ is not differomorphism invariant, but $e^{i\int_{B^2} L^{\text{bndry}}_{\text{eff}}(g_{\mu\nu})} e^{i\frac{2\pi c}{24}\int_{M^3,\partial M^3=B^2}\omega_3(g_{\mu\nu})}$ is.
  (2) Consider an 1+1D differomorphism $W: B^2 \to B^2$, $g_{\mu\nu} \to g^W_{\mu\nu}$.

$$\int_{B^2} L^{\text{bndry}}_{\text{eff}}(g^W_{\mu\nu}) - \int_{B^2} L^{\text{bndry}}_{\text{eff}}(g_{\mu\nu}) = \frac{2\pi c}{24}\int_{B^2\rtimes_W S^1} \omega^3(g_{\mu\nu})$$

Classify invertible bosonic topo. order (with no topo. exc.)

In 4+1D:

- $Z_{\text{vol-ind}}(M \rtimes_{\widehat{W}} S^1_t) = e^{i\pi \int_{M \rtimes_{\widehat{W}} S^1_t} w_2 w_3}$ where w_i is the i^{th} Stiefel-Whitney class $\to \mathbb{Z}_2$-class. We find $\int_{M \rtimes_{\widehat{W}} S^1_t} w_2 w_3 = 1$ when $M = \mathbb{C}P^2$ and $W: \ \mathbb{C}P^2 \to (\mathbb{C}P^2)^*$
- Global grav. anomaly: for $M = \mathbb{C}P^2$ and $W: \ \mathbb{C}P^2 \to (\mathbb{C}P^2)^*$

$$\int_M L^{\text{bndry}}_{\text{eff}}(g^W_{\mu\nu}) - \int_M L^{\text{bndry}}_{\text{eff}}(g_{\mu\nu}) = \int_{M \rtimes_W S^1} w_2 w_3$$

In 6+1D:

- Two independent grav. Chern-Simons terms:

$Z_{\text{vol-ind}}(M^7) = e^{2\pi i \int_{M^7} \left[k_1 \frac{\tilde\omega_7 - 2\omega_7}{5} + k_2 \frac{-2\tilde\omega_7 + 5\omega_7}{9} \right]}$
where $d\omega_7 = p_2$, $d\tilde\omega_7 = p_1 p_1 \to \mathbb{Z} \oplus \mathbb{Z}$-class (k_1, k_2). Kong-Wen 14

	$1+1D$	$2+1D$	$3+1D$	$4+1D$	$5+1D$	$6+1D$
Boson:	0	$\mathbb{Z}$ $_{E_8}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z} \oplus \mathbb{Z}$
Fermion:	$\mathbb{Z}_2$	$\mathbb{Z}$ $_{p+ip}$	?	?	?	?